UNIT - III

Free and Forced Vibrations

Free vibrations :-

When the bob of simple pendulum in displaced from its mean position and left, it executes simple harmonic motion. The time peroid of oscillation depends only on the length of the pendulum and the accelerations due to gravity at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. In such cases these is no loss of energy be friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude of swing remain's Constant.

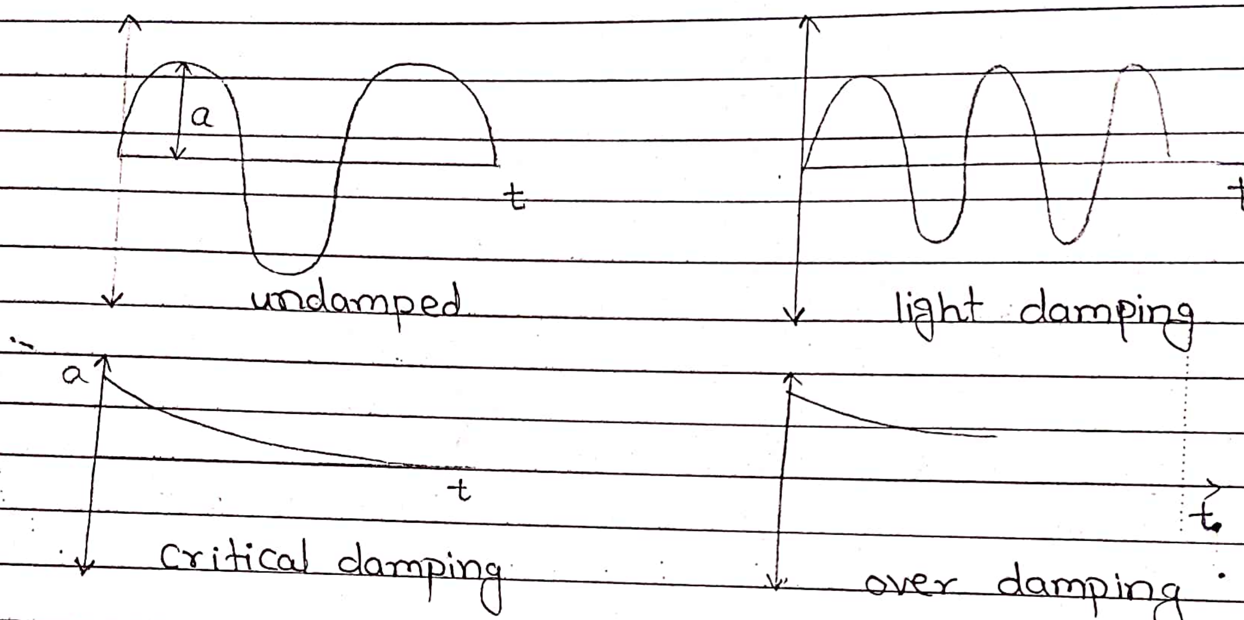
Forced Vibrations :-

The time peroid of a body executing simple harmonic motion depends on the dimensions of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, it continues to oscillate under the influence of such external forces. Such vibrations of the body are called forced vibrations.

Damped vibrations :-

In actual practice, When the pendulum vibrates in air medium, there are frictional forces and consequently energy energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. such vibrations are called free damped vibrations.

The decipition of energy is called damping effect.



- Consider a simple pendulum of vibrating string When displaced from mean position would return it due to restoring force. but the momentum acquire carries it from mean position to other side. till it kinetic energy is wholly converted into potential energy and the system comes to rest.
- Again the force brings its back and too and fro motion is repeated in all such cases. and interchanges of K.E to P.E and vice-versa is taking place.

• Here, assuming that neigther energy is received nor decipated into heat by internal friction we have low of conservation of energy that the some of kinetic energy and potential energy will be constant.

For a simple harmonically vibrating particle, the kinetic energy for displacement y. is given by

$$K.E = \frac{1}{2} m v^2$$

$$\boxed{K.E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2} \quad \text{-----(I)}$$

$$P.E = \frac{1}{2} k y^2$$

$$\boxed{P.E = \frac{1}{2} k y^2} \quad \text{-----(II)}$$

Total energy [I + II]

$$T.E = K.E + P.E$$

$$\boxed{T.E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \frac{1}{2} k y^2} \quad \text{---(III)}$$

T.E is the same instant, the potential energy of the particle is $\frac{1}{2} k y^2$.

Where,

k is the restoring force per unit displacement

For an undamped harmonic oscillator, this total energy remains constant.

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$$\Rightarrow \frac{1}{2} m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2} Ky^2 = \text{constant} \quad \text{---- (IV)}$$

$$\Rightarrow \boxed{m\left(\frac{dy}{dt}\right)^2 + ky^2 = 0} \quad \text{---- (V)}$$

eqn (V) Diff' w.r.t.o time, we get

$$\frac{dy}{dt} = 2m\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right) + 2ky\left(\frac{dy}{dt}\right) = 0$$

$$\frac{dy}{dt} = m\left(\frac{dy}{dt}\right)\left(\frac{d^2y}{dt^2}\right) + ky\left(\frac{dy}{dt}\right) = 0$$

$$\frac{dy}{dt} = m\left(\frac{d^2y}{dt^2}\right) + ky = 0$$

$$\boxed{\frac{dy}{dt} = \frac{d^2y}{dt^2} + \left(\frac{k}{m}\right)y = 0} \quad \text{---- (VI)}$$

Equation (VI) is similar to the equation

$$\boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0} \quad \text{---- (VII)}$$

Here $\omega^2 = \left(\frac{k}{m}\right)$

The solution for equation (VII) is

$$y = a\sin(\omega t - \alpha)$$

$$y = a\sin\left(\sqrt{\frac{k}{m}}\, t - \alpha\right)$$

frequancy $(n) = \frac{\omega}{2\pi}$

$$n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Thus, in the case of undamped free vibrations, the differential equation is

$$\boxed{\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right)y = 0} \quad \text{-----} \quad (VIII)$$

This is only an ideal case. In the first chapter, for the motion of a pendulum, loaded spring, LC circuit etc. it has been assumed that the vibrations are free and undamped.

Damped simple harmonic motion :

- In undamped vibration the vibratory system is not subject to any frictional forces envolving loss of energy and therefore it would continuous to vibrate for an idefinite time with a constant amplitude.
- In actual practice we find that this does not go on forever the from of tunnic fork vibrate with swing decreasing amplitude.
- The pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration.

- Consider the motion of simple pendulum is subjected to a restoring force proportional to displacement and resosing force or fricational force varing as the velocity such motion are called damped oscillations.

All vibrating body in the medium are subject to these forces.

- Let us, suppose that the restoring force is given by

$$F = -\varepsilon\left(\frac{dy}{dt}\right)$$

Where,

ε is constant depanding upon the diamention of the body.

Negative sign indecates that the force opposite the direction of y

The rate of change in workdone againsd in friction is

$$W.D = F\left(\frac{dy}{dt}\right)$$

$$= -\varepsilon\left(\frac{dy}{dt}\right)^2$$

The rate of total energy of the system (K.E + P.E) would be decipated.

$$\frac{d}{dt}\left[(K.E + P.E)\right] = W.D$$

$$\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{1}{2}Ky^2\right] = -\varepsilon\left(\frac{dy}{dt}\right)^2$$

$$m \cdot \frac{dy}{dt} \cdot \frac{d^2y}{dt^2} + Ky\frac{dy}{dt} = -r\left(\frac{dy}{dt}\right)^2$$

$$m\frac{d^2y}{dt^2} + Ky = -r\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} + \frac{Ky}{m} = \frac{-r}{m}\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y + \frac{r}{m}\frac{dy}{dt} = 0$$

put $\frac{k}{m} = n^2$ and $\frac{r}{m} = 2k$

$$\frac{d^2y}{dt^2} + n^2y + 2k\frac{dy}{dt} = 0$$

$$\boxed{\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2y = 0} \quad \text{-----} \ (1)$$

$$y = e^{\alpha t}$$

above eq^n diff' w.r.to time 't', we get

$$\frac{dy}{dt} = \alpha e^{\alpha t}$$

again diff' w.r.to time 't' we get

$$\frac{d^2y}{dt^2} = \alpha^2 e^{\alpha t}$$

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above value put in equation in 1

$$\alpha^2 e^{\alpha t} + 2k\alpha e^{\alpha t} + n^2 e^{\alpha t} = 0$$

$$\alpha^2 + 2k\alpha + n^2 = 0$$

Comparing eq^n $ax^2 + bx + c = 0$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2k \pm \sqrt{4k^2 - 4n^2}}{2}$$

$$= -k \pm \sqrt{k^2 - n^2}$$

$$= -k + \sqrt{k^2 - n^2} \quad \text{or} \quad -k - \sqrt{k^2 - n^2}$$

$$= \sqrt{k^2 - n^2} - k \quad \text{or} \quad -\sqrt{k^2 - n^2} - k$$

$$y = A' e^{(-k + \sqrt{k^2 - n^2})t}$$

$$y = B' e^{(-k - \sqrt{k^2 - n^2})t}$$

$$y - A' e^{(-k + \sqrt{k^2 - n^2})t}$$

$$y - B' e^{(-k - \sqrt{k^2 - n^2}).t}$$

Adding the above equation

$$2y - A' e^{(-K+\sqrt{K^2-n^2})t} - B' e^{(-K-\sqrt{K^2-n^2})t}$$

$$y - \frac{A'}{2} e^{(-K+\sqrt{K^2-n^2})t} - \frac{B'}{2} e^{(-K-\sqrt{K^2-n^2})t}$$

put $\frac{A'}{2} = A$

and $\frac{B'}{2} = B$

$$\boxed{y - A e^{(-K+\sqrt{K^2-n^2})t} - B e^{(-K-\sqrt{K^2-n^2})t}} \quad \text{----- (11)}$$

put,

$y = y_0$ When $t=0$

$\frac{dy}{dt} = 0$ When $t=0$

$$\boxed{y_0 = A + B} \quad \text{----- (a)}$$

$$y = A e^{(-K+\sqrt{K^2-n^2})t} + B e^{(-K-\sqrt{K^2-n^2})t}$$

$$\frac{dy}{dt} = A(-K+\sqrt{K^2-n^2}) e^{(-K+\sqrt{K^2-n^2})t}$$
$$+ B(-K-\sqrt{K^2-n^2}) e^{(-K-\sqrt{K^2-n^2})t}$$

$\frac{dy}{dt} = 0$, $t=0$

$$0 = (-k + \sqrt{k^2 - n^2})A + (-k - \sqrt{k^2 - n^2})B$$

$$(-k + \sqrt{k^2 - n^2})A = -(-k - \sqrt{k^2 - n^2})B$$

$$(-k + \sqrt{k^2 - n^2})A = (k + \sqrt{k^2 - n^2})B$$

$$A = \frac{(k + \sqrt{k^2 - n^2})B}{(-k + \sqrt{k^2 - n^2})}$$

$$-kA + \sqrt{k^2 - n^2}\,A = Bk + \sqrt{k^2 - n^2}\,B$$

$$-k(A+B) + \sqrt{k^2 - n^2}\,(A-B) = 0$$

$$-ky_0 + \sqrt{k^2 - n^2}\,(A-B) = 0$$

$$\boxed{A - B = \frac{ky_0}{\sqrt{k^2 - n^2}}} \quad \text{---- (b)}$$

from (a) & (b)

$$2A = y_0 + \frac{ky_0}{\sqrt{k^2 - n^2}}$$

$$A = \frac{y_0}{2}\left(1 + \frac{k}{\sqrt{k^2 - n^2}}\right)$$

$$2B = y_0 - \frac{ky_0}{\sqrt{k^2 - n^2}}$$

$$B = \frac{y_0}{2}\left(1 - \frac{1}{\sqrt{k^2 - n^2}}\right)$$

$$y = Ae^{(-k + \sqrt{k^2 - n^2})t} + Be^{(-k - \sqrt{k^2 - n^2})t}$$

$$\boxed{y = \frac{y_0}{2}\left[\left(1 + \frac{k}{\sqrt{k^2 - n^2}}\right)e^{(-k + \sqrt{k^2 - n^2})t} + \left(1 - \frac{1}{\sqrt{k^2 - n^2}}\right)e^{(-k - \sqrt{k^2 - n^2})t}\right.}$$

---- (iii)

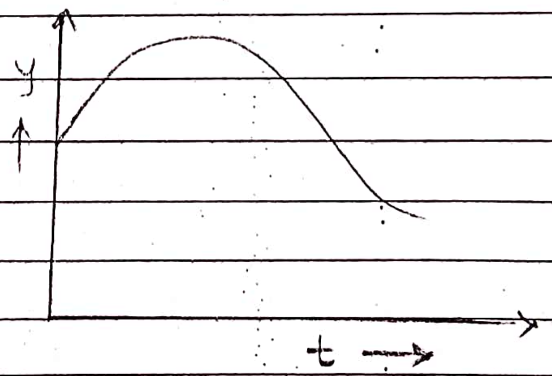
A periodic motion :-

This will happen when there are large frictional forces

ie $k > n$

or $k^2 > n^2$

The roots of equation (V) will be real and negative. The expression for shows that the displacement after passing its Ist maximum decays expontially to zero.
The retarding force is so great that the particle does not vibrate continuously.
ie The value of y does not change sign but falls from its maximum value to zero in an infinite site. Such motion is called a periodic or beat periodic motion.



Critically Damped oscillatory Motion :-

In critically damped motion

$k = n$

and $k^2 = n^2$

Therefore, In this case in equation (iv). The two of the coefficient becomes infinite. Here,

We assume that

$\sqrt{k^2-n^2}$ has not vanished but is reduced to small quantity 'h'.

From equation (iii)

$$y = A e^{(-k+h)t} + B e^{(-k-h)t}$$

$$y = e^{(-kt)} [A e^{ht} + B^{-ht}]$$

e^{ht} and e^{-ht} are written by series form

$$e^{ht} = 1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \text{-----}$$

$$e^{-ht} = 1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} \text{-----}$$

$$y = e^{-kt} \left[A \left(1 + ht + \frac{h^2t^2}{2!} + \text{----} \right) + B \left(1 - ht + \frac{h^2t^2}{2!} \text{---} \right) \right]$$

$$y = e^{-kt} [A(1+ht) + B(1-ht)]$$

$$y = e^{-kt} [(A+B) + (A-B) ht]$$

put $(A+B) = G$

and $h(A-B) = H$

$y = e^{-kt}[y_0 + ky_0]$

t is w.r.t. time

$$\boxed{y = e^{-kt} y_0 [1+k]} \quad \text{----} \; (v)$$

This is equation in critically damped oscillatory motion.

Oscillatory Motion :-

$k < n$

and

$k^2 < n^2$

In this case frictional force are small.

Therefore the root of equation (iv) in imaginary $\sqrt{k^2-n^2}$ is $i\beta$

$$\sqrt{k^2-n^2} = \sqrt{-1}\,\sqrt{n^2-k^2}$$

$\sqrt{k^2-n^2}$ - imagenary equation

$$\sqrt{k^2-n^2} = i\beta$$

Equation (vi)

$$y = \frac{y_0}{2}\left[\left(1+\frac{k}{\sqrt{k^2-n^2}}\right)e^{(-k+\sqrt{k^2-n^2})t} + \frac{y_0}{2}\left(1-\frac{k}{\sqrt{k^2-n^2}}\right)e^{(-k-\sqrt{k^2-n^2})t}\right]$$

$$y = e^{-kt}[G + Ht]$$

$$\boxed{y = e^{-kt}G + e^{-kt}Ht} \quad \text{---} \ (vi)$$

Diff' equation (vi) wrto time 't', we get

$$\frac{dy}{dt} = -ke^{-kt}G + \left[-ke^{-kt}Ht + He^{-kt}\right]$$

$$\frac{dy}{dt} = -kGe^{-kt} - ke^{-kt}Ht + He^{-kt}$$

put $\frac{dy}{dt} = 0$ when $t = 0$

$$0 = -kGe^{-k(0)} - ke^{-k(0)}H(0) + He^{-k(0)}$$

$$0 = -kG - 0 + H$$

$$0 = -kG + H$$

$$\boxed{-Gk + H = 0}$$

$$-ky_0 + H = 0$$

$$\boxed{H = ky_0}$$

$$\boxed{G = y_0}$$

This value put in equation (vi), we get

$$y = e^{-kt}y_0 + e^{-kt}ky_0t$$

$$y = e^{-kt}[y_0 + ky_0t]$$

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$$y = y_0\left[\frac{1}{2}\left(1+\frac{k}{i\beta}\right)e^{(-k+i\beta)t} + \frac{1}{2}\left(1-\frac{k}{i\beta}\right)e^{(-k-i\beta)t}\right]$$

$$= y_0 e^{-kt}\left[\left(\frac{1}{2}+\frac{k}{2i\beta}\right)e^{(i\beta)t} + \left(\frac{1}{2}-\frac{k}{2i\beta}\right)e^{(-i\beta)t}\right]$$

$$= y_0 e^{-kt}\left[\frac{e^{(i\beta)t}}{2} + \frac{k\,e^{(i\beta)t}}{2i\beta} + \frac{e^{(-i\beta)t}}{2} - \frac{k\,e^{(-i\beta)t}}{2i\beta}\right]$$

$$= y_0 e^{-kt}\left[\frac{e^{(i\beta)t}+e^{-(i\beta)t}}{2} + \frac{k}{\beta}\,\frac{e^{(i\beta)t}-e^{(-i\beta t)}}{2i}\right]$$

$$y = y_0 e^{-kt}\left[(\cos\beta t) + \frac{k}{\beta}(\sin\beta t)\right]$$

$$y = y_0 e^{-kt}\left[\frac{\cos\beta t + k\sin\beta t}{\beta}\right]$$

above equation put in values

$\beta = a'\sin\phi'$

and

$k = a'\cos\phi'$

$$y = y_0 e^{-kt}\left[\frac{a'\sin\phi'\cos\beta t + a'\cos\phi'\sin\beta t}{a'\sin\phi'}\right]$$

$$y = \frac{y_0 e^{-kt}}{a'\sin\phi'}\left[a'\sin\phi'\cos\beta t + a'\cos\phi'\sin\beta t\right]$$

$$= \frac{y_0 e^{(-kt)}}{\beta} \left[a' \sin\phi' \cos\beta t + a' \cos\phi' \sin\beta t \right]$$

$$= \frac{y_0 e^{(-kt)}}{\beta} . a' \left\{ \sin\phi' \cos\beta t + a' \cos\phi' \sin\beta t \right\}$$

$$y = \frac{a' y_0 e^{(-kt)}}{\beta} \left[\sin(\beta t + \phi') \right]$$

put $a = \frac{y_0 a'}{\beta}$

$$\boxed{y = a e^{(-kt)} \left[\sin(\beta t + \phi') \right]} \quad ---- \text{(VII)}$$

Where,

$$\beta = \sqrt{n^2 - k^2}$$

k - is the damping coefficient

~~Requi~~ eq^n (VII) Represents simple harmonic curve which the amplitude ae^{-kt} decreases exponentially to zero with the increasing time. whose time period is given by

$$T = \frac{2\pi}{\beta}$$

Since the term

$$\sin(\beta t + \phi') = \pm 1.$$

Therefore,

The displacement 'y' lies b/w

$$y = \pm a e^{(-kt)}$$

* Note :-

- $T = \frac{2\pi}{\beta}$, $N = \frac{\beta}{2\pi}$

Where,

$$\beta = \sqrt{n^2 - k^2}$$

- Damping co-efficient

$$k = \frac{f}{p} \log_{10}\left(\frac{a_o}{a_p}\right)$$

Where,

a_o = initial amplitude
a_p = decreasing amplitude
p = No. of oscillation
f = frequency.

- Energy of damped vibration

$$E = \frac{1}{2} m\beta^2 a^2 e^{-2kt}$$

Effect of damping on frequency :-

The existance of damping produces in two effects

(I) Decreasing amplitude
(II) Increasing time period.

In the case of undamped oscillation periodic time peroid and frequency is

$$T = \frac{2\pi}{n}$$

and $$N = \frac{n}{2\pi}$$

Where as in case of damped oscillation

$$T = \frac{2\pi}{\beta}$$

and $$N = \frac{\beta}{2\pi}$$

The differeance between k, n & $\sqrt{k^2-n^2}$ is very small quantity.

Resonance and Sharpness of Resonance :-

In this case forced vibrations, The general solⁿ for the displacement any instant is given by

$$y = ae^{-bt}\sin(\omega t - \alpha) + \frac{F}{\sqrt{\mu^2p^2 + (k - mp^2)^2}}\sin(pt - \alpha)$$

In the effect of viscocity of the medium is small, the amplitude,

$$\frac{F}{\sqrt{\mu^2p^2 + (k - mp^2)^2}}$$

under the action of the driving force is maximum when the denominator is minimum. It is possible

if $k - mp^2 = 0$

$$\text{or} \quad k = mp^2$$

$$k^2 = m^2p^4$$

$$\frac{k^2}{m^2} = p^4$$

$$\sqrt{\frac{k}{m}} = p$$

Further, the amplitude will be infinite if μ is also zero. The oscillations will have maximum amplitude and this state of vibration of a system is called resonance. It means that, when the forced frequency is equal to the natural frequency of vibration of the body, resonance takes place If friction is present, the amplitude at resonance

$$= \frac{F}{\mu p}$$

$$= \frac{F}{\mu \sqrt{k/m}}$$

or amplitude at resonance

$$= \frac{F}{\mu}\sqrt{\frac{m}{k}}$$

In the case of Sound, the study of resonance is of great importance. sharpness to the fall in amplitude with change in frequency on each side of the maximum amplitude.

The particular solution for displacement in the case of forced vibration is,

$$\boxed{y = \frac{F}{\sqrt{\mu^2 p^2 + (k - mp^2)^2}} \sin(pt - \alpha)} \quad \text{--- (1)}$$

Differentiating equation (1) wrto time

$$\boxed{\frac{dy}{dt} = \frac{FP}{\sqrt{\mu^2 p^2 + (k - mp^2)^2}} \cos(pt - \alpha)} \quad \text{----- (11)}$$

The velocity (dy/dt) is maximum when $\cos(pt-\alpha)$ is maximum.

ie the instant at which the particle crosses the mean position.

$$\boxed{\left(\frac{dy}{dt}\right)_{max} = \frac{FP}{\sqrt{\mu^2p^2+(k-mp^2)^2}}} \quad \text{-- (III)}$$

Kinetic energy of the vibrating particle at the instant of crossing the mean position is given by

$$K.E. = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2_{max}$$

$$\boxed{K.E = \frac{\frac{1}{2} mF^2p^2}{\mu^2p^2+(k-mp^2)^2}} \quad \text{----- (IV)}$$

The mean square of the driving porce per unit mass

$$= \frac{\left(\frac{0+F^2}{2}\right)}{m}$$

$$\boxed{= \frac{F^2}{2m}} \quad \text{-------- (V)}$$

Dividing equation (IV) by $\frac{F^2}{2m}$ we get kinetic energy per unit force which is called the response R

$$R = \frac{\frac{1}{2} mF^2p^2}{\mu^2p^2+(k-mp^2)^2} \Big/ \frac{F}{2m}$$

$$R = \frac{m^2 p^2}{\mu^2 p^2 + (k - mp^2)^2}$$

$$\boxed{R = \frac{p^2}{\frac{\mu^2 p^2}{m^2} + \left(\frac{k}{m} - p^2\right)^2}} \quad \text{------ (VI)}$$

The natural frequency of the system in the absence of damping is $\sqrt{\frac{k}{m}}$.

Therefore, the term $\left(\frac{k}{m} - p^2\right)$ in equation (VI) represents,

The extent to which the natural frequency of the system deviates from the forced frequency.

When, $\frac{k}{m} = p^2$

The natural frequency coincides with the forced frequency, and the value of R will be maximum. from equation (VI)

$$R = \frac{p^2}{\frac{\mu^2 p^2}{m^2}}$$

$$R = \frac{m^2}{\mu^2}$$

$$\boxed{R = \left(\frac{m}{\mu}\right)^2} \quad \text{------ (VII)}$$

The response

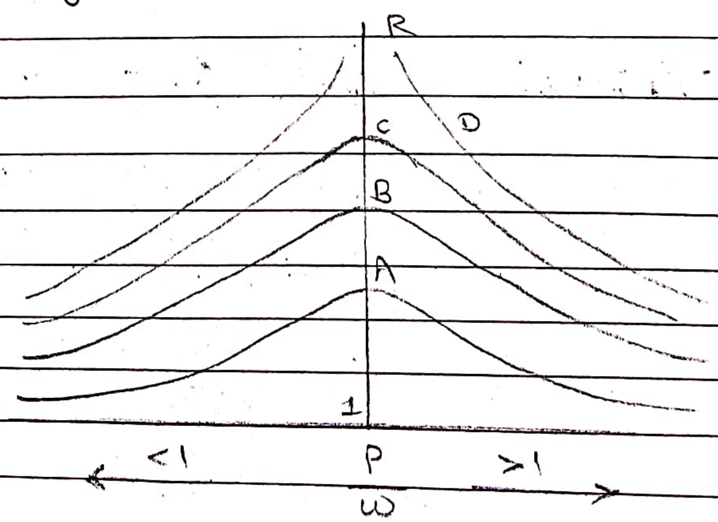
$$R \propto \frac{1}{\mu}$$

It means that the response R is inversely proportional to the fricitional force. In the absence of friction, the response is maximum.

The term $\left(\frac{k}{m} - p^2\right)$ in equation (vi), refers to mistuning. The larger is its value, the greater is the system away from resonance.

The graph b/w P/ω along the x-axis and the response R along the y-axis is diag

- P/ω is equal to 1 the response is maximum. For curve A, μ is large and for curve c, μ is less. The response decreases for values of P/ω greater than 1 or less than 1.



- When the frictional forces are absent, ie, $\mu = 0$, R is infinite and the sharpness of resonance is maximum.

- The sharpness of resonance decreases with increase in the value of u.
- The sharpness of resonance dies rapidly even for a very small change in the value of p/ω from 1, in the case, where u is the minimum.

In this case of the resonance tube, the damping force is large and the graph will be similar to the curve A.

Hence the results obtained with the resonance tube apparatus are not very accurate.

In this case of sonometer wire, the damping forces are small and the graph will be similar to the curve c. In this case sharpness of resonance is maximum in a very narrow region. Then, thus the results obtained with a sonometer are accurate.

Forced vibration and Resonance :-

The time period of the body executing simple harmonic motion depends on the dimension of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on a body, it continues to oscillate the under influence of such external forces. such vibrations of the body are called forced vibrations.

Initially, the amplitude of the swing increases then decreases with time, becomes minimum and again increases. This will be repeated if the external periodic force is constantly applied on the system. In such cases of the body will finally be forced to vibrate with the same frequency as that of the applied force. The frequency of the forced vibration is different from the natural frequency of vibration of the body. The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency of the applied force. The amplitude will be large if difference in frequenc is small and vice versa.

For the forced vibration, equation in modified in the form.

$$\boxed{m\frac{d^2y}{dt^2} + ky + \mu\frac{dy}{dt} = F\sin pt} \quad \text{---} (1)$$

Here,

p is the angular frequency of the periodic force

The perticular solution of equation (1) representing the forced vibrations is

$$\boxed{y = a\sin(pt-\alpha)} \quad \text{----} (11)$$

Diff' eq^n (11) wrto time 't', we get

$$\frac{dy}{dt} = a\cos(pt-\alpha).p$$

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$$\boxed{\frac{dy}{dt} = ap\cos(pt-\alpha)} \text{ --- (III)}$$

again diff' wrto time 't', we get

$$\frac{d^2y}{dt^2} = -ap^2\sin(pt-\alpha) \text{ --- (IV)}$$

$$\boxed{\frac{d^2y}{dt^2} = -p^2y}$$

Substing these value in equation (1)

$$-mp^2a\sin(pt-\alpha) + ka\sin(pt-\alpha) + \mu ap\cos(pt-\alpha) = F.\sin pt$$

$$-mp^2a[\sin pt\cos\alpha - \cos pt\sin\alpha] + ka[\sin pt\cos\alpha - \cos pt\sin\alpha] + \mu ap[\cos pt\cos\alpha + \sin pt\sin\alpha] - F\sin pt = 0$$

--- (V)

When, $\sin pt = 1$;
$\cos pt = 0$

$$\therefore -mp^2a\cos\alpha + ka\cos\alpha + \mu a\sin\alpha - F = 0$$

--- (VI)

When, $\sin pt = 0$
$\cos pt = 1$

$$+map^2\sin\alpha - ka\sin\alpha + \mu ap\cos\alpha \text{ --- (VII)}$$

Dividing eqn (VII) in $\cos\alpha$

$$-mp^2a\tan\alpha - ka\tan\alpha + \mu ap \cancel{\cot\alpha}$$

$$\boxed{\tan\alpha = \frac{\mu p}{K - mp^2} = \frac{A}{B}} \quad \text{--- (VIII)}$$

From equation (VIII), we get

$$\boxed{\sin\alpha = \frac{A}{\sqrt{A^2+B^2}}} \quad \text{---- (IX)}$$

$$\boxed{\cos\alpha = \frac{B}{\sqrt{A^2+B^2}}} \quad \text{---- (X)}$$

Dividing equation (VI) by $\cos\alpha$, we get

$$-mp^2a + Ka + \mu a\tan\alpha - \frac{F}{\cos\alpha} = 0$$

$$a\left[(k-mp^2) + \mu p\tan\alpha\right] = \frac{F}{\cos\alpha}$$

but,

$$(k-mp^2) = B$$

and $\mu p = A$

Substituting the value of $\tan\alpha$ and $\cos\alpha$

$$a\left[B + \frac{A^2}{b}\right] = \frac{F\sqrt{A^2+B^2}}{B}$$

$$a = \frac{F}{\sqrt{A^2+B^2}}$$

Substituting the value of A and B

$$\boxed{a = \frac{F}{\sqrt{\mu^2 p^2 + (k - mp^2)^2}}} \quad \text{---} \; (XI)$$

$$y = a\sin(pt - \alpha)$$

$$\boxed{y = \frac{F}{\sqrt{\mu^2 p^2 + (k - mp^2)^2}} \sin(pt - \alpha)} \quad \text{---} \; (XII)$$

Applying the boundary conditions, another solution is obtained when $F=0$. This corresponds to free vibration. In this case of free vibration the solution is

$$\boxed{y = a e^{-bt} \sin(\omega t - \alpha)} \quad \text{---} \; (XIII)$$

The general solution will include both are the particular solutions for free and forced vibrations:

$$\therefore \boxed{y = a e^{-bt} \sin(\omega t - \alpha) + \frac{F}{\sqrt{\mu^2 p^2 + (k - mp^2)^2}} \sin(pt - \alpha)} \quad \text{---} \; (XIV)$$

Here,

$$b = \frac{\mu}{2m}$$