e-Pathshala पाठशाला

e-Pathshala पाठशाला
A Gateway to All Post Graduate Courses

MHRD Govt. of India

An MHRD Project under its National Mission on Education throught ICT (NME-ICT)

Subject: Physics

Production of Courseware
e-Content for Post Graduate Courses

Paper No. : Quantum Mechanics-I
Module : Addition of Angular Momenta

e-Pathshala पाठशाला

MHRD Govt. of India

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Prof. V. S. Bhasin , Department of Physics and Astrophysics, University of Delhi,Delhi
Content Writer	Prof. V. S. Bhasin, Department of Physics and Astrophysics, University of Delhi,Delhi
Content Reviewer	Prof. Subash Chopra,Indian Institute of Technology, Delhi

Description of Module	
Subject Name	Physics
Paper Name	Quantum Mechanics-I
Module Name/Title	Addition of Angular momenta
Module Id	14

TABLE OF CONTENTS

1. Learning Outcomes

After studying this module, you shall be able to

- Know the vector addition of the two independent angular momenta
- Learn the possibility of the two possible representations, called uncoupled and coupled representations
- Learn how these representations are related to each other through unitary representations
- Know what the Clebsch-Gordon coefficients are
- Learn the rules for the allowed values, j, of the total angular momentum obtained by the coupling of two angular momenta with values j_1 *and* j_2 and the possible values that the magnetic quantum number m can have in terms of m_1 *and* m_2

2. Introduction

In this module, we study the coupling of two independent angular momentum operators which commute with each other. In the preceding module, we learnt that if $\vec{J}_1$ and $\vec{J}_2$ are the two angular momentum operators, then there exist the eigenstates $|j_1m_1\rangle$ *and* $|j_2m_2\rangle$ which are respectively the simultaneous eigenstates of $\hat{J}_1^2, \hat{J}_{1z}$ *and* $\hat{J}_2^2, \hat{J}_{2z}$. Here we shall learn that there are two possible representations in which the eigen states of the individual angular momentum operators can be coupled. One is the obvious uncoupled representation which is obtained by just taking the direct product of the two eigenstates. These states are obviously the simultaneous eigenstates of $\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}$. The second more useful representation is obtained when the two angular momentum operators, $\vec{J}_1$ *and* $\vec{J}_2$, are vectorially added to get the total angular momentum, $\vec{J}$, and we look for the representation in which the coupled state is the simultaneous eigenstate of $\hat{J}^2, \hat{J}_z, \hat{J}_1^2$ *and* $\hat{J}_2^2$. We shall show that these representations are related by the unitary transformation and the coefficients in terms of which the uncoupled states are related to the coupled states are called the Clebsch-Gordon coefficients. The present module and the one following it are devoted towards studying the important properties of these coefficients.

3. Angular Momentum Algebra

3.1Addition of Angular Momenta

Let there be two independent angular momentum operators, $\vec{\hat{J}}_1$ *and* $\vec{\hat{J}}_2$, which commute with each other, i.e.,

$$[\vec{\hat{J}}_1 , \vec{\hat{J}}_2] = 0 \qquad (14.1)$$

These angular momenta may refer to different particles or two independent systems, or $\vec{\hat{J}}_1$ may be the orbital angular momentum, $\vec{\hat{L}}$ and $\vec{\hat{J}}_2$ may be the spin, $\vec{\hat{S}}$, of the same particle. Let $|j_1, m_1\rangle$ be a normalized simultaneous eigen state of $\hat{J}_1^2$ *and* $\hat{J}_{1z}$, so that

$$\hat{J}_1^2 | j_1 m_1\rangle = \hbar^2 j_1(j_1+1) | j_1 m_1\rangle \qquad (14.2a)$$

and

$$\hat{J}_{1z} | j_1 m_1\rangle = m_1 \hbar | j_1 m_1\rangle \qquad (14.2b)$$

Similarly, let $| j_2 m_2\rangle$ be a normalized simultaneous eigen state of $\hat{J}_2^2$ *and* $\hat{J}_{2z}$:

$$\hat{J}_2^2 | j_2 m_2\rangle = \hbar^2 j_2(j_2+1) | j_2 m_2\rangle \qquad (14.3a)$$

and

$$\hat{J}_{2z} | j_2 m_2\rangle = m_2 \hbar | j_2 m_2\rangle \qquad (14.3b)$$

A normalized simultaneous eigen state of $\hat{J}_1^2$, $\hat{J}_2^2$, $\hat{J}_{1z}$ *and* $\hat{J}_{2z}$ with eigen values $\hbar^2 j_1(j_1+1), \hbar^2 j_2(j_2+1), m_1\hbar$ *and* $m_2\hbar$ respectively is obtained by taking the direct product of the states $| j_1 m_1\rangle$ *and* $| j_2 m_2\rangle$,i.e.,

$$| j_1\, j_2\, m_1\, m_2\rangle = | j_1 m_1\rangle | j_2 m_2\rangle \qquad (14.4)$$

However, in many physical situations of physical interest, we deal with systems whose Hamiltonian, $\hat{H}$ is invariant under rotations and therefore commutes with the total angular momentum operator $\vec{\hat{J}}$, which is obtained by the addition of two angular momentum operators, $\vec{\hat{J}}_1$ *and* $\vec{\hat{J}}_2$. In such cases, we look for the eigen states of $\hat{H}$ which are simultaneous eigen states of $\hat{J}^2$ *and* $\hat{J}_z$. Now the total angular momentum operator $\vec{\hat{J}}$ of the system is the vector sum of the two commuting angular momenta , $\vec{\hat{J}}_1$ **and** $\vec{\hat{J}}_2$,i.e.,

$$\vec{\hat{J}} = \vec{\hat{J}}_1 + \vec{\hat{J}}_2 \ . \qquad (14.5)$$

Consider the operator $\hat{J}^2$, which is

$$\hat{J}^2 = (\vec{\hat{J}}_1 + \vec{\hat{J}}_2)^2 = \hat{J}_1^2 + \hat{J}_2^2 + 2\vec{\hat{J}}_1.\vec{\hat{J}}_2$$
$$= \hat{J}_1^2 + \hat{J}_2^2 + 2\hat{J}_{1x}\hat{J}_{2x} + 2\hat{J}_{1y}\hat{J}_{2y} + 2\hat{J}_{1z}\hat{J}_{2z} \qquad (14.6)$$

Since all the components of , $\vec{\hat{\mathbf{J}}}_1$ commute with all those of , $\vec{\hat{\mathbf{J}}}_2$ and also $[\hat{J}_1^2, \vec{\hat{J}}_1] = [\hat{J}_2^2, \vec{\hat{J}}_2]$ =0, it follows that $\hat{J}^2$ commutes with $\hat{J}_1^2$ *and* $\hat{J}_2^2$. However, because $\hat{J}_{1z}$ does not commute with $\hat{J}_{1x}$ *or* $\hat{J}_{1y}$, we find $\hat{J}^2$ does not commute with $\hat{J}_{1z}$; for the same reason $\hat{J}^2$ does not commute with $\hat{J}_{2z}$. As a result, the simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_z$ are the eigenstates of $\hat{J}_1^2$ *and* $\hat{J}_2^2$ but not (in general) of $\hat{J}_{1z}$ *and* $\hat{J}_{2z}$.

3.2 The Clebsch-Gordon Coefficients

Thus there are two distict descriptions of the system,: (i) in terms of the eigenstates of $\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}$ *and* $\hat{J}_{2z}$, as stated above, and (ii) in terms of the eigenstates of $\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2$ *and* $\hat{J}_z$. Let us denote the basis vectors in the case (ii) by $|j_1\ j_2\ j\,m\rangle$ (*or sometimes by* $|jm\rangle$). We then have

$$\hat{J}_1^2 |j_1 j_2\, jm\rangle = \hbar^2 j_1(j_1+1)|j_1 j_2\, jm\rangle \qquad (14.7a)$$

$$\hat{J}_2^2 |j_1 j_2\, jm\rangle = \hbar^2 j_2(j_2+1)|j_1 j_2\, jm\rangle \qquad (14.7b)$$

$$\hat{J}^2 |j_1 j_2\, jm\rangle = \hbar^2 j(j+1)|j_1 j_2\, jm\rangle \qquad (14.7c)$$

$$\hat{J}_z |j_1 j_2\, jm\rangle = \hbar\, m|j_1 j_2\, jm\rangle \qquad (14.7d)$$

This representation is called the coupled representation. Since the two representations, $|j_1\ j_2\ j\,m\rangle$ and $|j_1 j_2 m_1 m_2\rangle$ are simply different orthonormal bases in the same Hilbert space, they are related by a unitary transformation, i.e.,

$$|j_1 j_2\, jm\rangle \equiv |jm\rangle = \sum_{m_1 m_2} \langle j_1 j_2 m_1 m_2 | jm\rangle |j_1 m_1\rangle |j_2 m_2\rangle \qquad (14.8)$$

As there are $(2\boldsymbol{j_1}+\mathbf{1})$ different values of $\boldsymbol{m_1}$ and for a given value of m_2, there are $(2\boldsymbol{j_2}+\mathbf{1})$ different values of m_2, the dimensionality of the represezntation is

$(2j_1+1)(2j_2+1)$. The summation in Eq.(14.8) must be performed over $\boldsymbol{m_1}$ ***and*** $\boldsymbol{m_2}$, since $\boldsymbol{j_1}$ ***and*** $\boldsymbol{j_2}$ are assumed to have fixed values. The coefficients $\langle j_1 j_2 m_1 m_2 | jm\rangle$, which depend on the six angular momentum quantum numbers, are called Clebsch-Gordon or vector addition coefficients.

Various symbols and names are used for it. We shall adopt the symbol $C^{j_1 j_2 j}_{m_1 m_2 m}$ and call it the Clebsch-Gordon or C-cofficients. In the abbreviated form, we write Eq.(14.8) as

$$|jm\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1 m_{12}\rangle |j_2 m_2\rangle \qquad (14.9)$$

The Eq.(14.9) is to be regarded as the defining equation for the C-coefficients.

3.3 Selection Rules for allowed values of j and m

In order to find out the allowed values of j for given values of j_1 ***and*** j_2 we proceed as follows: Using the relation, $\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$ and operating on the state $|jm\rangle$, we find

$$\hat{J}_z |jm\rangle = (\hat{J}_{1z} + \hat{J}_{2z})|jm\rangle$$

$$= m\hbar \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle = \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} (\hat{J}_{1z} + \hat{J}_{2z}) |j_1 j_2 m_1 m_2\rangle$$

$$= (m_1 + m_2)\hbar \sum_{m_1 m_2} C^{j_1 j_2 j}_{m_1 m_2 m} |j_1 j_2 m_1 m_2\rangle \qquad (14.10)$$

$$i.e., \quad \sum_{m_1 m_2} (m - m_1 - m_2) C^{j_1 j_2 j}_{m_1 m_2 m} |j_1 j_2 m_1 m_2\rangle = 0$$

Since the state vectors $|j_1 j_2 m_1 m_2\rangle$ are linearly indepe1ndent, this implies that

$$C^{j_1 j_2 j}_{m_1 m_2 m} = 0 \qquad if \quad m \neq m_1 + m_2$$

$$i.e., \quad C^{j_1 j_2 j}_{m_1 m_2 m} \neq 0, \quad if \quad m = m_1 + m_2 \qquad (14.11)$$

Since the maximum allowed values of m_1 *and* m_2 are respectively given by j_1 *and* j_2, from the relation $m = m_1 + m_2$, it follows that the maximum possible value of m is $j_1 + j_2$. Now, because m can only take on 2j+1 values, from –j, -j+1,………,j-1,j , it means that the maximum possible value of j is $j_1 + j_2$. For the case j= $j_1 + j_2$ and m= $j_1 + j_2$, there will only be one term in the summation on the right of Eq.(14.9), viz., corresponding to $m_1 = j_1$ and $m_2 = j_2$, we have

$$|j_1 + j_2, j_1 + j_2\rangle = C^{j_1 j_2 j_1 + j_2}_{j_1 j_2 j_1 + j_2} |j_1 j_1\rangle |j_2 j_2\rangle \qquad (14.12)$$

Using the orthonormality of the states involved, we have

$$\langle j_1 + j_2, j_1 + j_2 | j_1 + j_2, j_1 + j_2\rangle = 1 = \left| C^{j_1 j_2 j_1 + j_2}_{j_1 j_2 j_1 + j_2} \right|^2 \qquad (14.13)$$

According to Eq.(14.13),

$$C^{j_1 j_2 j_1+j_2}_{j_1 j_2 j_1+j_2} = \exp(i\delta) ,$$

where δ is real. We choose the phase of the state $|j_1+j_2, j_1+j_2\rangle$ in Eq.(14.12) such that δ is zero. Thus

$$C^{j_1 j_2 j_1+j_2}_{j_1 j_2 j_1+j_2} = 1 \qquad (14.14)$$

With this choice, we fix the phases of all the C-coefficients belonging to j=$\boldsymbol{j_1 + j_2}$.

Let us now consider a state for which j= $\boldsymbol{j_1 + j_2}$, but m=$\boldsymbol{j_1 + j_2 - 1}$. In this case we have two possible values for $\boldsymbol{m_1 \;\; and \; m_2}$: we can either have $m_1 = j_1 \quad and \quad m_2 = j_2 - 1$ or $m_1 = j_1 - 1 \quad and \quad m_2 = j_2$. Thus the state $|jm\rangle$ must be a linear combination of the two linearly independent eigenstates; $|j_1 j_1\rangle| j_2 j_2 - 1\rangle \quad and \quad |j_1 j_1 - 1\rangle| j_2 j_2\rangle$. Note there are two such linear combinations, one of them belongs to the set of egenstates wih j= $j_1 + j_2$ while the other orthogonal combination is a member of a set of eigenstates for which the maximum value of m is $j_1 + j_2 - 1$; this latter set must be such that $j = j_1 + j_2 - 1$. Proceeding further in this way to the state j= $\boldsymbol{j_1 + j_2}$, but m=$\boldsymbol{j_1 + j_2 - 2}$, we shall see there exist three linearly ind--ependent states corresponding to the values $j = j_1 + j_2$, $j = j_1 + j_2 - 1$ *and* $j = j_1 + j_2 - 2$, respectively. Each time m is reduced by unity by proceeding this way, one of the coupled states

which arises belongs to a j-value reduced by unity and in this state m has its maximum value. The lower limit j_{min} is reached when the number of coupled state has matched with all the uncoupled states. We know

$$\sum_{j_{min}}^{j_{max}} (2j+1) = (2j_1+1)(2j_2+1) \qquad (14.15)$$

We can determine the minimum value of j by noting that $j_{max} = j_1 + j_2$ and the left hand side of Eq.(14.15) can be evaluated by using

$$\sum_{\alpha}^{j_{max}} j = \frac{1}{2}[j_{max}(j_{max}+1) - \alpha(\alpha-1)] \qquad (14.16)$$

Using this relation in Eq.(14.15), we find

$$j^2_{min} = (j_1 - j_2)^2 .$$

Since $j_{min} \geq 0$,

$$j_{min} = |j_1 - j_2| \qquad (14.17)$$

We thus arrive at an important result that in the addition of two angular momenta, $\vec{\hat{J}} = \vec{\hat{J}}_1 + \vec{\hat{J}}_2$, if the eigen values of $\hat{J}_1^2$ are $j_1(j_1+1)$ and the eigen values of $\hat{J}_2^2$ are $j_2(j_2+1)$, then the eigen values of $\hat{J}^2$ are j(j+1) , where

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots \ldots \; |j_1 - j_2| \qquad (14.18)$$

The three angular momentum quantum numbers j_1, j_2 *and* j satisfy the triangular condition

$$|j_1 - j_2| \le j \le j_1 + j_2 \qquad (14.19)$$

And for each value of j there are 2j+1 values of m, given by

m = - j , -j+1 , -j+2, ... j – 2, j – 1, j (14.20)

3.3.1 Example

Let us consider an example to illustrate the allowed values of j and m for the case : $j_1 = 3/2$ *and* $j_2 = 2$*. Note that for given* $j_1 = 3/2$ *,* $\boldsymbol{m_1}$ *can take the values -3/2, -1/2, 1/2 and 3/2 and for* $j_2 = 2$*,* $\boldsymbol{m_2}$ *can have the values -2,-1,0,1 and 2.*

j	m	$\boldsymbol{m_1}$	$\boldsymbol{m_2}$
7/2	7/2	3/2	2
7/2, 5/2	5/2	3/2 ½	1 2
7/2, 5/2, 3/2	3/2	3/2 1/2 -1/2	0 1 2
7/2, 5/2, 3/2, 1/2	1/2	3/2 1/2 -1/2 -3/2	-1 0 1 2

7/2, 5/2, 3/2, 1/2	-1/2	-3/2 -1/2 1/2 3/2	1 0 -1 -2
7/2, 5/2, 3/2	-3/2	-3/2 -1/2 1/2	0 -1 -2
7/2, 5/2	-5/2	-3/2 -1/2	-1 -2
7/2	-7/2	-3/2	-2

It should be checked from the above table that in the uncoupled representation, total number of states is $(2j_1+1)(2j_2+1)=4\times5=20$, which matches with the number of states in the coupled representation, viz., $\sum_{|j_1-j_2|=1/2}^{j_1+j_2=7/2}(2j+1)=20$.

4. Summary

After studying this module, you would be able to

- Know the vector addition of the two independent angular momenta
- Learn the possibility of the two possible representations, called uncoupled and coupled representations
- Learn how these representations are related to each other through unitary representations
- Know what the Clebsch-Gordon coefficients are
- Learn the rules for the allowed values, j, of the total angular momentum obtained by the coupling of two angular momenta with values j_1 and j_2 and the possible values that the magnetic quantum number m can have in terms of m_1 and m_2