Chapter 5

Dynamic Programming

Dynamic programming is a name, coined by Richard Bellman in 1955. Dynamic programming, as greedy method, is a powerful algorithm design technique that can be used when the solution to the problem may be viewed as the result of a sequence of decisions. In the greedy method we make irrevocable decisions one at a time, using a greedy criterion. However, in dynamic programming we examine the decision sequence to see whether an optimal decision sequence contains optimal decision subsequence.

When optimal decision sequences contain optimal decision subsequences, we can establish recurrence equations, called *dynamic-programming recurrence equations,* that enable us to solve the problem in an efficient way.

Dynamic programming is based on the principle of optimality (also coined by Bellman). The principle of optimality states that no matter whatever the initial state and initial decision are, the remaining decision sequence must constitute an optimal decision sequence with regard to the state resulting from the first decision. The principle implies that an optimal decision sequence is comprised of optimal decision subsequences. Since the principle of optimality may not hold for some formulations of some problems, it is necessary to verify that it does hold for the problem being solved. Dynamic programming cannot be applied when this principle does not hold.

The steps in a dynamic programming solution are:

- Verify that the principle of optimality holds
- Set up the dynamic-programming recurrence equations
- Solve the dynamic-programming recurrence equations for the value of the optimal solution.
- Perform a trace back step in which the solution itself is constructed.

Dynamic programming differs from the greedy method since the greedy method produces only one feasible solution, which may or may not be optimal, while dynamic programming produces all possible sub-problems at most once, one of which guaranteed to be optimal. Optimal solutions to sub-problems are retained in a table, thereby avoiding the work of recomputing the answer every time a sub-problem is encountered

The divide and conquer principle solve a large problem, by breaking it up into smaller problems which can be solved independently. In dynamic programming this principle is carried to an extreme: when we don't know exactly which smaller problems to solve, we simply solve them all, then store the answers away in a table to be used later in solving larger problems. Care is to be taken to avoid recomputing previously computed values, otherwise the recursive program will have prohibitive complexity. In some cases, the solution can be improved and in other cases, the dynamic programming technique is the best approach.

Two difficulties may arise in any application of dynamic programming:

1. It may not always be possible to combine the solutions of smaller problems to form the solution of a larger one.

2. The number of small problems to solve may be un-acceptably large.

There is no characterized precisely which problems can be effectively solved with dynamic programming; there are many hard problems for which it does not seen to be applicable, as well as many easy problems for which it is less efficient than standard algorithms.

5.1 MULTI STAGE GRAPHS

A multistage graph G = (V, E) is a directed graph in which the vertices are partitioned into $k \geq 2$ disjoint sets V_i, $1 \leq i \leq k$. In addition, if <u, v> is an edge in E, then $u \in V_i$ and $v \in V_{i+1}$ for some i, $1 \leq i < k$.

Let the vertex 's' is the source, and 't' the sink. Let c (i, j) be the cost of edge <i, j>. The cost of a path from 's' to 't' is the sum of the costs of the edges on the path. The multistage graph problem is to find a minimum cost path from 's' to 't'. Each set V_i defines a stage in the graph. Because of the constraints on E, every path from 's' to 't' starts in stage 1, goes to stage 2, then to stage 3, then to stage 4, and so on, and eventually terminates in stage k.

A dynamic programming formulation for a k-stage graph problem is obtained by first noticing that every s to t path is the result of a sequence of k – 2 decisions. The i^{th} decision involves determining which vertex in v_{i+1}, $1 \leq i \leq k - 2$, is to be on the path. Let c (i, j) be the cost of the path from source to destination. Then using the forward approach, we obtain:

$$\text{cost}(i, j) = \min_{\substack{l \in V_{i+1} \\ <j, l> \in E}} \{c(j, l) + \text{cost}(i + 1, l)\}$$

ALGORITHM:

```
Algorithm Fgraph (G, k, n, p)
// The input is a k-stage graph G = (V, E) with n vertices
// indexed in order or stages. E is a set of edges and c [i, j]
// is the cost of (i, j). p [1 : k] is a minimum cost path.
{
        cost [n] := 0.0;
        for j:= n - 1 to 1 step – 1 do
        {                                       // compute cost [j]
                let r be a vertex such that (j, r) is an edge
                of G and c [j, r] + cost [r] is minimum;
                cost [j] := c [j, r] + cost [r];
                d [j] := r:
        }
        p [1] := 1; p [k]  := n;                // Find a minimum cost path.
        for j := 2 to k - 1 do p [j] := d [p [j - 1]];
}
```

The multistage graph problem can also be solved using the backward approach. Let bp(i, j) be a minimum cost path from vertex s to j vertex in V_i. Let Bcost(i, j) be the cost of bp(i, j). From the backward approach we obtain:

$$\text{Bcost}(i, j) = \min_{\substack{l \in V_{i-1} \\ \langle l, j\rangle \in E}} \{ \text{Bcost}(i-1, l) + c(l, j)\}$$

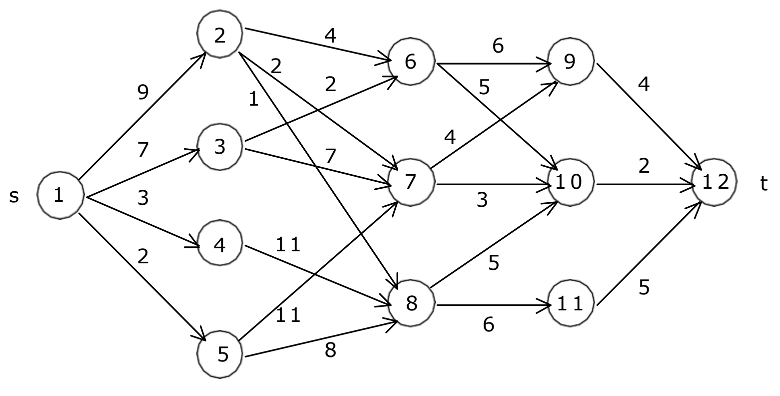
```
Algorithm Bgraph (G, k, n, p)
// Same function as Fgraph
{
        Bcost [1] := 0.0;
        for j := 2 to n do
        {                                       // Compute Bcost [j].
                Let r be such that (r, j) is an edge of
                G and Bcost [r] + c [r, j] is minimum;
                Bcost [j] := Bcost [r] + c [r, j];
                D [j] := r;
        }                                       //find a minimum cost path
        p [1] := 1; p [k] := n;
        for j:= k - 1 to 2 do p [j] := d [p [j + 1]];
}
```

Complexity Analysis:

The complexity analysis of the algorithm is fairly straightforward. Here, if G has $|E|$ edges, then the time for the first for loop is $\Phi(|V| + |E|)$.

EXAMPLE 1:

Find the minimum cost path from s to t in the multistage graph of five stages shown below. Do this first using forward approach and then using backward approach.



FORWARD APPROACH:

We use the following equation to find the minimum cost path from s to t:

$$\text{cost}(i, j) = \min_{l \in V_{i+1}} \{c(j, l) + \text{cost}(i + 1, l)\}$$

$$\text{cost}(1, 1) = \min_{\langle j, l\rangle \in E} \{c(1, 2) + \text{cost}(2, 2), c(1, 3) + \text{cost}(2, 3), c(1, 4) + \text{cost}(2, 4), c(1, 5) + \text{cost}(2, 5)\}$$
$$= \min\{9 + \text{cost}(2, 2), 7 + \text{cost}(2, 3), 3 + \text{cost}(2, 4), 2 + \text{cost}(2, 5)\}$$

Now first starting with,

cost (2, 2) = min{c (2, 6) + cost (3, 6), c (2, 7) + cost (3, 7), c (2, 8) + cost (3, 8)}
= min {4 + cost (3, 6), 2 + cost (3, 7), 1 + cost (3, 8)}

---

cost (3, 6) = min {c (6, 9) + cost (4, 9), c (6, 10) + cost (4, 10)}
= min {6 + cost (4, 9), 5 + cost (4, 10)}

cost (4, 9) = min {c (9, 12) + cost (5, 12)} = min {4 + 0) = 4

cost (4, 10) = min {c (10, 12) + cost (5, 12)} = 2

Therefore, cost (3, 6) = min {6 + 4, 5 + 2} = 7

---

cost (3, 7) = min {c (7, 9) + cost (4, 9) , c (7, 10) + cost (4, 10)}
= min {4 + cost (4, 9), 3 + cost (4, 10)}

cost (4, 9) = min {c (9, 12) + cost (5, 12)} = min {4 + 0} = 4

Cost (4, 10) = min {c (10, 2) + cost (5, 12)} = min {2 + 0} = 2

Therefore, cost (3, 7) = min {4 + 4, 3 + 2} = min {8, 5} = 5

---

cost (3, 8) = min {c (8, 10) + cost (4, 10), c (8, 11) + cost (4, 11)}
= min {5 + cost (4, 10), 6 + cost (4 + 11)}

cost (4, 11) = min {c (11, 12) + cost (5, 12)} = 5

Therefore, cost (3, 8) = min {5 + 2, 6 + 5} = min {7, 11} = 7

Therefore, cost (2, 2) = min {4 + 7, 2 + 5, 1 + 7} = min {11, 7, 8} = 7

Therefore, cost (2, 3) = min {c (3, 6) + cost (3, 6), c (3, 7) + cost (3, 7)}
= min {2 + cost (3, 6), 7 + cost (3, 7)}
= min {2 + 7, 7 + 5} = min {9, 12} = 9

cost (2, 4) = min {c (4, 8) + cost (3, 8)} = min {11 + 7} = 18
cost (2, 5) = min {c (5, 7) + cost (3, 7), c (5, 8) + cost (3, 8)}
= min {11 + 5, 8 + 7} = min {16, 15} = 15

Therefore, cost (1, 1) = min {9 + 7, 7 + 9, 3 + 18, 2 + 15}
= min {16, 16, 21, 17} = 16

The minimum cost path is 16.

The path is 1 — 2 — 7 — 10 — 12

or

1 — 3 — 6 — 10 — 12

BACKWARD APPROACH:

We use the following equation to find the minimum cost path from t to s:

$$\text{Bcost}(i, J) = \min_{\substack{l \,\varepsilon\, v_{i-1} \\ <l,\, j> \,\varepsilon\, E}} \{\text{Bcost}(i - 1, l) + c(l, J)\}$$

Bcost (5, 12) = min {Bcost (4, 9) + c (9, 12), Bcost (4, 10) + c (10, 12),
Bcost (4, 11) + c (11, 12)}
= min {Bcost (4, 9) + 4, Bcost (4, 10) + 2, Bcost (4, 11) + 5}

Bcost (4, 9) = min {Bcost (3, 6) + c (6, 9), Bcost (3, 7) + c (7, 9)}
= min {Bcost (3, 6) + 6, Bcost (3, 7) + 4}

Bcost (3, 6) = min {Bcost (2, 2) + c (2, 6), Bcost (2, 3) + c (3, 6)}
= min {Bcost (2, 2) + 4, Bcost (2, 3) + 2}

Bcost (2, 2) = min {Bcost (1, 1) + c (1, 2)} = min {0 + 9} = 9

Bcost (2, 3) = min {Bcost (1, 1) + c (1, 3)} = min {0 + 7} = 7

Bcost (3, 6) = min {9 + 4, 7 + 2} = min {13, 9} = 9

Bcost (3, 7) = min {Bcost (2, 2) + c (2, 7), Bcost (2, 3) + c (3, 7),
Bcost (2, 5) + c (5, 7)}

Bcost (2, 5) = min {Bcost (1, 1) + c (1, 5)} = 2

Bcost (3, 7) = min {9 + 2, 7 + 7, 2 + 11} = min {11, 14, 13} = 11

Bcost (4, 9) = min {9 + 6, 11 + 4} = min {15, 15} = 15

Bcost (4, 10) = min {Bcost (3, 6) + c (6, 10), Bcost (3, 7) + c (7, 10),
Bcost (3, 8) + c (8, 10)}

Bcost (3, 8) = min {Bcost (2, 2) + c (2, 8), Bcost (2, 4) + c (4, 8),
Bcost (2, 5) + c (5, 8)}
Bcost (2, 4) = min {Bcost (1, 1) + c (1, 4)} = 3

Bcost (3, 8) = min {9 + 1, 3 + 11, 2 + 8} = min {10, 14, 10} = 10

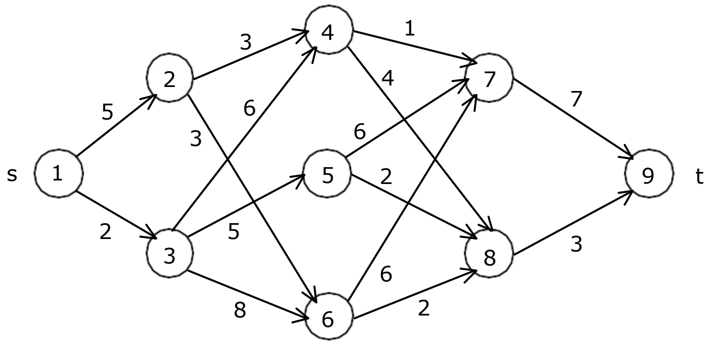
Bcost (4, 10) = min {9 + 5, 11 + 3, 10 + 5} = min {14, 14, 15) = 14

Bcost (4, 11) = min {Bcost (3, 8) + c (8, 11)} = min {Bcost (3, 8) + 6}
= min {10 + 6} = 16

Bcost (5, 12) = min {15 + 4, 14 + 2, 16 + 5} = min {19, 16, 21} = 16.

EXAMPLE 2:

Find the minimum cost path from s to t in the multistage graph of five stages shown below. Do this first using forward approach and then using backward approach.



SOLUTION:

FORWARD APPROACH:

$$\text{cost }(i, J) = \min_{\substack{l \,\varepsilon\, V_{i+1} \\ \langle J, l\rangle \in E}} \{c(j, l) + \text{cost }(i + 1, l)\}$$

cost (1, 1) = min {c (1, 2) + cost (2, 2), c (1, 3) + cost (2, 3)}
= min {5 + cost (2, 2), 2 + cost (2, 3)}

cost (2, 2) = min {c (2, 4) + cost (3, 4), c (2, 6) + cost (3, 6)}
= min {3+ cost (3, 4), 3 + cost (3, 6)}

cost (3, 4) = min {c (4, 7) + cost (4, 7), c (4, 8) + cost (4, 8)}
= min {(1 + cost (4, 7), 4 + cost (4, 8)}

cost (4, 7) = min {c (7, 9) + cost (5, 9)} = min {7 + 0) = 7

cost (4, 8) = min {c (8, 9) + cost (5, 9)} = 3

Therefore, cost (3, 4) = min {8, 7} = 7

cost (3, 6) = min {c (6, 7) + cost (4, 7), c (6, 8) + cost (4, 8)}
= min {6 + cost (4, 7), 2 + cost (4, 8)} = min {6 + 7, 2 + 3} = 5

Therefore, cost (2, 2) = min {10, 8} = 8

cost (2, 3) = min {c (3, 4) + cost (3, 4), c (3, 5) + cost (3, 5), c (3, 6) + cost (3,6)}

cost (3, 5) = min {c (5, 7) + cost (4, 7), c (5, 8) + cost (4, 8)}= min {6 + 7, 2 + 3} = 5

Therefore, cost (2, 3) = min {13, 10, 13} = 10

cost (1, 1) = min {5 + 8, 2 + 10} = min {13, 12} = 12

BACKWARD APPROACH:

$$\text{Bcost}(i, J) = \min_{\substack{l \in v_{i-1} \\ <l, j> \in E}} \{\text{Bcost}(i - 1, l) = c(l, J)\}$$

Bcost (5, 9) = min {Bcost (4, 7) + c (7, 9), Bcost (4, 8) + c (8, 9)}
= min {Bcost (4, 7) + 7, Bcost (4, 8) + 3}

Bcost (4, 7) = min {Bcost (3, 4) + c (4, 7), Bcost (3, 5) + c (5, 7), Bcost (3, 6) + c (6, 7)}
= min {Bcost (3, 4) + 1, Bcost (3, 5) + 6, Bcost (3, 6) + 6}

Bcost (3, 4) = min {Bcost (2, 2) + c (2, 4), Bcost (2, 3) + c (3, 4)}
= min {Bcost (2, 2) + 3, Bcost (2, 3) + 6}

Bcost (2, 2) = min {Bcost (1, 1) + c (1, 2)} = min {0 + 5} = 5

Bcost (2, 3) = min (Bcost (1, 1) + c (1, 3)} = min {0 + 2} = 2

Therefore, Bcost (3, 4) = min {5 + 3, 2 + 6} = min {8, 8} = 8

Bcost (3, 5) = min {Bcost (2, 3) + c (3, 5)} = min {2 + 5} = 7

Bcost (3, 6) = min {Bcost (2, 2) + c (2, 6), Bcost (2, 3) + c (3, 6)}
= min {5 + 5, 2 + 8} = 10

Therefore, Bcost (4, 7) = min {8 + 1, 7 + 6, 10 + 6} = 9

Bcost (4, 8) = min {Bcost (3, 4) + c (4, 8), Bcost (3, 5) + c (5, 8), Bcost (3, 6) + c (6, 8)}
= min {8 + 4, 7 + 2, 10 + 2} = 9

Therefore, Bcost (5, 9) = min {9 + 7, 9 + 3} = 12

All pairs shortest paths

In the all pairs shortest path problem, we are to find a shortest path between every pair of vertices in a directed graph G. That is, for every pair of vertices (i, j), we are to find a shortest path from i to j as well as one from j to i. These two paths are the same when G is undirected.

When no edge has a negative length, the all-pairs shortest path problem may be solved by using Dijkstra's greedy single source algorithm n times, once with each of the n vertices as the source vertex.

The all pairs shortest path problem is to determine a matrix A such that A (i, j) is the length of a shortest path from i to j. The matrix A can be obtained by solving n single-source problems using the algorithm shortest Paths. Since each application of this procedure requires O (n^2) time, the matrix A can be obtained in O (n^3) time.

The dynamic programming solution, called Floyd's algorithm, runs in O (n^3) time. Floyd's algorithm works even when the graph has negative length edges (provided there are no negative length cycles).

The shortest i to j path in G, i ≠ j originates at vertex i and goes through some intermediate vertices (possibly none) and terminates at vertex j. If k is an intermediate vertex on this shortest path, then the subpaths from i to k and from k to j must be shortest paths from i to k and k to j, respectively. Otherwise, the i to j path is not of minimum length. So, the principle of optimality holds. Let A^k (i, j) represent the length of a shortest path from i to j going through no vertex of index greater than k, we obtain:

$$A^k(i, j) = \{\min \{\min_{1 \le k \le n} \{A^{k-1}(i, k) + A^{k-1}(k, j)\}, c(i, j)\}$$

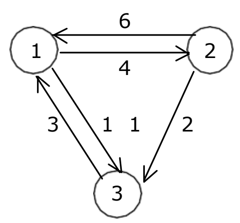
Algorithm All Paths (Cost, A, n)

```
// cost [1:n, 1:n] is the cost adjacency matrix of a graph which
// n vertices; A [I, j] is the cost of a shortest path from vertex
// i to vertex j. cost [i, i] = 0.0, for 1 < i < n.
{
        for i := 1 to n do
                for j:= 1 to n do
                        A [i, j] := cost  [i, j];                  // copy cost into A.
        for k := 1 to n do
                for i := 1 to n do
                        for j := 1 to n do
                                A [i, j] := min (A [i, j], A [i, k] + A [k, j]);
}
```

Complexity Analysis: A Dynamic programming algorithm based on this recurrence involves in calculating n+1 matrices, each of size n x n. Therefore, the algorithm has a complexity of O (n^3).

Example 1:

Given a weighted digraph G = (V, E) with weight. Determine the length of the shortest path between all pairs of vertices in G. Here we assume that there are no cycles with zero or negative cost.



Cost adjacency matrix (A^0) = $\begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & \infty & 0 \end{bmatrix}$

General formula: $\min_{1 \le k \le n} \{A^{k-1}(i, k) + A^{k-1}(k, j)\}, c(i, j)\}$

Solve the problem for different values of k = 1, 2 and 3

Step 1: Solving the equation for, k = 1;

$A^1(1, 1) = \min \{(A^o(1, 1) + A^o(1, 1)), c(1, 1)\} = \min \{0 + 0, 0\} = 0$

$A^1(1, 2) = \min \{(A^o(1, 1) + A^o(1, 2)), c(1, 2)\} = \min \{(0 + 4), 4\} = 4$

$A^1(1, 3) = \min \{(A^o(1, 1) + A^o(1, 3)), c(1, 3)\} = \min \{(0 + 11), 11\} = 11$

$A^1(2, 1) = \min \{(A^o(2, 1) + A^o(1, 1)), c(2, 1)\} = \min \{(6 + 0), 6\} = 6$

$A^1(2, 2) = \min \{(A^o(2, 1) + A^o(1, 2)), c(2, 2)\} = \min \{(6 + 4), 0)\} = 0$

$A^1(2, 3) = \min \{(A^o(2, 1) + A^o(1, 3)), c(2, 3)\} = \min \{(6 + 11), 2\} = 2$

$A^1(3, 1) = \min \{(A^o(3, 1) + A^o(1, 1)), c(3, 1)\} = \min \{(3 + 0), 3\} = 3$

$A^1(3, 2) = \min \{(A^o(3, 1) + A^o(1, 2)), c(3, 2)\} = \min \{(3 + 4), \propto\} = 7$

$A^1(3, 3) = \min \{(A^o(3, 1) + A^o(1, 3)), c(3, 3)\} = \min \{(3 + 11), 0\} = 0$

$$A^{(1)} = \begin{bmatrix} 0 & 4 & 11 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

Step 2: Solving the equation for, K = 2;

$A^2(1, 1) = \min \{(A^1(1, 2) + A^1(2, 1), c(1, 1)\} = \min \{(4 + 6), 0\} = 0$

$A^2(1, 2) = \min \{(A^1(1, 2) + A^1(2, 2), c(1, 2)\} = \min \{(4 + 0), 4\} = 4$

$A^2(1, 3) = \min \{(A^1(1, 2) + A^1(2, 3), c(1, 3)\} = \min \{(4 + 2), 11\} = 6$

$A^2(2, 1) = \min \{(A(2, 2) + A(2, 1), c(2, 1)\} = \min \{(0 + 6), 6\} = 6$

$A^2(2, 2) = \min \{(A(2, 2) + A(2, 2), c(2, 2)\} = \min \{(0 + 0), 0\} = 0$

$A^2(2, 3) = \min \{(A(2, 2) + A(2, 3), c(2, 3)\} = \min \{(0 + 2), 2\} = 2$

$A^2(3, 1) = \min \{(A(3, 2) + A(2, 1), c(3, 1)\} = \min \{(7 + 6), 3\} = 3$

$A^2(3, 2) = \min \{(A(3, 2) + A(2, 2), c(3, 2)\} = \min \{(7 + 0), 7\} = 7$

$A^2(3, 3) = \min \{(A(3, 2) + A(2, 3), c(3, 3)\} = \min \{(7 + 2), 0\} = 0$

$$A^{(2)} = \begin{bmatrix} 0 & 4 & 6 \\ 6 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

Step 3: Solving the equation for, k = 3;

$A^3(1, 1) = \min \{A^2(1, 3) + A^2(3, 1), c(1, 1)\} = \min \{(6 + 3), 0\} = 0$

$A^3(1, 2) = \min \{A^2(1, 3) + A^2(3, 2), c(1, 2)\} = \min \{(6 + 7), 4\} = 4$

$A^3(1, 3) = \min \{A^2(1, 3) + A^2(3, 3), c(1, 3)\} = \min \{(6 + 0), 6\} = 6$

$A^3(2, 1) = \min \{A^2(2, 3) + A^2(3, 1), c(2, 1)\} = \min \{(2 + 3), 6\} = 5$

$A^3(2, 2) = \min \{A^2(2, 3) + A^2(3, 2), c(2, 2)\} = \min \{(2 + 7), 0\} = 0$

$A^3(2, 3) = \min \{A^2(2, 3) + A^2(3, 3), c(2, 3)\} = \min \{(2 + 0), 2\} = 2$

$A^3(3, 1) = \min \{A^2(3, 3) + A^2(3, 1), c(3, 1)\} = \min \{(0 + 3), 3\} = 3$

$A^3(3, 2) = \min \{A^2(3, 3) + A^2(3, 2), c(3, 2)\} = \min \{(0 + 7), 7\} = 7$

$A^3 (3, 3) = \min \{A^2 (3, 3) + A^2 (3, 3), c (3, 3)\} = \min \{(0 + 0), 0\} = 0$

$$A^{(3)} = \begin{bmatrix} 0 & 4 & 6 \\ 5 & 0 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

TRAVELLING SALESPERSON PROBLEM

Let G = (V, E) be a directed graph with edge costs C_{ij}. The variable c_{ij} is defined such that $c_{ij} > 0$ for all I and j and $c_{ij} = \alpha$ if $< i, j> \notin E$. Let $|V| = n$ and assume $n > 1$. A tour of G is a directed simple cycle that includes every vertex in V. The cost of a tour is the sum of the cost of the edges on the tour. The traveling sales person problem is to find a tour of minimum cost. The tour is to be a simple path that starts and ends at vertex 1.

Let g (i, S) be the length of shortest path starting at vertex i, going through all vertices in S, and terminating at vertex 1. The function g (1, V – {1}) is the length of an optimal salesperson tour. From the principal of optimality it follows that:

$$g(1, V - \{1\}) = \min_{2 \le k \le n} \{c_{1k} + g(k, V - \{1, k\})\} \qquad -- \qquad 1$$

Generalizing equation 1, we obtain (for $i \notin S$)

$$g(i, S) = \min_{j \in S} \{c_{ij} + g(i, S - \{j\})\} \qquad -- \qquad 2$$

The Equation can be solved for g (1, V – 1}) if we know g (k, V – {1, k}) for all choices of k.

Complexity Analysis:

For each value of |S| there are n – 1 choices for i. The number of distinct sets S of size k not including 1 and i is $\binom{n-2}{k}$.

Hence, the total number of g (i, S)'s to be computed before computing g (1, V – {1}) is:

$$\sum_{k=0}^{n-1} (n-1)\binom{n-2}{k}$$

To calculate this sum, we use the binominal theorem:

$$(n-1)\left[\binom{n-2}{0} + \binom{n-2}{1} + \binom{n-2}{2} + ---- + \binom{n-2}{n-2}\right]$$

According to the binominal theorem:

$$\left[\binom{n-2}{0} + \binom{n-2}{1} + \binom{n-2}{2} + ---- + \binom{n-2}{n-2}\right] = 2^{n-2}$$

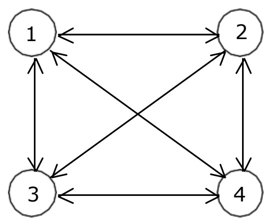
Therefore,

$$\sum_{k=0}^{n-1} (n-1)\binom{n-2}{k} = (n-1)\, 2^{n-2}$$

This is Φ (n 2^{n-2}), so there are exponential number of calculate. Calculating one g (i, S) require finding the minimum of at most n quantities. Therefore, the entire algorithm is Φ (n^2 2^{n-2}). This is better than enumerating all n! different tours to find the best one. So, we have traded on exponential growth for a much smaller exponential growth. The most serious drawback of this dynamic programming solution is the space needed, which is O (n 2^n). This is too large even for modest values of n.

Example 1:

For the following graph find minimum cost tour for the traveling salesperson problem:



The cost adjacency matrix = $\begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{bmatrix}$

Let us start the tour from vertex 1:

$$g(1, V - \{1\}) = \min_{2 \le k \le n} \{c_{1k} + g(k, V - \{1, K\})\} \qquad - \qquad (1)$$

More generally writing:

$$g(i, s) = \min \{c_{ij} + g(J, s - \{J\})\} \qquad - \qquad (2)$$

Clearly, $g(i, \Phi) = c_{i1}$, $1 \le i \le n$. So,

$g(2, \Phi) = C_{21} = 5$

$g(3, \Phi) = C_{31} = 6$

$g(4, \Phi) = C_{41} = 8$

Using equation – (2) we obtain:

$g(1, \{2, 3, 4\}) = \min \{c_{12} + g(2, \{3, 4\}, c_{13} + g(3, \{2, 4\}), c_{14} + g(4, \{2, 3\})\}$

$g(2, \{3, 4\}) = \min \{c_{23} + g(3, \{4\}), c_{24} + g(4, \{3\})\}$
$= \min \{9 + g(3, \{4\}), 10 + g(4, \{3\})\}$

$g(3, \{4\}) = \min \{c_{34} + g(4, \Phi)\} = 12 + 8 = 20$

$g(4, \{3\}) = \min \{c_{43} + g(3, \Phi)\} = 9 + 6 = 15$

Therefore, g (2, {3, 4}) = min {9 + 20, 10 + 15} = min {29, 25} = 25

g (3, {2, 4}) = min {(c_{32} + g (2, {4}), (c_{34} + g (4, {2})}

g (2, {4}) = min {c_{24} + g (4, Φ)} = 10 + 8 = 18

g (4, {2}) = min {c_{42} + g (2, Φ)} = 8 + 5 = 13

Therefore, g (3, {2, 4}) = min {13 + 18, 12 + 13} = min {41, 25} = 25

g (4, {2, 3}) = min {c_{42} + g (2, {3}), c_{43} + g (3, {2})}

g (2, {3}) = min {c_{23} + g (3, Φ} = 9 + 6 = 15

g (3, {2}) = min {c_{32} + g (2, Φ} = 13 + 5 = 18

Therefore, g (4, {2, 3}) = min {8 + 15, 9 + 18} = min {23, 27} = 23

g (1, {2, 3, 4}) = min {c_{12} + g (2, {3, 4}), c_{13} + g (3, {2, 4}), c_{14} + g (4, {2, 3})}
= min {10 + 25, 15 + 25, 20 + 23} = min {35, 40, 43} = 35

The optimal tour for the graph has length = 35

The optimal tour is: 1, 2, 4, 3, 1.

OPTIMAL BINARY SEARCH TREE

Let us assume that the given set of identifiers is {a_1, . . . , a_n} with $a_1 < a_2 < \ldots < a_n$. Let p (i) be the probability with which we search for a_i. Let q (i) be the probability that the identifier x being searched for is such that $a_i < x < a_{i+1}$, $0 \leq i \leq n$ (assume $a_0 = -\infty$ and $a_{n+1} = +\infty$). We have to arrange the identifiers in a binary search tree in a way that minimizes the expected total access time.
In a binary search tree, the number of comparisons needed to access an element at depth 'd' is d + 1, so if 'a_i' is placed at depth 'd_i', then we want to minimize:

$$\sum_{i=1}^{n} P_i \ (1 + d_i) \ .$$

Let P (i) be the probability with which we shall be searching for 'a_i'. Let Q (i) be the probability of an un-successful search. Every internal node represents a point where a successful search may terminate. Every external node represents a point where an unsuccessful search may terminate.

The expected cost contribution for the internal node for 'a_i' is:

$$P(i) \ * \ level \ (a_i) \ .$$

Unsuccessful search terminate with I = 0 (i.e at an external node). Hence the cost contribution for this node is:

$$Q \ (i) \ * \ level \ ((E_i) - 1)$$

The expected cost of binary search tree is:

$$\sum_{i=1}^{n} P(i) * level\ (a_i) + \sum_{i=0}^{n} Q\ (i) * level\ ((E_i) - 1)$$

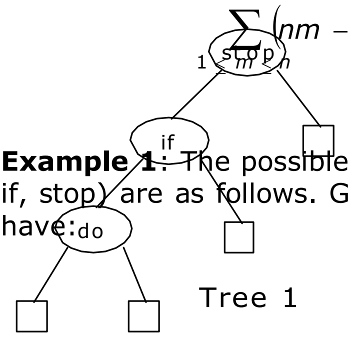
Given a fixed set of identifiers, we wish to create a binary search tree organization. We may expect different binary search trees for the same identifier set to have different performance characteristics.

The computation of each of these c(i, j)'s requires us to find the minimum of m quantities. Hence, each such c(i, j) can be computed in time O(m). The total time for all c(i, j)'s with j – i = m is therefore O(nm – m^2).

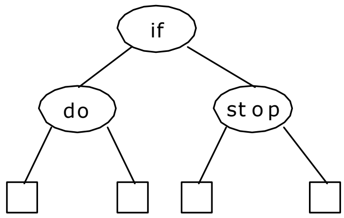
The total time to evaluate all the c(i, j)'s and r(i, j)'s is therefore:

$$\sum_{1 \le m \le n} \left(nm - m^2\right) = O\left(n^3\right)$$

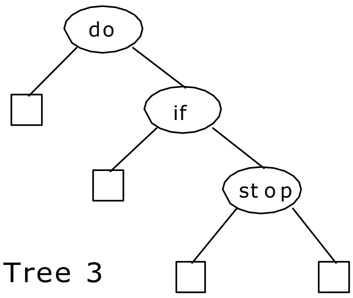
Example 1: The possible binary search trees for the identifier set (a_1, a_2, a_3) = (do, if, stop) are as follows. Given the equal probabilities p (i) = Q (i) = 1/7 for all i, we have:



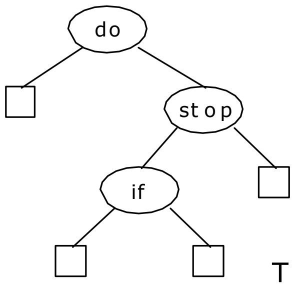
Tree 1



Tree 2



Tree 3



Tree 4

Cost (tree # 1) = $\left(\frac{1}{7} x\ 1 + \frac{1}{7}\ x\ 2 + \frac{1}{7}\ x\ 3\right) + \left(\frac{1}{7} x\ 1 + \frac{1}{7}\ x\ 2 + \frac{1}{7}\ x\ 3 + \frac{1}{7} x\ 3\right)$

$= \frac{1+2+3}{7} + \frac{1+2+3+3}{7} = \frac{6+9}{7} = \frac{15}{7}$

Cost (tree # 2) = $\left(\frac{1}{7} x1 + \frac{1}{7}\ x2 + \frac{1}{7}\ x2\right) + \left(\frac{1}{7} x2 + \frac{1}{7}\ x\ 2 + \frac{1}{7}\ x\ 2 + \frac{1}{7}\ x2\right)$

$= \frac{1+2+2}{7} + \frac{2+2+2+2}{7} = \frac{5+8}{7} = \frac{13}{7}$

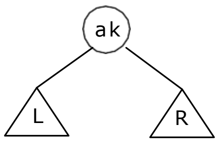
Cost (tree # 3) = $\left(\frac{1}{7} x\ 1 + \frac{1}{7} x\ 2 + \frac{1}{7} x\ 3\right) + \left(\frac{1}{7} x\ 1 + \frac{1}{7} x\ 2 + \frac{1}{7} x\ 3 + \frac{1}{7} x\ 3\right)$

$$= \frac{1+2+3}{7} + \frac{1+2+3+3}{7} = \frac{6+9}{7} = \frac{15}{7}$$

Cost (tree # 4) = $\left(\frac{1}{7} x\ 1 + \frac{1}{7} x\ 2 + \frac{1}{7} x\ 3\right) + \left(\frac{1}{7} x\ 1 + \frac{1}{7} x\ 2 + \frac{1}{7} x\ 3 + \frac{1}{7} x\ 3\right)$

$$= \frac{1+2+3}{7} + \frac{1+2+3+3}{7} = \frac{6+9}{7} = \frac{15}{7}$$

Huffman coding tree solved by a greedy algorithm has a limitation of having the data only at the leaves and it must not preserve the property that all nodes to the left of the root have keys, which are less etc. Construction of an optimal binary search tree is harder, because the data is not constrained to appear only at the leaves, and also because the tree must satisfy the binary search tree property and it must preserve the property that all nodes to the left of the root have keys, which are less.
A dynamic programming solution to the problem of obtaining an optimal binary search tree can be viewed by constructing a tree as a result of sequence of decisions by holding the principle of optimality. A possible approach to this is to make a decision as which of the a_i's be arraigned to the root node at 'T'. If we choose 'a_k' then is clear that the internal nodes for a_1, a_2, ... a_{k-1} as well as the external nodes for the classes E_0, E_1, ... E_{k-1} will lie in the left sub tree, L, of the root. The remaining nodes will be in the right subtree, R. The structure of an optimal binary search tree is:



Cost (L) = $\sum_{i=1}^{K} P(i) * level\ (a_i) + \sum_{i=0}^{K} Q(i) * (level\ (E_i) - 1)$

Cost (R) = $\sum_{i=K}^{n} P(i) * level\ (a_i) + \sum_{i=K}^{n} Q(i) * (level\ (E_i) - 1)$

The C (i, J) can be computed as:

C (i, J) = $\min_{i<k\leq J}$ {C (i, k-1) + C (k, J) + P (K) + w (i, K-1) + w (K, J)}

= $\min_{i<k\leq J}$ {C (i, K-1) + C (K, J)} + w (i, J) -- (1)

Where W (i, J) = P (J) + Q (J) + w (i, J-1) -- (2)

Initially C (i, i) = 0 and w (i, i) = Q (i) for $0 \leq i \leq n$.

Equation (1) may be solved for C (0, n) by first computing all C (i, J) such that J - i = 1
Next, we can compute all C (i, J) such that J - i = 2, Then all C (i, J) with J - i = 3 and so on.

C (i, J) is the cost of the optimal binary search tree 'T_{ij}' during computation we record the root R (i, J) of each tree 'T_{ij}'. Then an optimal binary search tree may be constructed from these R (i, J). R (i, J) is the value of 'K' that minimizes equation (1).

We solve the problem by knowing W (i, i+1), C (i, i+1) and R (i, i+1), 0 ≤ i < 4; Knowing W (i, i+2), C (i, i+2) and R (i, i+2), 0 ≤ i < 3 and repeating until W (0, n), C (0, n) and R (0, n) are obtained.

The results are tabulated to recover the actual tree.

Example 1:

Let n = 4, and (a_1, a_2, a_3, a_4) = (do, if, need, while) Let P (1: 4) = (3, 3, 1, 1) and Q (0: 4) = (2, 3, 1, 1, 1)

Solution:

Table for recording W (i, j), C (i, j) and R (i, j):

Column Row	**0**	**1**	**2**	**3**	**4**
0	2, 0, 0	3, 0, 0	1, 0, 0	1, 0, 0,	1, 0, 0
1	8, 8, 1	7, 7, 2	3, 3, 3	3, 3, 4	
2	12, 19, 1	9, 12, 2	5, 8, 3		
3	14, 25, 2	11, 19, 2			
4	16, 32, 2				

This computation is carried out row-wise from row 0 to row 4. Initially, W (i, i) = Q (i) and C (i, i) = 0 and R (i, i) = 0, 0 ≤ i < 4.

Solving for C (0, n):

First, computing all C (i, j) such that j - i = 1; j = i + 1 and as 0 ≤ i < 4; i = 0, 1, 2 and 3; i < k ≤ J. Start with i = 0; so j = 1; as i < k ≤ j, so the possible value for k = 1

W (0, 1) = P (1) + Q (1) + W (0, 0) = 3 + 3 + 2 = 8
C (0, 1) = W (0, 1) + min {C (0, 0) + C (1, 1)} = 8
R (0, 1) = 1 (value of 'K' that is minimum in the above equation).

Next with i = 1; so j = 2; as i < k ≤ j, so the possible value for k = 2

W (1, 2) = P (2) + Q (2) + W (1, 1) = 3 + 1 + 3 = 7
C (1, 2) = W (1, 2) + min {C (1, 1) + C (2, 2)} = 7
R (1, 2) = 2

Next with i = 2; so j = 3; as i < k ≤ j, so the possible value for k = 3

W (2, 3) = P (3) + Q (3) + W (2, 2) = 1 + 1 + 1 = 3
C (2, 3) = W (2, 3) + min {C (2, 2) + C (3, 3)} = 3 + [(0 + 0)] = 3
R (2, 3) = 3

Next with i = 3; so j = 4; as i < k ≤ j, so the possible value for k = 4

W (3, 4) = P (4) + Q (4) + W (3, 3) = 1 + 1 + 1 = 3
C (3, 4) = W (3, 4) + min {[C (3, 3) + C (4, 4)]} = 3 + [(0 + 0)] = 3
R (3, 4) = 4

Second, Computing all C (i, j) such that j - i = 2; j = i + 2 and as 0 ≤ i < 3; i = 0, 1, 2; i < k ≤ J. Start with i = 0; so j = 2; as i < k ≤ J, so the possible values for k = 1 and 2.

W (0, 2) = P (2) + Q (2) + W (0, 1) = 3 + 1 + 8 = 12
C (0, 2) = W (0, 2) + min {(C (0, 0) + C (1, 2)), (C (0, 1) + C (2, 2))}
= 12 + min {(0 + 7, 8 + 0)} = 19
R (0, 2) = 1
Next, with i = 1; so j = 3; as i < k ≤ j, so the possible value for k = 2 and 3.

W (1, 3) = P (3) + Q (3) + W (1, 2) = 1 + 1+ 7 = 9
C (1, 3) = W (1, 3) + min {[C (1, 1) + C (2, 3)], [C (1, 2) + C (3, 3)]}
= W (1, 3) + min {(0 + 3), (7 + 0)} = 9 + 3 = 12
R (1, 3) = 2

Next, with i = 2; so j = 4; as i < k ≤ j, so the possible value for k = 3 and 4.

W (2, 4) = P (4) + Q (4) + W (2, 3) = 1 + 1 + 3 = 5
C (2, 4) = W (2, 4) + min {[C (2, 2) + C (3, 4)], [C (2, 3) + C (4, 4)]
= 5 + min {(0 + 3), (3 + 0)} = 5 + 3 = 8
R (2, 4) = 3

Third, Computing all C (i, j) such that J - i = 3; j = i + 3 and as 0 ≤ i < 2; i = 0, 1; i < k ≤ J. Start with i = 0; so j = 3; as i < k ≤ j, so the possible values for k = 1, 2 and 3.

W (0, 3) = P (3) + Q (3) + W (0, 2) = 1 + 1 + 12 = 14
C (0, 3) = W (0, 3) + min {[C (0, 0) + C (1, 3)], [C (0, 1) + C (2, 3)],
[C (0, 2) + C (3, 3)]}
= 14 + min {(0 + 12), (8 + 3), (19 + 0)} = 14 + 11 = 25
R (0, 3) = 2

Start with i = 1; so j = 4; as i < k ≤ j, so the possible values for k = 2, 3 and 4.

W (1, 4) = P (4) + Q (4) + W (1, 3) = 1 + 1 + 9 = 11
C (1, 4) = W (1, 4) + min {[C (1, 1) + C (2, 4)], [C (1, 2) + C (3, 4)],
[C (1, 3) + C (4, 4)]}
= 11 + min {(0 + 8), (7 + 3), (12 + 0)} = 11 + 8 = 19
R (1, 4) = 2

Fourth, Computing all C (i, j) such that j - i = 4; j = i + 4 and as 0 ≤ i < 1; i = 0; i < k ≤ J.

Start with i = 0; so j = 4; as i < k ≤ j, so the possible values for k = 1, 2, 3 and 4.

W (0, 4) = P (4) + Q (4) + W (0, 3) = 1 + 1 + 14 = 16
C (0, 4) = W (0, 4) + min {[C (0, 0) + C (1, 4)], [C (0, 1) + C (2, 4)],
[C (0, 2) + C (3, 4)], [C (0, 3) + C (4, 4)]}
= 16 + min [0 + 19, 8 + 8, 19+3, 25+0] = 16 + 16 = 32
R (0, 4) = 2

From the table we see that C (0, 4) = 32 is the minimum cost of a binary search tree for (a_1, a_2, a_3, a_4). The root of the tree 'T_{04}' is 'a_2'.

Hence the left sub tree is 'T_{01}' and right sub tree is T_{24}. The root of 'T_{01}' is 'a_1' and the root of 'T_{24}' is a_3.

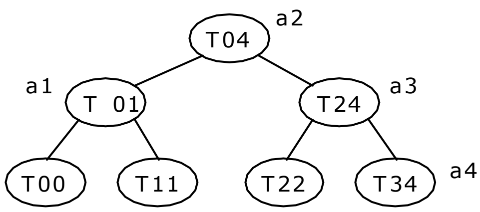
The left and right sub trees for 'T_{01}' are 'T_{00}' and 'T_{11}' respectively. The root of T_{01} is 'a_1'

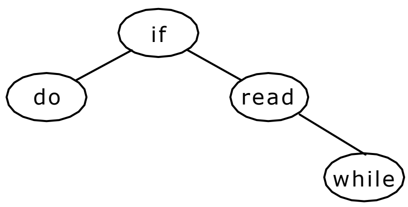
The left and right sub trees for T_{24} are T_{22} and T_{34} respectively.

The root of T_{24} is 'a_3'.

The root of T_{22} is null

The root of T_{34} is a_4.





Example 2:

Consider four elements a_1, a_2, a_3 and a_4 with Q_0 = 1/8, Q_1 = 3/16, Q_2 = Q_3 = Q_4 = 1/16 and p_1 = 1/4, p_2 = 1/8, p_3 = p_4 =1/16. Construct an optimal binary search tree. Solving for C (0, n):

First, computing all C (i, j) such that j - i = 1; j = i + 1 and as 0 $\leq$ i < 4; i = 0, 1, 2 and 3; i < k ≤ J. Start with i = 0; so j = 1; as i < k ≤ j, so the possible value for k = 1

W (0, 1) = P (1) + Q (1) + W (0, 0) = 4 + 3 + 2 = 9
C (0, 1) = W (0, 1) + min {C (0, 0) + C (1, 1)} = 9 + [(0 + 0)] = 9
R (0, 1) = 1 (value of 'K' that is minimum in the above equation).

Next with i = 1; so j = 2; as i < k ≤ j, so the possible value for k = 2

W (1, 2) = P (2) + Q (2) + W (1, 1) = 2 + 1 + 3 = 6
C (1, 2) = W (1, 2) + min {C (1, 1) + C (2, 2)} = 6 + [(0 + 0)] = 6
R (1, 2) = 2

Next with i = 2; so j = 3; as i < k ≤ j, so the possible value for k = 3

W (2, 3) = P (3) + Q (3) + W (2, 2) = 1 + 1 + 1 = 3
C (2, 3) = W (2, 3) + min {C (2, 2) + C (3, 3)} = 3 + [(0 + 0)] = 3

R (2, 3) = 3

Next with i = 3; so j = 4; as i < k ≤ j, so the possible value for k = 4

W (3, 4) = P (4) + Q (4) + W (3, 3) = 1 + 1 + 1 = 3
C (3, 4) = W (3, 4) + min {[C (3, 3) + C (4, 4)]} = 3 + [(0 + 0)] = 3
R (3, 4) = 4

Second, Computing all C (i, j) such that j - i = 2; j = i + 2 and as 0 ≤ i < 3; i = 0, 1, 2; i < k ≤ J

Start with i = 0; so j = 2; as i < k ≤ j, so the possible values for k = 1 and 2.

W (0, 2) = P (2) + Q (2) + W (0, 1) = 2 + 1 + 9 = 12
C (0, 2) = W (0, 2) + min {(C (0, 0) + C (1, 2)), (C (0, 1) + C (2, 2))}
= 12 + min {(0 + 6, 9 + 0)} = 12 + 6 = 18
R (0, 2) = 1
Next, with i = 1; so j = 3; as i < k ≤ j, so the possible value for k = 2 and 3.

W (1, 3) = P (3) + Q (3) + W (1, 2) = 1 + 1+ 6 = 8
C (1, 3) = W (1, 3) + min {[C (1, 1) + C (2, 3)], [C (1, 2) + C (3, 3)]}
= W (1, 3) + min {(0 + 3), (6 + 0)} = 8 + 3 = 11
R (1, 3) = 2

Next, with i = 2; so j = 4; as i < k ≤ j, so the possible value for k = 3 and 4.

W (2, 4) = P (4) + Q (4) + W (2, 3) = 1 + 1 + 3 = 5
C (2, 4) = W (2, 4) + min {[C (2, 2) + C (3, 4)], [C (2, 3) + C (4, 4)]
= 5 + min {(0 + 3), (3 + 0)} = 5 + 3 = 8
R (2, 4) = 3

Third, Computing all C (i, j) such that J - i = 3; j = i + 3 and as 0 ≤ i < 2; i = 0, 1; i < k ≤ J. Start with i = 0; so j = 3; as i < k ≤ j, so the possible values for k = 1, 2 and 3.

W (0, 3) = P (3) + Q (3) + W (0, 2) = 1 + 1 + 12 = 14
C (0, 3) = W (0, 3) + min {[C (0, 0) + C (1, 3)], [C (0, 1) + C (2, 3)],
[C (0, 2) + C (3, 3)]}
= 14 + min {(0 + 11), (9 + 3), (18 + 0)} = 14 + 11 = 25
R (0, 3) = 1

Start with i = 1; so j = 4; as i < k ≤ j, so the possible values for k = 2, 3 and 4.

W (1, 4) = P (4) + Q (4) + W (1, 3) = 1 + 1 + 8 = 10
C (1, 4) = W (1, 4) + min {[C (1, 1) + C (2, 4)], [C (1, 2) + C (3, 4)],
[C (1, 3) + C (4, 4)]}
= 10 + min {(0 + 8), (6 + 3), (11 + 0)} = 10 + 8 = 18
R (1, 4) = 2

Fourth, Computing all C (i, j) such that J - i = 4; j = i + 4 and as 0 ≤ i < 1; i = 0; i < k ≤ J. Start with i = 0; so j = 4; as i < k ≤ j, so the possible values for k = 1, 2, 3 and 4.

W (0, 4) = P (4) + Q (4) + W (0, 3) = 1 + 1 + 14 = 16
C (0, 4) = W (0, 4) + min {[C (0, 0) + C (1, 4)], [C (0, 1) + C (2, 4)],
[C (0, 2) + C (3, 4)], [C (0, 3) + C (4, 4)]}

= 16 + min [0 + 18, 9 + 8, 18 + 3, 25 + 0] = 16 + 17 = 33
R (0, 4) = 2

Table for recording W (i, j), C (i, j) and R (i, j)

Column Row	**0**	**1**	**2**	**3**	**4**
0	2, 0, 0	1, 0, 0	1, 0, 0	1, 0, 0,	1, 0, 0
1	9, 9, 1	6, 6, 2	3, 3, 3	3, 3, 4	
2	12, 18, 1	8, 11, 2	5, 8, 3		
3	14, 25, 2	11, 18, 2			
4	16, 33, 2				

From the table we see that C (0, 4) = 33 is the minimum cost of a binary search tree for (a_1, a_2, a_3, a_4)

The root of the tree 'T_{04}' is 'a_2'.

Hence the left sub tree is 'T_{01}' and right sub tree is T_{24}. The root of 'T_{01}' is 'a_1' and the root of 'T_{24}' is a_3.

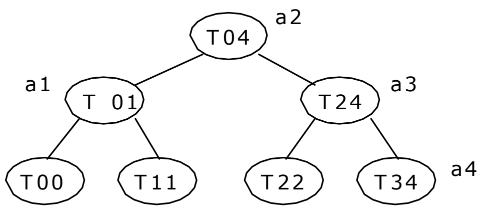
The left and right sub trees for 'T_{01}' are 'T_{00}' and 'T_{11}' respectively. The root of T_{01} is 'a_1'

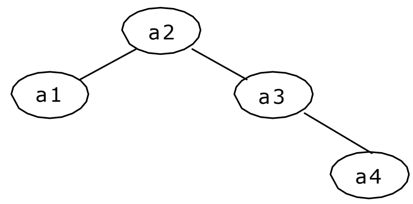
The left and right sub trees for T_{24} are T_{22} and T_{34} respectively.

The root of T_{24} is 'a_3'.

The root of T_{22} is null.

The root of T_{34} is a_4.





Example 3:

WORD	PROBABILITY
A	4
B	2
C	1
D	3
E	5
F	2
G	1

and all other elements have zero probability.

Solving c(0,n):

First computing all c(i, j) such that j- i = 1;j = i +1 and as 0 ≤ i < 7; i = 0, 1, 2, 3, 4, 5 and 6; i < k ≤ j. Start with i = 0 ; so j = 1; as i < k ≤ j, so the possible value for k = 1

W(0, 1) = P(1) + Q(1)+W(0, 0) = 4+0+0 = 4
C(0, 1) = W(0, 1)+ min {C (0, 0) + C(1, 1) }=4 + [(0 + 0)] = 4
R(0, 1) = 1

next with i = 1 ; so j = 2; as i < k ≤ j, so the possible value for k = 2

W(1, 2) = P(2) + Q(2)+W(1, 1) = 2+0+0 = 2
C(1, 2) = W(1, 2)+ min {C (1, 1) + C(2, 2) }=2 + [(0 + 0)] = 2
R(1, 2) = 2
next with i = 2 ; so j = 3; as i < k ≤ j, so the possible value for k = 3

W(2, 3) = P(3) + Q(3)+W(2, 2) = 1+0+0 = 1
C(2, 3) = W(2, 3)+ min {C (2, 2) + C(3, 3) }=1 + [(0 + 0)] = 1
R(2, 3) = 3

next with i = 3 ; so j = 4; as i < k ≤ j, so the possible value for k = 4

W(3, 4) = P(4) + Q(4)+W(3, 3) = 3+0+0 = 3
C(3, 4) = W(3, 4)+ min {C (3, 4) + C(4, 4) }=3 + [(0 + 0)] = 3
R(3, 4) = 4

next with i = 4 ; so j = 5; as i < k ≤ j, so the possible value for k = 5

W(4, 5) = P(5) + Q(5)+W(4, 4) = 5+0+0 = 5
C(4, 5) = W(4, 5)+ min {C (4, 4) + C(5, 5) }=5 + [(0 + 0)] = 5
R(4, 5) = 5

next with i = 5; so j = 6; as i < k ≤ j, so the possible value for k = 6

W(5, 6) = P(6) + Q(6)+W(5, 5) = 2+0+0 = 2
C(5, 6) = W(5, 6)+ min {C (5, 5) + C(6, 6) }=2 + [(0 + 0)] = 2
R(5, 6) = 6

next with i = 6; so j = 7; as i < k ≤ j, so the possible value for k = 7

W(6, 7) = P(7) + Q(7)+W(6, 6) = 1+0+0 = 1
C(6, 7) = W(6, 7)+ min {C (6, 6) + C(7, 7) }=1 + [(0 + 0)] = 1
R(6, 7) = 7

Second, computing all c(i, j) such that j - i = 2 ;j = i + 2 and as 0 ≤ i < 6; i = 0, 1, 2, 3, 4 and 5; i < k ≤ j; Start with i = 0 ; so j = 2; as i < k ≤ j, so the possible values for k = 1 and 2.

W(0, 2) = P(2) + Q(2)+W(0, 1) = 2 + 0 + 4 = 6
C(0, 2) = W(0, 2)+ min {C (0, 0) + C(1, 2) ,C(0, 1) + C(2, 2)}
= 6 +min{ 0 + 2, 4 + 0} = 8
R(0, 2) = 1

next with i = 1 ; so j = 3; as i < k ≤ j, so the possible values for k = 2 and 3.

W(1, 3) = P(3) + Q(3) +W(1, 2) = 1+ 0 + 2 = 3
C(1, 3) = W(1, 3)+ min {C (1, 1) + C(2,3) ,C(1, 2) + C(3, 3)}
= 3 +min{ 0 + 1, 2 + 0} = 4
R(1, 3) = 2

next with i = 2 ; so j = 4; as i < k ≤ j, so the possible values for k = 3 and 4.

W(2, 4) = P(4) + Q(4) +W(2, 3) = 3+ 0 + 1 = 4
C(2, 4) = W(2, 4)+ min {C (2, 2) + C(3,4) ,C(2, 3) + C(4, 4)}
= 4 +min{ 0 + 3, 1 + 0} = 5
R(2, 4) = 4
next with i = 3 ; so j = 5; as i < k ≤ j, so the possible values for k = 4 and 5.

W(3, 5) = P(5) + Q(5)+W(3, 4) = 5+ 0 + 3 =8
C(3, 5) = W(3, 5)+ min {C (3, 3) + C(4,5) ,C(3,4) + C(5, 5)}
= 8 +min{ 0 + 5, 3 + 0} = 11
R(3, 5) = 5
next with i = 4 ; so j = 6; as i < k ≤ j, so the possible values for k = 5 and 6.

W(4, 6) = P(6) + Q(6)+W(4, 5) = 2+ 0 + 5 = 7
C(4, 6) = W(4, 6)+ min {C (4, 4) + C(5,6) ,C(4, 5) + C(6, 6)}
= 7 +min{ 0 + 2, 5 + 0} = 9
R(4, 6) = 5

next with i = 5 ; so j = 7; as i < k ≤ j, so the possible values for k = 6 and 7.

W(5, 7) = P(7) + Q(7)+W(5, 6) = 1+ 0 + 2 = 3
C(5, 7) = W(5, 7)+ min {C (5, 5) + C(6,7) ,C(5, 6) + C(7, 7)}
= 3 +min{ 0 + 1, 2 + 0} = 4
R(5, 7) = 6

Third, computing all c(i, j) such that j – i = 3 ;j = i + 3 and as 0 ≤ i < 5 ; i = 0, 1, 2, 3, 4 and I < k ≤ j.

Start with i = 0 ; so j = 3; as i < k ≤ j, so the possible values for k = 1,2 and 3.

W(0, 3) = P(3) + Q(3)+W(0, 2) = 1+ 0 + 6 = 7
C(0, 3) = W(0, 3)+ min {C (0, 0) + C(1,3) ,C(0, 1) + C(2, 3),C(0, 2) + C(3, 3)}
= 7 +min{ 0 + 4, 4 + 1, 8 + 0} = 7
R(0, 3) = 1

next with i = 1 ; so j = 4; as i < k ≤ j, so the possible values for k = 2,3 and 4.

W(1, 4) = P(4) + Q(4)+W(1, 3) = 3+ 0 + 3 = 6
C(1, 4) = W(1, 4)+ min {C (1, 1) + C(2, 4) ,C(1, 2) + C(3, 4),C(1, 3) + C(4, 4)}
= 6 +min{ 0 + 5, 2 + 3, 4 + 0} = 10
R(1, 4) = 4

next with i = 2 ; so j = 5; as i < k ≤ j, so the possible values for k = 3, 4 and 5.

W(2, 5) = P(5) + Q(5)+W(2, 4) = 5+ 0 + 4 = 9
C(2, 5) = W(2, 5)+ min {C (2, 2) + C(3, 5) ,C(2, 3) + C(4, 5),C(2, 4) + C(5, 5)}
= 9 +min{ 0 + 11, 1 + 5 ,5 + 0} = 14
R(2, 5) = 5

next with i = 3 ; so j = 6; as i < k ≤ j, so the possible values for k = 4, 5 and 6.

W(3, 6) = P(6) + Q(6)+W(3, 5) = 2+ 0 + 8 = 10
C(3, 6) = W(3, 6)+ min {C (3, 3) + C(4, 6) ,C(3 ,4) + C(5, 6),C(3, 5) + C(6, 6)}
= 10 +min{ 0 + 9 , 3 + 2 ,11 + 0} = 15
R(3, 6) = 5

next with i = 4 ; so j = 7; as i < k ≤ j, so the possible values for k = 5, 6 and 7.

W(4, 7) = P(7) + Q(7)+W(4, 6) = 1+ 0 + 7 = 8
C(4, 7) = W(4, 7)+ min {C (4, 4) + C(5, 7) ,C(4 ,5) + C(6, 7),C(4, 6) + C(7, 7)}
= 8 +min{ 0 + 4 , 5 + 1 ,9 + 0} = 12
R(4, 7) = 5

Fourth, computing all c(i, j) such that j – i = 4 ;j = i + 4 and as 0 ≤ i < 4 ; i = 0, 1, 2, 3 for i < k ≤ j. Start with i = 0 ; so j = 4; as i < k ≤ j, so the possible values for k = 1,2 ,3 and 4.

W(0, 4) = P(4) + Q(4)+W(0, 3) = 3+ 0 + 7 = 10
C(0, 4) = W(0, 4)+ min {C (0, 0) + C(1,4) ,C(0, 1) + C(2, 4),C(0, 2) + C(3, 4),
C(0, 3) + C(4, 4)}
= 10 +min{ 0 + 10, 4 + 5,8 + 3,11 + 0} = 19
R(0, 4) = 2

next with i = 1 ; so j = 5; as i < k ≤ j, so the possible values for k = 2,3 ,4 and 5.

W(1, 5) = P(5) + Q(5)+W(1, 4) = 5+ 0 + 6 = 11
C(1, 5) = W(1, 5)+ min {C (1, 1) + C(2, 5) ,C(1, 2) + C(3, 5),C(1, 3) + C(4, 5)
C(1, 4) + C(5, 5)}
= 11 +min{ 0 + 14, 2 + 11,4 + 5,10 +0} = 20
R(1, 5) = 4

next with i = 2 ; so j = 6; as i < k ≤ j, so the possible values for k = 3,4,5 and 6.

W(2, 6) = P(6) + Q(6)+W(2, 5) = 2+ 0 + 9 = 11
C(2, 6) = W(2, 6)+ min {C (2, 2) + C(3, 6) ,C(2, 3) + C(4, 6),C(2, 4) + C(5, 6)
C(2, 5) + C (6, 6)} = 11 +min{ 0 + 15, 1 + 9 ,5 + 2,14 + 0} = 18
R(2, 6) = 5

next with i = 3 ; so j = 7; as i < k ≤ j, so the possible values for k = 4,5,6 and 7.

W(3, 7) = P(7) + Q(7)+W(3, 6) = 1+ 0 +11 = 12
C(3, 7) = W(3, 7)+ min {C (3, 3) + C(4, 7) ,C(3, 4) + C(5, 7),C(3, 5) + C(6, 7)
C(3, 6) + C (7, 7)} = 12 +min{ 0 + 12, 3 +4 ,11 +1,15 + 0} = 19
R(3, 7) = 5

Fifth, computing all c(i, j) such that j – i = 5; j = i + 5 and as 0 ≤ i < 3;
i = 0, 1, 2, i < k ≤ j. Start with i = 0 ; so j = 4; as i < k ≤ j, so the possible values for k = 1,2 ,3,4 and 5.

W(0, 5) = P(5) + Q(5)+W(0, 4) = 5+ 0 + 10 = 15
C(0, 5) = W(0, 5)+ min {C (0, 0) + C(1,5) ,C(0, 1) + C(2, 5),C(0, 2) + C(3, 5),
C(0, 3) + C(4, 5),C(0, 4) + C(5, 5)}
= 10 +min{ 0 + 20, 4 + 14, 8 + 11 ,19 + 0} = 28

R(0, 5) = 2

next with i = 1 ; so j = 6; as i < k ≤ j, so the possible values for k = 2, 3 ,4, 5 & 6.

W(1, 6) = P(6) + Q(6)+W(1, 5) = 2+ 0 + 11 = 13
C(1, 6) = W(1, 6)+ min {C (1, 1) + C(2, 6) ,C(1, 2) + C(3, 6),C(1, 3) + C(4, 6)
C(1, 4) + C(5, 6),C(1, 5)+C(6, 6)}
= 13 +min{ 0 + 18, 2 + 15, 4 + 9, 10 +2, 20 + 0} = 25
R(1, 6) = 5

next with i = 2 ; so j = 7; as i < k ≤ j, so the possible values for k = 3,4,5,6 and 7.

W(2, 7) = P(7) + Q(7)+W(2, 6) = 1+ 0 + 11 = 12
C(2, 7) = W(2, 7)+ min {C (2, 2) + C(3, 7) ,C(2, 3) + C(4, 7),C(2, 4) + C(5, 7)
C(2, 5) + C (6, 7),C(2, 6) + C(7,7)}
= 12 +min{ 0 + 18, 1 + 12 , 5 + 4, 14 + 1, 18 + 0} = 21
R(2, 7) = 5

Sixth, computing all c(i, j) such that j – i = 6 ;j = i + 6 and as 0 ≤ i < 2 ; i = 0, 1 i < k ≤ j. Start with i = 0; so j = 6; as i < k ≤ j, so the possible values for k = 1, 2, 3, 4 5 & 6.
W(0, 6) = P(6) + Q(6)+W(0, 5) = 2+ 0 + 15 = 17
C(0, 6) = W(0,6)+ min {C (0, 0) + C(1,6) ,C(0, 1) + C(2, 6),C(0, 2) + C(3, 6),
C(0, 3) + C(4, 6),C(0, 4) + C(5, 6),C(0, 5) + C(6, 6)}
= 17 +min{ 0 + 25, 4 + 18, 8 + 15,19 + 2, 31 + 0} = 37
R(0, 6) = 4

next with i = 1 ; so j = 7; as i < k ≤ j, so the possible values for k = 2, 3, 4, 5, 6 and 7.

W(1, 7) = P(7) + Q(7)+W(1, 6) = 1+ 0 + 13 = 14
C(1, 7) = W(1, 7)+ min {C (1, 1) + C(2, 7) ,C(1, 2) + C(3, 7),C(1, 3) + C(4, 7)
C(1, 4) + C(5, 7),C(1, 5)+C(6, 7),C(1, 6) +C(7, 7)}
= 14 +min{ 0 + 21, 2 + 18, 4 + 12, 10 + 4, 20 + 1, 21 + 0} = 28
R(1, 7) = 5

Seventh, computing all c(i, j) such that j – i = 7 ;j = i + 7 and as 0 ≤ i < 1 ; i = 0 i < k ≤ j. Start with i = 0 ; so j = 7; as i < k ≤ j, so the possible values for k = 1, 2, 3, 4, 5, 6 and 7.

W(0, 7) = P(7) + Q(7)+W(0, 6) = 1+ 0 + 17 = 18
C(0, 7) = W(0, 7)+ min {C (0, 0) + C(1, 7) ,C(0, 1) + C(2, 7),C(0, 2) + C(3, 7),
C(0, 3) + C(4, 7),C(0, 4) + C(5, 6),C(0, 5) + C (6, 7),C(0, 6) + C(7, 7)}
= 18 +min{ 0 + 28, 4 + 21, 8 + 18,19 +4, 31 + 1, 37 + 0} = 41
R(0, 7) = 4

0/1 – KNAPSACK

We are given n objects and a knapsack. Each object i has a positive weight w_i and a positive value V_i. The knapsack can carry a weight not exceeding W. Fill the knapsack so that the value of objects in the knapsack is optimized.

A solution to the knapsack problem can be obtained by making a sequence of decisions on the variables $x_1, x_2, \ldots, x_n$. A decision on variable x_i involves determining which of the values 0 or 1 is to be assigned to it. Let us assume that decisions on the x_i are made in the order $x_n, x_{n-1}, \ldots x_1$. Following a decision on x_n, we may be in one of two possible states: the capacity remaining in $m - w_n$ and a profit of p_n has accrued. It is clear that the remaining decisions $x_{n-1}, \ldots, x_1$ must be optimal with respect to the problem state resulting from the decision on x_n. Otherwise, $x_n, \ldots, x_1$ will not be optimal. Hence, the principal of optimality holds.

$$F_n(m) = \max\{f_{n-1}(m), f_{n-1}(m - w_n) + p_n\} \qquad \text{--} \qquad 1$$

For arbitrary $f_i(y)$, $i > 0$, this equation generalizes to:

$$F_i(y) = \max\{f_{i-1}(y), f_{i-1}(y - w_i) + p_i\} \qquad \text{--} \qquad 2$$

Equation-2 can be solved for $f_n(m)$ by beginning with the knowledge $f_o(y) = 0$ for all y and $f_i(y) = -\propto$, $y < 0$. Then $f_1, f_2, \ldots f_n$ can be successively computed using equation–2.

When the w_i's are integer, we need to compute $f_i(y)$ for integer y, $0 \leq y \leq m$. Since $f_i(y) = -\propto$ for $y < 0$, these function values need not be computed explicitly. Since each f_i can be computed from $f_i - 1$ in $\Theta(m)$ time, it takes $\Theta(m\ n)$ time to compute f_n. When the wi's are real numbers, $f_i(y)$ is needed for real numbers y such that $0 \leq y \leq m$. So, f_i cannot be explicitly computed for all y in this range. Even when the w_i's are integer, the explicit $\Theta(m\ n)$ computation of f_n may not be the most efficient computation. So, we explore **an alternative method for both cases.**

The $f_i(y)$ is an ascending step function; i.e., there are a finite number of y's, $0 = y_1 < y_2 < \ldots < y_k$, such that $f_i(y_1) < f_i(y_2) < \ldots < f_i(y_k)$; $f_i(y) = -\propto$, $y < y_1$; $f_i(y) = f(y_k)$, $y \geq y_k$; and $f_i(y) = f_i(y_j)$, $y_j \leq y \leq y_{j+1}$. So, we need to compute only $f_i(y_j)$, $1 \leq j \leq k$. We use the ordered set $S^i = \{(f(y_j), y_j) \mid 1 \leq j \leq k\}$ to represent $f_i(y)$. Each number of S^i is a pair (P, W), where $P = f_i(y_j)$ and $W = y_j$. Notice that $S^0 = \{(0, 0)\}$. We can compute S^{i+1} from S_i by first computing:

$$S^i_1 = \{(P, W) \mid (P - p_i, W - w_i)\ \varepsilon\ S^i\}$$

Now, S^{i+1} can be computed by merging the pairs in S^i and S^i_1 together. Note that if S^{i+1} contains two pairs (P_j, W_j) and (P_k, W_k) with the property that $P_j \leq P_k$ and $W_j \geq W_k$, then the pair (P_j, W_j) can be discarded because of equation-2. Discarding or purging rules such as this one are also known as dominance rules. Dominated tuples get purged. In the above, (P_k, W_k) dominates (P_j, W_j).

Example 1:

Consider the knapsack instance n = 3, $(w_1, w_2, w_3) = (2, 3, 4)$, $(P_1, P_2, P_3) = (1, 2, 5)$ and M = 6.

Solution:

Initially, $f_o(x) = 0$, for all x and $f_i(x) = -\propto$ if $x < 0$.

$F_n(M) = \max\{f_{n-1}(M), f_{n-1}(M - w_n) + p_n\}$

$F_3(6) = \max(f_2(6), f_2(6 - 4) + 5\} = \max\{f_2(6), f_2(2) + 5\}$

$F_2(6) = \max(f_1(6), f_1(6 - 3) + 2\} = \max\{f_1(6), f_1(3) + 2\}$

$F_1(6) = \max(f_0(6), f_0(6-2)+1\} = \max\{0, 0+1\} = 1$

$F_1(3) = \max(f_0(3), f_0(3-2)+1\} = \max\{0, 0+1\} = 1$

Therefore, $F_2(6) = \max(1, 1+2\} = 3$

$F_2(2) = \max(f_1(2), f_1(2-3)+2\} = \max\{f_1(2), -\infty + 2\}$

$F_1(2) = \max(f_0(2), f_0(2-2)+1\} = \max\{0, 0+1\} = 1$

$F_2(2) = \max\{1, -\infty + 2\} = 1$

Finally, $f_3(6) = \max\{3, 1+5\} = 6$

Other Solution:

For the given data we have:

$S^0 = \{(0, 0)\}$; $S^0_1 = \{(1, 2)\}$

$S^1 = (S^0 \cup S^0_1) = \{(0, 0), (1, 2)\}$

$X - 2 = 0 \Rightarrow x = 2.$	$y - 3 = 0 \Rightarrow y = 3$
$X - 2 = 1 \Rightarrow x = 3.$	$y - 3 = 2 \Rightarrow y = 5$

$S^1_1 = \{(2, 3), (3, 5)\}$

$S^2 = (S^1 \cup S^1_1) = \{(0, 0), (1, 2), (2, 3), (3, 5)\}$

$X - 5 = 0 \Rightarrow x = 5.$	$y - 4 = 0 \Rightarrow y = 4$
$X - 5 = 1 \Rightarrow x = 6.$	$y - 4 = 2 \Rightarrow y = 6$
$X - 5 = 2 \Rightarrow x = 7.$	$y - 4 = 3 \Rightarrow y = 7$
$X - 5 = 3 \Rightarrow x = 8.$	$y - 4 = 5 \Rightarrow y = 9$

$S^2_1 = \{(5, 4), (6, 6), (7, 7), (8, 9)\}$

$S^3 = (S^2 \cup S^2_1) = \{(0, 0), (1, 2), (2, 3), (3, 5), (5, 4), (6, 6), (7, 7), (8, 9)\}$

By applying Dominance rule,

$S^3 = (S^2 \cup S^2_1) = \{(0, 0), (1, 2), (2, 3), (5, 4), (6, 6)\}$

From (6, 6) we can infer that the maximum Profit $\sum p_i x_i = 6$ and weight $\sum x_i w_i = 6$

Reliability Design

The problem is to design a system that is composed of several devices connected in series. Let r_i be the reliability of device D_i (that is r_i is the probability that device i will function properly) then the reliability of the entire system is Πr_i. Even if the individual devices are very reliable (the r_i's are very close to one), the reliability of the system may not be very good. For example, if n = 10 and $r_i = 0.99$, $i \le i \le 10$, then $\Pi r_i = .904$. Hence, it is desirable to duplicate devices. Multiply copies of the same device type are connected in parallel.

If stage i contains mi copies of device D_i. Then the probability that all m_i have a malfunction is $(1 - r_i)^{m_i}$. Hence the reliability of stage i becomes $1 - (1 - r_i)^{m_i}$.

The reliability of stage 'i' is given by a function ϕ_i (m_i).

Our problem is to use device duplication. This maximization is to be carried out under a cost constraint. Let c_i be the cost of each unit of device i and let c be the maximum allowable cost of the system being designed.

We wish to solve:

$$\text{Maximize} \prod_{1 \leq i \leq n} \phi_i (m_i)$$

$$\text{Subject to} \sum_{1 \leq i \leq n} C_i \, m_i < C$$

$m_i \geq 1$ and interger, $1 \leq i \leq n$

Assume each $C_i > 0$, each m_i must be in the range $1 \leq m_i \leq u_i$, where

$$u_i = \left\lfloor \left(C + C_i - \sum_{1}^{n} C_J \right) \Big/ C_i \right\rfloor$$

The upper bound u_i follows from the observation that $m_j \geq 1$

An optimal solution $m_1, m_2 \ldots\ldots m_n$ is the result of a sequence of decisions, one decision for each m_i.

Let f_i (x) represent the maximum value of $\prod_{1 \leq j \leq i} \phi (m_J)$

Subject to the constrains:

$$\sum_{1 \leq J \leq i} C_J \, m_J \leq x \quad \text{and} \quad 1 \leq m_j \leq u_J, \; 1 \leq j \leq i$$

The last decision made requires one to choose m_n from $\{1, 2, 3, \ldots\ldots u_n\}$

Once a value of m_n has been chosen, the remaining decisions must be such as to use the remaining funds $C - C_n m_n$ in an optimal way.

The principle of optimality holds on

$$f_n(C) = \max_{1 \leq m_n \leq u_n} \{ \phi_n (m_n) \, f_{n-1} (C - C_n \, m_n) \}$$

for any f_i (x_i), $i > 1$, this equation generalizes to

$$f_n(x) = \max_{1 \le m_i \le u_i} \{\phi_i(m_i)\, f_{i-1}(x - C_i\, m_i)\}$$

clearly, $f_0(x) = 1$ for all x, $0 \le x \le C$ and $f(x) = -\infty$ for all $x < 0$.

Let S^i consist of tuples of the form (f, x), where $f = f_i(x)$.

There is atmost one tuple for each different 'x', that result from a sequence of decisions on $m_1, m_2, \ldots m_n$. The dominance rule (f_1, x_1) dominate (f_2, x_2) if $f_1 \ge f_2$ and $x_1 \le x_2$. Hence, dominated tuples can be discarded from S^i.

Example 1:

Design a three stage system with device types D_1, D_2 and D_3. The costs are \$30, \$15 and \$20 respectively. The Cost of the system is to be no more than \$105. The reliability of each device is 0.9, 0.8 and 0.5 respectively.

Solution:

We assume that if if stage I has mi devices of type i in parallel, then $\phi_i(m_i) = 1 - (1-r_i)^{m_i}$

Since, we can assume each $c_i > 0$, each m_i must be in the range $1 \le m_i \le u_i$. Where:

$$u_i = \left\lfloor \left(C + C_i - \sum_1^n C_J \right) \Big/ C_i \right\rfloor$$

Using the above equation compute u_1, u_2 and u_3.

$$u_1 = \frac{105 + 30 - (30 + 15 + 20)}{30} = \frac{70}{30} = 2$$

$$u_2 = \frac{105 + 15 - (30 + 15 + 20)}{15} = \frac{55}{15} = 3$$

$$u_3 = \frac{105 + 20 - (30 + 15 + 20)}{20} = \frac{60}{20} = 3$$

We use $S_j^i \rightarrow i: stage\ number\ and\ J:\ no.\ of\ devices\ in\ stage\ i = m_i$

$S^o = \{f_o(x), x\}$ $\quad$ $initially\ f_o(x) = 1\ and\ x = 0,\ so,\ S^o = \{1, 0\}$

Compute S^1, S^2 and S^3 as follows:

S^1 = depends on u_1 value, as $u_1 = 2$, so

$$S^1 = \{S_1^1, S_2^1\}$$

S^2 = depends on u_2 value, as $u_2 = 3$, so

$S^2 = \{S^2_1, S^2_2, S^2_3\}$

S^3 = depends on u_3 value, as $u_3 = 3$, so

$S^3 = \{S^3_1, S^3_2, S^3_3\}$

Now find $S_1^1 = \{(f_1(x), x)\}$

$f_1(x) = \{\phi_1(1) f_o(), \phi_1(2) f_0()\}$ With devices $m_1 = 1$ and $m_2 = 2$

Compute $\phi_1(1)$ and $\phi_1(2)$ using the formula: $\phi_i(mi)) = 1 - (1 - r_i)^{mi}$

$\phi_1(1) = 1 - (1 - r_1)^{m1} = 1 - (1 - 0.9)^1 = 0.9$

$\phi_1(2) = 1 - (1 - 0.9)^2 = 0.99$

$S^1_1 = \{f_1(x), x\} = = (0.9, 30)$

$S_2^1 = \{0.99, 30 + 30\} = (0.99, 60)$

Therefore, $S^1 = \{(0.9, 30), (0.99, 60)\}$

Next find $S_1^2 = \{(f_2(x), x)\}$

$f_2(x) = \{\phi_2(1) * f_1(), \phi_2(2) * f_1(), \phi_2(3) * f_1()\}$

$\phi_2(1) = 1 - (1 - r_I)^{mi} = 1 - (1 - 0.8)^1 = 1 - 0.2 = 0.8$

$\phi_2(2) = 1 - (1 - 0.8)^2 = 0.96$

$\phi_2(3) = 1 - (1 - 0.8)^3 = 0.992$

$S_1^2 = \{(0.8(0.9), 30 + 15), (0.8(0.99), 60 + 15)\} = \{(0.72, 45), (0.792, 75)\}$

$S_2^2 = \{(0.96(0.9), 30 + 15 + 15), (0.96(0.99), 60 + 15 + 15)\}$
$= \{(0.864, 60), (0.9504, 90)\}$

$S_3^2 = \{(0.992(0.9), 30 + 15 + 15 + 15), (0.992(0.99), 60 + 15 + 15 + 15)\}$
$= \{(0.8928, 75), (0.98208, 105)\}$

$S^2 = \{S^2_1, S^2_2, S^2_3\}$

By applying Dominance rule to S^2:

Therefore, $S^2 = \{(0.72, 45), (0.864, 60), (0.8928, 75)\}$

<u>Dominance Rule</u>:

If S^i contains two pairs (f_1, x_1) and (f_2, x_2) with the property that $f_1 \geq f_2$ and $x_1 \leq x_2$, then (f_1, x_1) dominates (f_2, x_2), hence by dominance rule (f_2, x_2) can be discarded. Discarding or pruning rules such as the one above is known as dominance rule. Dominating tuples will be present in S^i and Dominated tuples has to be discarded from S^i.

Case 1: if $f_1 \leq f_2$ and $x_1 > x_2$ then discard (f_1, x_1)

Case 2: if $f_1 \geq f_2$ and $x_1 < x_2$ the discard (f_2, x_2)

Case 3: otherwise simply write (f_1, x_1)

$S_2 = \{(0.72, 45), (0.864, 60), (0.8928, 75)\}$

$\phi_3\ (1) = 1 - (1 - r_I)^{mi} = 1 - (1 - 0.5)^1 = 1 - 0.5 = 0.5$

$\phi_3\ (2) = 1 - (1 - 0.5)^2 = 0.75$

$\phi_3\ (3) = 1 - (1 - 0.5)^3 = 0.875$

$S_1^3 = \{(0.5\ (0.72), 45 + 20), (0.5\ (0.864), 60 + 20), (0.5\ (0.8928), 75 + 20)\}$

$S_1^3 = \{(0.36, 65), (0.437, 80), (0.4464, 95)\}$

$S_2^3 = \{(0.75\ (0.72), 45 + 20 + 20), (0.75\ (0.864), 60 + 20 + 20),$
$(0.75\ (0.8928), 75 + 20 + 20)\}$

$= \{(0.54, 85), (0.648, 100), (0.6696, 115)\}$

$S_3^3 = \{\ (0.875\ (0.72), 45 + 20 + 20 + 20), (0.875\ (0.864), 60 + 20 + 20 + 20),$
$(0.875\ (0.8928), 75 + 20 + 20 + 20)\ \}$

$S_3^3 = \{(0.63, 105), (1.756, 120), (0.7812, 135)\}$

If cost exceeds 105, remove that tuples

$S^3 = \{(0.36, 65), (0.437, 80), (0.54, 85), (0.648, 100)\}$

The best design has a reliability of 0.648 and a cost of 100. Tracing back for the solution through S^i's we can determine that $m_3 = 2$, $m_2 = 2$ and $m_1 = 1$.

Other Solution:

According to the principle of optimality:

$$f_n(C) = \max_{1 \leq m_n \leq u_n} \{\phi_n (m_n).\ f_{n-1} (C - C_n m_n) \text{ with } f_o (x) = 1 \text{ and } 0 \leq x \leq C;$$

Since, we can assume each $c_i > 0$, each mi must be in the range $1 \leq m_i \leq u_i$. Where:

$$u_i = \left\lfloor \left(C + C_i - \sum_{i}^{n} C_J \right) / C_i \right\rfloor$$

Using the above equation compute u_1, u_2 and u_3.

$$u_1 = \frac{105 + 30 - (30 + 15 + 20)}{30} = \frac{70}{30} = 2$$

$$u_2 = \frac{105 + 15 - (30 + 15 + 20)}{15} = \frac{55}{15} = 3$$

$$u_3 = \frac{105 + 20 - (30 + 15 + 20)}{20} = \frac{60}{20} = 3$$

$f_3(105) = \max\limits_{1 \le m3 \le u3} \{\phi_3(m_3).\ f_2(105 - 20m_3)\}$

$= \max \{\phi_3(1)\ f_2(105 - 20),\ \phi_3(2)\ f_2(105 - 20x2),\ \phi_3(3)\ f_2(105 - 20x3)\}$

$= \max \{0.5\ f_2(85),\ 0.75\ f_2(65),\ 0.875\ f_2(45)\}$

$= \max \{0.5 \text{ x } 0.8928,\ 0.75 \text{ x } 0.864,\ 0.875 \text{ x } 0.72\} = 0.648.$

$f_2\ (85) = \max\limits_{1 \le m2 \le u2} \{\phi_2(m_2).\ f_1(85 - 15m_2)\}$

$= \max \{\phi_2(1).f_1(85 - 15),\ \phi_2(2).f_1(85 - 15x2),\ \phi_2(3).f_1(85 - 15x3)\}$

$= \max \{0.8\ f_1(70),\ 0.96\ f_1(55),\ 0.992\ f_1(40)\}$

$= \max \{0.8 \text{ x } 0.99,\ 0.96 \text{ x } 0.9,\ 0.99 \text{ x } 0.9\} = 0.8928$

$f_1\ (70) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(70 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(70 - 30),\ \phi_1(2)\ f_0(70 - 30x2)\}$

$= \max \{\phi_1(1) \text{ x } 1,\ \phi_1(2) \text{ x } 1\} = \max \{0.9,\ 0.99\} = 0.99$

$f_1\ (55) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(55 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(50 - 30),\ \phi_1(2)\ f_0(50 - 30x2)\}$

$= \max \{\phi_1(1) \text{ x } 1,\ \phi_1(2) \text{ x } -\infty\} = \max \{0.9,\ -\infty\} = 0.9$

$f_1\ (40) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(40 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(40 - 30),\ \phi_1(2)\ f_0(40 - 30x2)\}$

$= \max \{\phi_1(1) \text{ x } 1,\ \phi_1(2) \text{ x } -\infty\} = \max\{0.9,\ -\infty\} = 0.9$

$f_2\ (65) = \max\limits_{1 \le m2 \le u2} \{\phi_2(m_2).\ f_1(65 - 15m_2)\}$

$= \max \{\phi_2(1)\ f_1(65 - 15),\ \underline{\phi_2(2)\ f_1(65 - 15x2)},\ \phi_2(3)\ f_1(65 - 15x3)\}$

$= \max \{0.8\ f_1(50),\ 0.96\ f_1(35),\ 0.992\ f_1(20)\}$

$= \max \{0.8 \times 0.9,\ 0.96 \times 0.9,\ -\infty\} = 0.864$

$f_1\ (50) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(50 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(50 - 30),\ \phi_1(2)\ f_0(50 - 30x2)\}$

$= \max \{\phi_1(1) \times 1,\ \phi_1(2) \times -\infty\} = \max\{0.9, -\infty\} = 0.9$

$f_1\ (35) = \max\limits_{1 \le m1 \le u1} \phi_1(m_1).\ f_0(35 - 30m_1)\}$

$= \max \{\underline{\phi_1(1).f_0(35\text{-}30)},\ \phi_1(2).f_0(35\text{-}30x2)\}$

$= \max \{\phi_1(1) \times 1,\ \phi_1(2) \times -\infty\} = \max\{0.9, -\infty\} = 0.9$

$f_1\ (20) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(20 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(20 - 30),\ \phi_1(2)\ f_0(20 - 30x2)\}$

$= \max \{\phi_1(1) \times -\infty,\ \phi_1(2) \times -\infty\} = \max\{-\infty, -\infty\} = -\infty$

$f_2\ (45) = \max\limits_{1 \le m2 \le u2} \{\phi_2(m_2).\ f_1(45 - 15m_2)\}$

$= \max \{\phi_2(1)\ f_1(45 - 15),\ \phi_2(2)\ f_1(45 - 15x2),\ \phi_2(3)\ f_1(45 - 15x3)\}$

$= \max \{0.8\ f_1(30),\ 0.96\ f_1(15),\ 0.992\ f_1(0)\}$

$= \max \{0.8 \times 0.9,\ 0.96 \times -\infty,\ 0.99 \times -\infty\} = 0.72$

$f_1\ (30) = \max\limits_{1 \le m1 \le u1} \{\phi_1(m_1).\ f_0(30 - 30m_1)\}$

$= \max \{\phi_1(1)\ f_0(30 - 30),\ \phi_1(2)\ f_0(30 - 30x2)\}$

$= \max \{\phi_1(1) \times 1,\ \phi_1(2) \times -\infty\} = \max\{0.9, -\infty\} = 0.9$

Similarly, $f_1\,(15) = -\infty$, $f_1\,(0) = -\infty$.

The best design has a reliability = 0.648 and

Cost = 30 x 1 + 15 x 2 + 20 x 2 = 100.

Tracing back for the solution through S^i's we can determine that:

$m_3 = 2$, $m_2 = 2$ and $m_1 = 1$.