Theorem 2.4

$A \in \mathrm{F}(X)$

(x) ${}^{\alpha}A = \bigcap_{\beta<\alpha} {}^{\beta}A = \bigcap_{\beta<\alpha} {}^{\beta+}A$

(1) $\forall \beta < \alpha, \underset{\text{Theorem 2.1}}{\Rightarrow} {}^{\alpha}A \subseteq {}^{\beta}A \Leftrightarrow {}^{\alpha}A \subseteq \bigcap_{\beta<\alpha} {}^{\beta}A \longrightarrow \text{(A)}$

(2) $\forall x \in \bigcap_{\beta<\alpha} {}^{\beta}A, \Rightarrow x \in {}^{\beta}A, \forall \beta < \alpha$

let $\beta = \alpha - \varepsilon, \varepsilon > 0, \Rightarrow X \in {}^{\beta}A = {}^{\alpha-\varepsilon}A$

$\therefore A(x) \geq \alpha - \varepsilon, \because \varepsilon$ can be infinitely small.

let $\varepsilon \to 0 \Rightarrow A(x) \geq \alpha,$ i.e. $x \in {}^{\alpha}A$

$\therefore \bigcap_{\beta<\alpha} {}^{\beta}A \subseteq {}^{\alpha}A \longrightarrow \text{(B)}$

(A),(B) $\Rightarrow {}^{\alpha}A = \bigcap_{\beta<\alpha} {}^{\beta}A$

Proof: (xi) ${}^{\alpha+}A = \bigcup_{\alpha<\beta} {}^{\beta}A = \bigcup_{\alpha<\beta} {}^{\beta+}A$

i, $\Leftarrow$

for any $\alpha < \beta \Rightarrow {}^{\beta+}A \subseteq {}^{\beta}A \subseteq {}^{\alpha+}A$

$\Rightarrow \bigcup_{\alpha<\beta} {}^{\beta+}A \subseteq \bigcup_{\alpha<\beta} {}^{\beta}A \subseteq {}^{\alpha}A$

ii, $\Rightarrow \forall \mathrm{x} \in {}^{\alpha+}A \Rightarrow A(x) > \alpha$

Let $\beta = \alpha + \varepsilon, \varepsilon > 0, \alpha = \beta - \varepsilon$

$\because A(x) > \alpha = \beta - \varepsilon$

as $\varepsilon \to 0, A(x) > \beta, \therefore x \in {}^{\beta+}A$ *and* $x \in \bigcup_{\alpha<\beta} {}^{\beta+}A$

$\because {}^{\beta+}A \subset {}^{\beta}A \ \therefore \bigcup_{\alpha<\beta} {}^{\beta+}A \subset \bigcup_{\alpha<\beta} {}^{\beta}A \quad \therefore x \in \bigcup_{\alpha<\beta} {}^{\beta}A$

2.2 Representations of fuzzy sets

◎ Representations of fuzzy sets by crisp sets (decomposition)

e.g. $A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1.0}{x_5}$

This can be represented by itsα-cut

α-cuts

$$^{0.2}A = \{x_1, x_2, x_3, x_4, x_5\}$$
$$^{0.4}A = \{x_2, x_3, x_4, x_5\}$$
$$^{0.6}A = \{x_3, x_4, x_5\}$$
$$^{0.8}A = \{x_4, x_5\}$$
$$^{1.0}A = \{x_5\}$$

Define a fuzzy set $_{\alpha}A$ for eachα-cut as

$$_{\alpha}A = \sum_{x \in {}^{\alpha}A} \frac{\alpha}{x} \qquad \text{fuzzy}\alpha\text{-cut}$$

$$_{0.2}A = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5}$$
$$_{0.4}A = \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5}$$
$$_{0.6}A = \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$$
$$_{0.8}A = \frac{0.8}{x_4} + \frac{0.8}{x_5}$$
$$_{1.0}A = \frac{1.0}{x_5}$$
$$\because A = \bigcup_{\alpha \subset \Lambda} {}_{\alpha}A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1.0}{x_5}$$

◎ Decomposition theorems of fuzzy sets

- **Theorem 2.5** (First decomposition Theorem)

$$A = \bigcup_{\alpha\in[0,1]} {}_{\alpha}A, \ where \ {}_{\alpha}A = \sum_{x\in {}^{\alpha}A} \frac{\alpha}{x}$$

proof: $\forall x \in X$, Let $A(x) = a$

$$\Rightarrow (\bigcup_{\alpha\in[0,1]} {}_{\alpha}A)(x) = \sup_{\alpha\in[0,1]} {}_{\alpha}A(x)$$

$$\Rightarrow \max[\sup_{\alpha\in[0,a]} {}_{\alpha}A(x), \sup_{\alpha\in[a,1]} {}_{\alpha}A(x)]$$

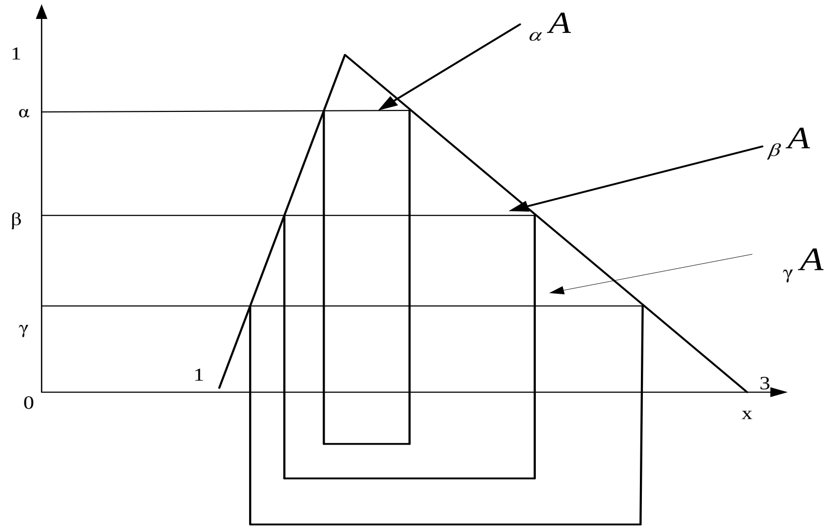
$$\begin{pmatrix} \forall\alpha \in (a,1], A(x) = a < 2, \therefore x \notin {}^{\alpha}A \Rightarrow {}_{\alpha}A(x) = 0 \\ \forall\alpha \in (0,a], A(x) = a \geq 2, \therefore x \in {}^{\alpha}A \Rightarrow {}_{\alpha}A(x) = \alpha \end{pmatrix}$$

$$\Rightarrow \max[\sup_{\alpha\in[0,a]} \alpha, 0] = \max[a, 0] = a$$

$$\therefore \bigcup_{\alpha\in[0,1]} {}_{\alpha}A = A$$

Example :

A: a fuzzy set with membership function



$$A(x)=\begin{cases} x-1 & x\in[1,2] \\ 3-x & x\in[2,3] \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \forall\alpha\in(0,1],$$

$$\alpha-cut\quad {}_{\alpha}A=\begin{cases} 2 & x\in[\alpha+1,3-\alpha] \\ 0 & \text{otherwise} \end{cases}$$

according to theorem 2.5

$$A=\bigcup_{\alpha\in[0,1]} {}_{\alpha}A$$

- **Theorem 2.6** (Second decomposition Theorem)

$$A=\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A,\quad {}_{\alpha+}A=\sum_{x\in{}_{\alpha+}A}\frac{\alpha}{x}$$

proof: $\forall x \in X$, Let $A(x) = a$

$$\Rightarrow \left(\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A\right)(x) = \sup_{\alpha\in[0,1]} {}_{\alpha+}A(x)$$

$$\Rightarrow \max\left[\sup_{\alpha\in[0,a]} {}_{\alpha+}A(x),\ \sup_{\alpha\in[a,1]} {}_{\alpha+}A(x)\right]$$

$$\sup_{\alpha\in[0,a]} \alpha = a = A(x)$$

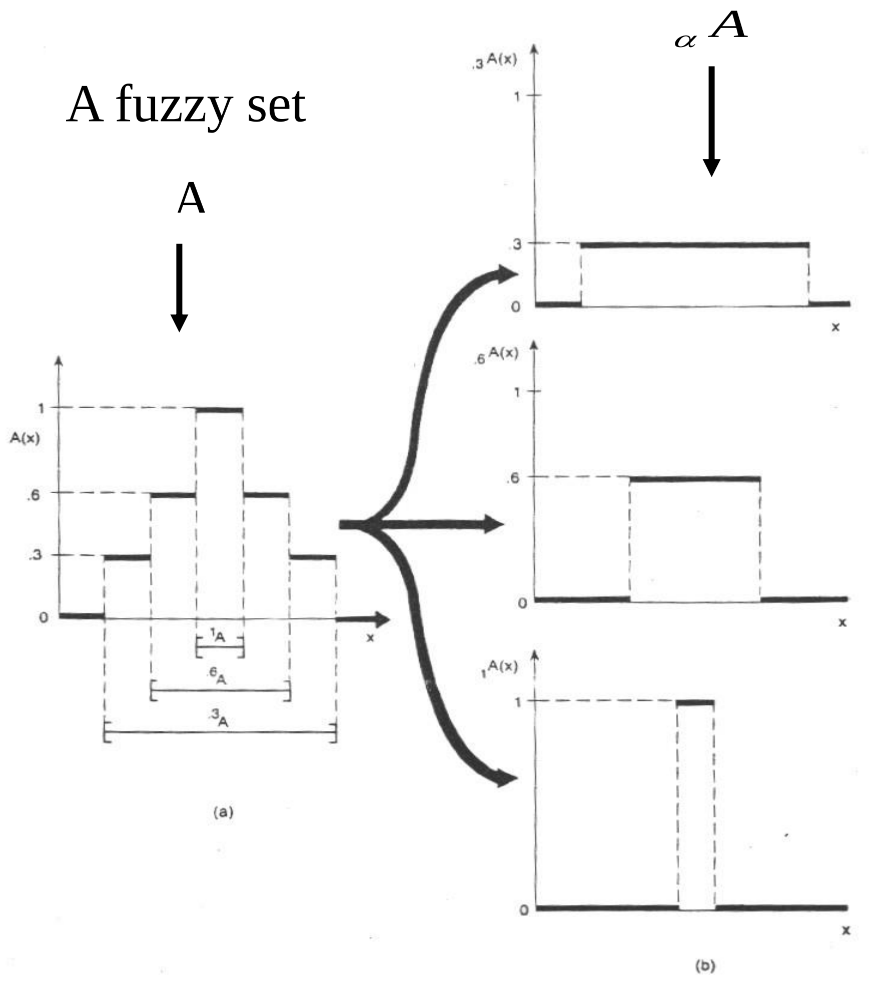
。**Theorem 2.7** (Third decomposition Theorem)

$$A = \bigcup_{\alpha\subset\Lambda} {}_{\alpha}A, \quad \Lambda(A): \text{level set}$$

Theorems 2.5, 2.6 for continuous membership function

Theorem 2.7 for discrete membership function

◦ Example :



2.3 Ex

--- a

concerning sets to power sets

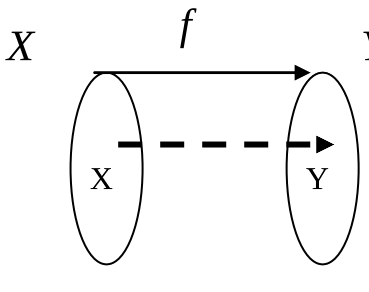
⊙Crisp case:

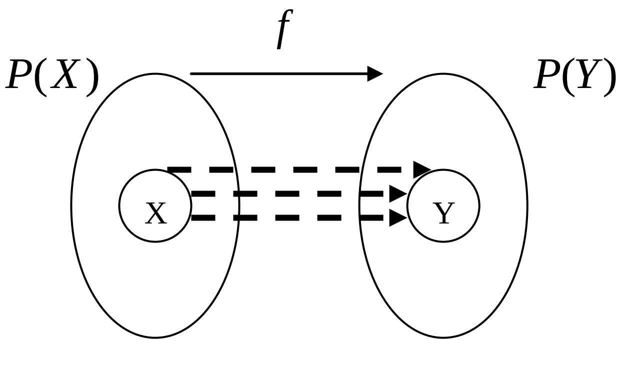
a crisp function-

f: X --> Y , X,Y : crisp sets defined on univesal sets U,V

an extension

$$\begin{cases} f : P(X) \rightarrow P(Y) \\ \quad P(X),P(Y)\text{: Crisp power set of X,Y} \\ f^{-1} : P(Y) \rightarrow P(X) \end{cases}$$



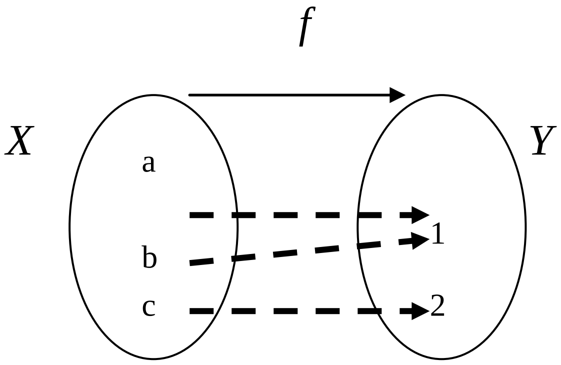


$$Let\ A \in P(X) \Rightarrow B = f(A) = \{y \mid y = f(x), x \in A\}$$

$$Let\ B \in P(Y) \Rightarrow A = f^{-1}(B) = \{x \mid f(x) \in B\}$$

Example:

X={a,b,c} , Y={1,2}

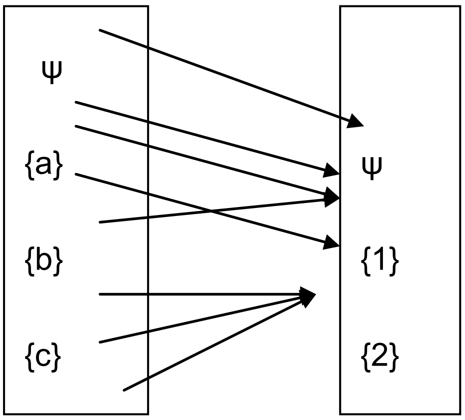


Extension f: p(X) p(Y)

Where

$$p(X) = \{\Phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

$$p(y) = \{\Phi, \{1\}, \{2\}, \{1,2\}\}$$



F(A)={y | y=f(x), $x \in A$}

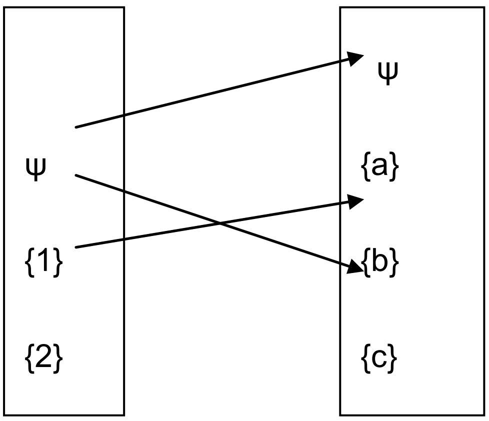
e.g

A={a,c}

⇨ $f(A) = f(\{a,c\}) = \{1,2\}$

A={a,b}

=> $F(A) = f(\{a,b\}) = \{1\}$



$$F^{-1}(B) = \{x \mid f(x) \in B\}$$

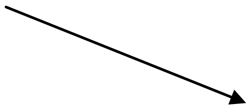
e.g

B={1}

⇨ $f^{-1}(A) = f^{-1}(\{1\}) = \{a,b\}$

B={1,2}

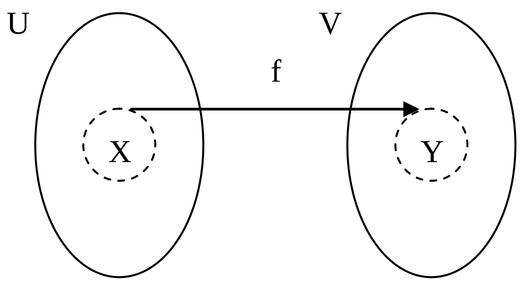
=> $f^{-1}(B) = f^{-1}(\{1,2\}) = \{a,b,c\}$



Fuzzy case:

Given a fuzzy function f: X => Y

X,Y: fuzzy sets defined on crisp universal sets U,V



An extension

$$f : f(X) -> F(Y)$$
$$f^{-1} : F(Y) -> F(X)$$

F(X),F(Y): Fuzzy power sets of X,Y

$\forall a \in \mathrm{F}(X),\ Let\ \mathrm{B} = \mathrm{f(A)} \in \mathrm{f(Y)}$

The membership function of fuzzy set B

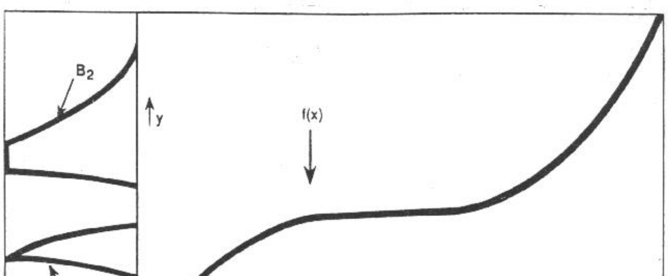
$$B(Y) = [f(A)](y) = \sup_{x \mid f(x)=y} \mathrm{A(X)}$$

The membership function of fuzzy set A

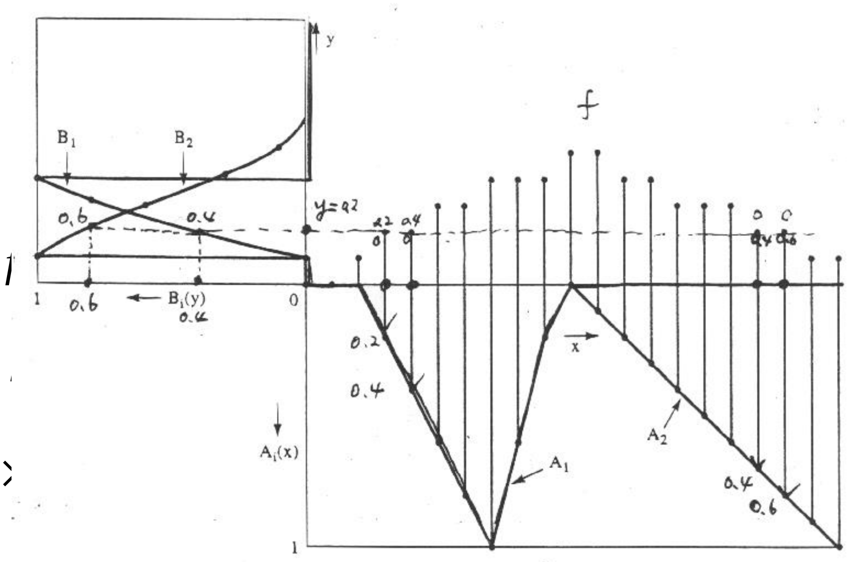
$$A(x) = [f^{-1}(B)](x) = B(\mathrm{f(x)})$$

Example : Function Extension

(a) Continuous case



(b) Discrete case



$B_1(y) = [$

$B_2(y) = [$

$f : X_1 \times$

X_1, X_2, X_n : crisp set

Let fuzzy set $A_1, A_2 ....A_n$ defined on

$X_1, X_2, ...X_n$ respectively

if $f(x_1^k, x_2^k, ...) = y \quad k = 1...m$

$$\frac{\mu}{y} = \frac{\sup_k \min\{A_1(x_1^k), A2(x_2^k), ..., A_n(x_n^k)\}}{y}$$

- Example: Fuzzy Mapping (Multivariants)

$X_1 = \{a, b, c\}, \quad X_2 = \{x, y\}, \quad Y = \{p, q, r\},$

$f : X_1 \times X_2 \rightarrow Y$

Where

$$f: \quad \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} x \quad\; y \\ \begin{bmatrix} \textcircled{p} & \textcircled{p} \\ q & r \\ r & \textcircled{p} \end{bmatrix} \end{array}$$

Let $A_1,\ A_2$; Fuzzy sets defined on $X_1,\ X_2$

$$A_1 = \frac{0.3}{a} + \frac{0.9}{b} + \frac{0.5}{c} \quad A_2 = \frac{0.5}{x} + \frac{1.0}{y} \qquad F(Y)$$

Let $B = f(A_1, A_2) \in F(Y)$

(*a*,*x*) (*a*,*y*) (*c*,*y*)

B(*p*) = *max*{*min*{0.3,0.5},*min*{0.3,0.5},*min*{0.3,0.5}}

= *max*{0.3,0.3,0.5} = 0.5

B(*q*) = *max*{*min*{0.9,0.5} = 0.5

B(*r*) = *max*{*min*{0.9,1}, *min*{0.5,0.5}}

= *max*{0.9,0.5} = 0.9

$$B = f(A_1, A_2) = \frac{0.5}{p} + \frac{0.5}{q} + \frac{0.9}{r}$$

- **Theorem 2.8** : $f: X \to Y$ crisp function

$$\forall A_i \in F(X),\ B_i \in F(Y),$$

$\Rightarrow$ (i) $f(A) = \phi$ iff $A = \phi$

(ii) $if\ \ A_1 \subseteq A_2,\ f(A_1) \subseteq f(A_2)$

(iii) $f(\bigcup_i A_i) = \bigcup_i f(A_i),\quad f(\bigcap_i A_i) \subseteq \bigcup_i f(A_i)$

(iv) $if\ B_1 \subseteq B_2,\ f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(v) $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i), \quad f^{-1}(\bigcap_i B_i) \subseteq \bigcup_i f^{-1}(B_i)$

(vi) $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

(vii) $A \subseteq f^{-1}(f(A)), \quad B \supseteq f^{-1}(f(B))$

Proof:

(i) (a) If $A = \phi$, then $f(A) = \phi$

(b) If $f(A) = \phi$, then $A = \phi$

(ii) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

$\because A_1 \subseteq A_2, \quad \therefore \forall x \in X \qquad A_1(x) \le A_2(x)$

$$\forall y,\ [f(A_1)](y) = \sup_{x \mid y=f(x)} A_1(x) \le \sup_{x \mid y=f(x)} A_2(x) = [f(A_2)](y)$$

$\therefore f(A_1) \subseteq f(A_2)$

(iii) $f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$

$$\forall y,\ [f(\bigcup_{i\in I} A_1)](y) = \sup_{x \mid y=f(x)} \left(\bigcup_{i\in I} A_i\right)(x)$$

$$= \sup_{x \mid y=f(x)} \max_{i\in I}\{A_i(x)\} = \max_{i\in I}\left\{\sup_{x \mid y=f(x)} A_i(x)\right\}$$

$$= \max_{i\in I}\{f(A_i)(y)\} = \left[\bigcup_{i\in I} f(A_i)\right](y)$$

$\therefore f(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} f(A_i)$

(iv) $f(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} f(A_i)$

$$\forall y,\ [f(\bigcap_{i\in I} A_i)](y) = \sup_{x|y=f(x)} \left(\bigcap_{i\in I} A_i\right)(x)$$

$C_1=\{\ldots\ldots\ldots..\}$

$C_2=\{\ldots\ldots\ldots..\}$

$\vdots$

$C_n=\{\ldots\ldots\ldots..\}$ $\Rightarrow$

$$= \sup_{x|y=f(x)} \min_{i\in I} \{A_i(x)\} \le \min_{i\in I} \left\{ \sup_{x|y=f(x)} A_i(x) \right\}$$

$$= \min_{i\in I} \{f(A_i)(y)\} = \left[\bigcap_{i\in I} f(A_i)\right](y)$$

$$f(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} f(A_i)$$

(v) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

$\because B_1 \subseteq B_2,\ \therefore B_1(y) \le B_2(y)$

$$\forall x \in \mathrm{X} \quad \left[f^{-1}(B_1)\right](x) = B_1(f(x)) \le B_2(f(x)) = \left[f^{-1}(B_2)\right](x)$$

$$\therefore f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

(vi) $f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$

$\forall x \in \mathrm{X}$

$$\left[f^{-1}\left(\bigcup_{i\in I} B_i\right)(x)\right] = \left(\bigcup_{i\in I} B_i\right)(f(x)) = \sup_{i\in I} B_i(f(x))$$

$$= \left[\sup_{i\in I} f^{-1}(B_i)(x)\right] = \left[\bigcup_{i\in I} f^{-1}(B_i)\right](x)$$

$$\therefore f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$$

(vii) $f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i)$

$\forall x \in \mathrm{X}$

$$\left[f^{-1}\left(\bigcap_{i\in I} B_i \right) \right](x) = \bigcap_{i\in I} B_i(f(x)) = \inf_{i\in I} B_i(f(x))$$

$$= \left[\inf_{i\in I} f^{-1}(B_i)(x) \right] = \left[\bigcap_{i\in I} f^{-1}(B_i) \right](x)$$

$$\therefore f^{-1}\left(\bigcap_{i\in I} B_i \right) = \bigcap_{i\in I} f^{-1}(B_i)$$

(viii) $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

$$\forall x, \because \left[f^{-1}(B) \right](x) = B(f(x))$$

$$\therefore \overline{[f^{-1}(B)](x)} = 1 - \left[f^{-1}(B) \right](x) = 1 - B(f(x)) = \overline{B}(f(x))$$

$$\because \left[f^{-1}(B) \right](x) = \overline{B}(f(x))$$

$$\therefore \overline{[f^{-1}(B)](x)} = \overline{B}(f(x)) \qquad \left[f^{-1}(B) \right](x)$$

$$\therefore \overline{f^{-1}(B)} = f^{-1}(\overline{B})$$

(ix) $A \subseteq f^{-1}(f(A))$

$\forall x \in \mathrm{X},$

Let $\quad f(x) = y$

$$\because \left[f^{-1}(f(A)) \right](x) = [f(A)](f(x)) = [f(A)](y)$$

$$= \sup_{x' | y = f(x')} A(x') \geq A(x)$$

$$\therefore A \subseteq f^{-1}(f(A))(\mathrm{x})$$

$\forall\alpha,$ assume $y \in {}^{\alpha}[f(f^{-1}(B))]$

$\Rightarrow [f(f^{-1}(B))](y) \geq \alpha$

$\Rightarrow \sup_{x|y=f(x)} [f^{-1}(B)](x) \geq \alpha$

$\Rightarrow \exists x_0 \in X,$ s.t. $y = f(x_0),\ \underline{[f^{-1}(B)](x) \geq \alpha}$

$\Downarrow$

$\underline{B[f(x_0)] \geq \alpha}$

$\Downarrow$

$\underline{f(x_0) \in {}^{\alpha}B}$

$\Downarrow$

$y = f(x_0) \in {}^{\alpha}B$

$\therefore B \supseteq (f^{-1}(B))$

- **Theorem 2.9** : $f: X \to Y$ Crisp function

$$\forall A_i \in F(X),\ B_i \in F(Y),$$

(xi) ${}^{\alpha^+}[f(A)] = f({}^{\alpha^+}A)$

(xii) ${}^{\alpha}[f(A)] \supseteq f({}^{\alpha}A)$

proof:

(xi) $\forall y \in {}^{\alpha^+}[f(A)]$

$$\Leftrightarrow [f(A)](y) > \alpha$$

$$\overset{(2.10)}{\Leftrightarrow} \sup_{x \mid y = f(x)} A(x) > \alpha$$

$$\Leftrightarrow \exists x_0,\ \text{s.t.}\, y = f(x_0),\ A(x_0) > \alpha$$

$$\Leftrightarrow x_0 \in {}^{\alpha^+}A \overset{1-1}{\Leftrightarrow} f(x_0) \in f({}^{\alpha^+}A)$$

$$\Leftrightarrow y \in [f({}^{\alpha^+}A)]$$

$$\therefore {}^{\alpha^+}[f(A)] = [f({}^{\alpha^+}A)]$$

(xii) $\forall y \in [f({}^{\alpha}A)],\ \exists\, x_0 \in {}^{\alpha}A$ s.t. $y = f(x_0)$

$$\because [f(A)](y) = \sup_{x \mid y = f(x)} A(x) \geq A(x_0) \geq \alpha$$

$$\therefore y \in {}^{\alpha}[f(A)]$$

$$\therefore f({}^{\alpha}A) \subseteq {}^{\alpha}[f(A)]$$

● Example Let $X = N,\ Y = \{a, b\}$

$$f(n) = \begin{cases} a & n \le 10 \\ b & n > 10 \end{cases}$$

Let $A(n) = 1 - \frac{1}{n}$: fuzzy set

$$\Rightarrow [f(A)](a) = \sup_{n | f(n)=a} A(n) = A(10) = 1 - \frac{1}{10}$$

$$= \frac{1}{10}$$

$$[f(A)](b) = \sup_{n | f(n)=a} A(n) = 1$$

$\therefore$ Let $\alpha = 1$, ${}^{1}[f(A)] = \{b\}$

$$\because f({}^{1}A) = f(\phi) = \phi$$

$$\therefore f({}^{\alpha}A) \underset{\ne}{\subseteq} {}^{\alpha}[f(A)]$$

※ ${}^{\alpha}[f(A)] = f({}^{\alpha}A), \ \forall \alpha \in [0,1]$

whenever X : finite

- **Theorem 2.10 :** $f : \mathrm{X} \to \mathrm{Y}$ Crisp function

$$\forall A_i \in F(\mathrm{X}), \ f(A) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha^+}A)$$

proof:

From Theorem 2.6

$$f(A) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha^+}A)$$

where $$_{\alpha^+}[f(A)] = \sum_{Y \in {}^{\alpha^+}[f(A)]} \frac{\alpha}{Y} \overset{Thm\,2.9(\text{xi})}{\overset{\downarrow}{=}} \sum_{Y \in [f({}^{\alpha+}A)]} \frac{\alpha}{Y} = f({}_{\alpha^+}A)$$

$$\therefore f(A) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha^+}A)$$

- Procedure for calculation of $f(A)$

$$\begin{cases} 1.\ \text{Calculate all images of strong } \alpha\text{-cuts under } f,\ i.e.,\ f({}^{\alpha^+}A); \\ 2.\ \text{Convert them to } f({}_{\alpha^+}A); \\ 3.\ f(A) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha^+}A). \end{cases}$$