Unit III

Introduction :-

If T is an operator on Hilbert space H then the simplest thing T can do and vectors is it to transform scalar multiple of x i.e. $Tx = \lambda x$ —— (1)
A non-zero vector x such that eq^n (1) is true for some scalar λ is called eigen vector of T and scalar λ in eq^n (1) is said to be eigen value of T.

Each eigen value has one or more eigen vectors associated with and to ~~show~~ ~~th~~ such that eigen value. If H has no non-zero vector at all then T certainly has no eigen vector. In this case the whole theory called into trivially.

So we assume that H is non-empty $H \neq \{0\}$ let λ be an eigen value of T and consider the set M of all its corresponding eigen vector together with the vector zero (note that '0' is not an eigen vector). Thus M is the set of all vector x which satisfies the eq^n
$(T - \lambda I)x = 0$ & it is clearly a non-zero closed linear subspace of H we call M is the eigen space of T corresponding to λ.

Def^n :-

Let T be an operator on H. A scalar λ is called an eigen value of T if their exist a non-zero vectors x in H such that $Tx = \lambda x$ such x is called an eigen vector of T corresponding to λ.

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Remark :-

1) If x is an eigen vector of T then αx is also an eigen vector corresponding to same λ

as $T(\alpha x) = \alpha Tx = \alpha \lambda x = \lambda(\alpha x)$

$$= \alpha \lambda x \quad \because (Tx = \lambda x)$$

$$= \lambda(\alpha x)$$

Def^n :-

Let λ be an eigen value of T then the set M_λ of all its corresponding eigen vectors together with the zero vector is called the eigen space of T.

i.e. $M_\lambda = \{x \mid Ix = \lambda x\}$

(Here zero also satisfies this condition)

Theorem :-

If T is normal then x is an eigen vector of T with eigen value λ iff x is an eigen vector of T^* with eigen value $\bar{\lambda}$

Proof :-

since T is normal, $T - \lambda I$ is also normal.

Now $TT^* = T^*T$, $(\because (T-\lambda I)(T-\lambda I)^*)$

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - (\lambda I)^*)$$

$$= (T-\lambda I)(T^* - \bar{\lambda} I^*)$$

$$\because ((\alpha T)^* = \bar{\alpha} T^*)$$

$$= TT^* - T\bar{\lambda} I^* - \lambda I T^* + |\lambda|^2 I I^*$$

$$(\because I I^* = I = I^* I)$$

$$= TT^* - \bar{\lambda} T I^* - \lambda I T^* + \lambda \bar{\lambda} I^* I$$

$$(\because T \text{ is normal})$$

$$= T(T^* - \bar{\lambda} I^*) - \lambda I (T^* - \bar{\lambda} I^*)$$
$$= (T^* - \bar{\lambda} I^*)(T - \lambda I)$$
$$= (T - \lambda I)^* (T - \lambda I)$$

$\therefore \|(T-\lambda I)x\| = \|(T-\lambda I)^* x\| \quad \forall x \in H$

$= \|(T^* - \bar{\lambda} I^*)x\| \quad \forall x$

$= \|(T^* - \bar{\lambda} I)x\|$

$= \|T^* x - \bar{\lambda} x\|$

x is an eigen vector corresponding to λ

i.e. $Tx = \lambda x$ iff $(Tx)^* = (\lambda x)^*$

iff $T^* x = \bar{\lambda} x$

$\therefore$ x is eigen vector corresponding to $\bar{\lambda}$ of T^*

Theorem :-

If T is normal then M_i's are pairwise orthogonal.

Proof :-

Let T is normal i.e. $TT^* = T^*T$

let $x_i \in M_i$, $x_j \in M_j$ $i \neq j$

$\therefore Tx_i = \lambda_i x_i$ & $T^* x_j = \bar{\lambda}_j x_j$

$\therefore \lambda_i \langle x_i, x_j \rangle = \langle \lambda_i x_i, x_j \rangle$

$(\because \langle \alpha x, y \rangle = \alpha \langle x, y \rangle)$

$= \langle Tx_i, x_j \rangle$ $\quad (\because Tx_i = \lambda_i x_i)$

$= \langle x_i, T^* x_j \rangle$ $\quad (\langle Tx, y \rangle = \langle x, T^* y \rangle \ \forall x, y \in H)$

$= \langle x_i, \bar{\lambda}_j x_j \rangle$ $\quad (T^* x_j = \bar{\lambda}_j x_j)$

$= \lambda_j \langle x_i, x_j \rangle$

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$\implies \lambda_i \langle x_i, x_j \rangle - \lambda_j \langle x_i, x_j \rangle = 0$

$\implies (\lambda_i - \lambda_j) \langle x_i, x_j \rangle = 0$

$\because \lambda_i \neq \lambda_j \implies (\lambda_i - \lambda_j) \neq 0$

$\therefore \langle x_i, x_j \rangle = 0$

$\implies x_i \perp x_j \quad \forall x_i \in M_i \ \& \ x_j \in M_j$

$\implies M_i \perp M_j \quad i \neq j$

$\therefore$ M_i's are pairwise orthogonal.

Theorem :-

If T is normal then each M_i reduces T.

Proof :-

Clearly, M_i is invarient under T

$\{ \because T(x_i) = \lambda x_i \implies T(M_i) = \lambda_i M_i \subseteq M_i$

$\implies T(M_i) \subseteq M_i$

Now we show that each M_i is invarient under T^*. i.e. $T^*(M_i) \subseteq M_i$

Let $x_i \in M_i$ then $T(x_i) = \lambda_i x_i$

$\implies T^* x_i = \bar{\lambda}_i x_i$

$\implies T^*(M_i) = \bar{\lambda}_i M_i \subseteq M_i$

$\implies T^*(M_i) \subseteq M_i \quad \forall x_i \in M_i$

$\therefore T(M_i) \subseteq M_i$ and $T^*(M_i) \subseteq M_i$

$\therefore M_i$ is reduces T

Hence proved.

Theorem :-

If T is normal then M_i's are span of H.

Proof :-

Since M_i's are pairwise orthogonal we get $M = M_1 + M_2 + \cdots + M_m$ is closed linear subspace of H and $P = P_1 + P_2 + \cdots + P_m$ is the projection on M

Since each M_i reduces T

$TP_i = P_i T$

$\therefore$ $TP = T(P_1 + P_2 + \cdots + P_m)$

$= TP_1 + TP_2 + \cdots + TP_m$

$= P_1 T + P_2 T + \cdots + P_m T$

$= (P_1 + P_2 + \cdots + P_m) T$

$TP = PT$

$\therefore$ M reduces T and so $M^\perp$ is also invarient under T

i.e. $T(M) \subseteq M$ and $T(M^\perp) \subseteq M^\perp$

Now $M^\perp \neq \{0\}$

then the restricted operation i.e.

$U : M^\perp \longrightarrow M^\perp$ has an eigen value say λ then their exist $x \in M^\perp$ which is non-zero such that $Ux = \lambda x$

i.e. $\Longrightarrow$ $Tx = \lambda x$ $(\because M^\perp$ is invarient under $T)$

$\Longrightarrow$ x is an eigen vector of T

$\Longrightarrow$ $Tx \in M$

$\Longrightarrow$ $x \in M$

which is contradiction to $x \in M^\perp$

$\therefore$ our assumption is wrong

$\therefore$ $M^\perp = \{0\}$ and we know that

$H = M \oplus M^\perp$

$\Longrightarrow$ $H = M + M^\perp$ and $M \cap M^\perp = \{0\} \Rightarrow M^\perp = \{0\}$

$\Longrightarrow$ $H = M + \{0\}$

$\Longrightarrow$ $H = M$, $M^\perp = \{0\}$

$\therefore$ M_i's are span (H).

Theorem :- Spectral Theorem.

Let T be an operator on finite dimensional Hilbert space H with distinct eigen values $\lambda_1, \lambda_2 ---- \lambda_m$ and let $M_1, M_2, ---M_m$ be their corresponding eigen spaces & let $P_1, P_2, ---- P_m$ be the projection on this eigen spaces then the following are equivalent.

1) The M_i's are pair wise orthogonal and span (H)

2) The P_i's are pairwise orthogonal &
$I = \sum_{i=1}^{m} P_i$ and $T = \sum_{i=1}^{m} \lambda_i P_i$

3) T is normal

Proof :-

1) $\implies$ 2)

Since M_i's are pairwise orthogonal i.e.

$M_i \perp M_j$, $i \neq j$

$\implies P_i P_j = 0 \quad \forall \ i \neq j$

$\therefore$ P_i's are pairwise orthogonal ~~i.e~~ and we know that M_i's are span (H) for any $x \in H$ and $\exists$ $x_i \in M_i$ such that $x = x_1 + x_2 + -- + x_m = \sum_{j=1}^{n} x_j$

$\therefore I(x) = I(x_1 + x_2 + --- + x_m)$
$= x_1 + x_2 + --- + x_m$

Also $P_i(x) = P_i(x_1 + x_2 + --- + x_m)$
$= P_i x_i + P_i x_2 + ---- + P_i x_m$
$= \sum_{j=1}^{m} P_i x_j$

We know that $P_j(x_j) = x_j$

$\therefore P_i(x) = \sum_{j=1}^{m} P_i(P_j x_j)$

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$$= P_i(P_1 x_1) + P_i(P_2 x_2) + \cdots + P_i(P_m x_m)$$

$$= P_1^2 x + P_2^2 x + \cdots + P_m^2 (x)$$

$$= P_1 x + P_2 x + \cdots + P_m x$$

$$= (P_1 + P_2 + \cdots + P_m)(x)$$

$$= P(x)$$

Now $I(x) = x = x_1 + x_2 + \cdots + x_m$

$$= P_1 x_1 + P_2 x_2 + \cdots + P_m x_m$$

$(\because (P_i(x_i) = x_i)$

$$= P_1(x) + P_2(x) + \cdots + P_m(x)$$

$$I(x) = (P_1 + P_2 + \cdots + P_m)(x)$$

$$I = P_1 + P_2 + \cdots + P_m$$

$$I = \sum_{i=1}^{m} P_i$$

Now $T(x) = T(x_1 + x_2 + \cdots + x_m)$

$$= \sum_{j=1}^{m} T x_j$$

$$= \sum_{j=1}^{m} \lambda_j x_j$$

because $x_1, x_2 \ldots x_m$ are eigen vectors corresponding to eigen values $\lambda_1, \lambda_2, \ldots \lambda_m$

Now replace j by i

$$Tx = \sum_{i=1}^{m} \lambda_i x_i$$

$$= \sum_{i=1}^{m} \lambda_i P_i(x_i) \qquad \because P_i(x_i) = x_i$$

$$Tx = \sum_{i=1}^{m} \lambda_i P_i(x) \qquad \because P_i(x_i) = P_i(x)$$

$$T = \sum_{i=1}^{m} \lambda_i P_i$$

2) $\Rightarrow$ 3)

since $T = \sum_{i=1}^{m} \lambda_i P_i$ and $T^* = \sum_{j=1}^{m} \bar{\lambda}_j P_j^*$

$= \sum_{j=1}^{m} \bar{\lambda}_j P_j \quad (\because P_j^* = P_j)$

$TT^* = \left(\sum_{i=1}^{m} \lambda_i P_i\right)\left(\sum_{j=1}^{m} \bar{\lambda}_j P_j\right)$

$= \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_i \bar{\lambda}_j P_i P_j$

$= \sum_{i=1}^{m} \lambda_i \bar{\lambda}_i P_i P_i$

$TT^* = \sum_{i=1}^{m} |\lambda_i|^2 P_i^2$

$TT^* = \sum_{i=1}^{m} |\lambda_i|^2 P_i$ ——— (1)

similarly $T^*T = \left(\sum_{j=1}^{m} \bar{\lambda}_j P_j\right)\left(\sum_{i=1}^{m} \lambda_i P_i\right)$

$= \sum_{j=1}^{m} \sum_{i=1}^{m} \bar{\lambda}_j \lambda_i P_j P_i$

$= \sum_{j=1}^{m} \sum_{i=1}^{m} \lambda_i \bar{\lambda}_j P_i P_j$

$= \sum_{i=1}^{m} \lambda_i \bar{\lambda}_i P_i P_i$ ($\because P_i$'s are pair wise orthogonal i.e. $P_i P_j = 0$ if $i \neq j$)

$= \sum_{i=1}^{m} |\lambda_i|^2 P_i^2$

$T^*T = \sum_{i=1}^{m} |\lambda_i|^2 P_i$ ——— (2) $\quad \because P_i^2 = P_i$

By eqn (1) & (2)

$TT^* = T^*T$

$\therefore T$ is normal.

3) $\Rightarrow$ 1)

If T is normal then we show that M_i's are pair wise orthogonal.

Let $x_i \in M_i$ and $x_j \in M_j$

where $i \neq j$ we consider $Tx_i = \lambda_i x_i$ and

$T^* x_j = \bar{\lambda}_j x_j$

$\therefore$ we consider ~~$Tx_i = \lambda_i x_i$~~ ~~and~~ ~~$T^* x_j =$~~

$$\lambda_i \langle x_i, x_j \rangle = \langle \lambda_i x_i, x_j \rangle$$
$$= \langle Tx_i, x_j \rangle$$
$$= \langle x_i, T^* x_j \rangle \qquad \because \langle Tx, y \rangle = \langle x, T^* y \rangle$$
$$= \langle x_i, \bar{\lambda}_j x_j \rangle$$
$$= \bar{\bar{\lambda}}_j \langle x_i, x_j \rangle$$
$$= \lambda_j \langle x_i, x_j \rangle$$

$$\lambda_i \langle x_i, x_j \rangle - \lambda_j \langle x_i, x_j \rangle = 0$$
$$(\lambda_i - \lambda_j) \langle x_i, x_j \rangle = 0$$

where $i \neq j$ then the eigen values of T are λ_i and T^* is λ_j

$\therefore \lambda_i \neq \lambda_j$

$\Rightarrow (\lambda_i - \lambda_j) \neq 0$

$\Rightarrow \langle x_i, x_j \rangle = 0$

$\Rightarrow x_i \perp x_j \qquad \forall x_i \in M_i, x_j \in M_j$

$\Rightarrow M_i \perp M_j$

$\therefore M_i$'s are ~~span of~~ pair wise orthogonal

claim :- M_i's are span of H.

Given that T is normal i.e.

$TT^* = T^*T$.

Since M_i's are pairwise orthogonal we get

$M = M_1 + M_2 + \cdots + M_m$ is closed linear subspace and $P = P_1 + P_2 + \cdots + P_m$ is the

projections on M ($P_i^2 = P_i$ and $P_i^* = P_i$)
since each M_i reduces P then $TP_i = P_iT$
$\forall P_i$

since $TP = T(P_1 + P_2 + \dots + P_m)$

$= T(P_1) + TP_2 + \dots + TP_m$ $\because T$ is linear

$= P_1T + P_2T + \dots + P_mT$

$= (P_1 + P_2 + \dots + P_m)T$

$TP = PT$

Now M reduces P and so $M^\perp$ is also invarient under T i.e. $T(M) \subseteq M$ and $T(M^\perp) \subseteq M^\perp$

Now consider $M^\perp \neq \{0\}$ then the restricted operator $U : M^\perp \longrightarrow M^\perp$ has a eigen value λ then their exist $x \in M^\perp$ such that

$Ux = \lambda x \implies Tx = \lambda x$

x is an eigen vector.

$\therefore Tx \in M$

$\implies x \in M$

which is contradiction to $x \in M^\perp$

$\therefore$ our assumption is wrong

$M^\perp = \{0\}$

We know that every Hilbert space

$H = M \oplus M^\perp$

$H = M + M^\perp$

$H = M$

$\therefore$ H is a Hilbert space and M_i's are pairwise orthogonal and which is span of H.