What is a Relation?

"Relation" is a mathematical term for the way we express relationships between objects.

There are many kinds of relationships that we may wish to model, such as

- parent - child
- teacher - student
- student - course

Mathematical Relations

There are also purely mathematical relationships such as

- is less than
- is a subset of
- is greater than or equal to
- is logically equivalent to

All of the preceding relationships are examples of binary relations, because they relate the elements of one set with those of another (possibly the same set).

Regardless of the type of relationship that we are representing, we can use the same mathematical structure

Binary Relations

Definition 2.1
Let A and B be non-empty sets. A binary relation from A to B is a subset of the Cartesian product $A \times B$.
A binary relation from a set A to itself will simply be refered to as a binary relation on A.

If R is a relation from A to B and $(a, b) \in R$ then we can write

$$a \; R \; b$$

To mean that a relates to b

An Example of a Binary Relation

Example 2.2

Suppose we have a three students John, Sue and Emily and we are interested in which subjects they are enrolled in. To keep things simple suppose that the only subjects we care about are MATH1510, MATH1110, SENG1110 and INFO1010.

They are enrolled as follows

Student	Subjects
John	MATH1510, SENG1110 and INFO1010
Sue	MATH1510, MATH1110 and SENG1110
Emily	MATH1510 and MATH1110

Example 2.2 (continued)

We have a set of students

$$\{\text{John}, \text{Sue}, \text{Emily}\}$$

and a set of subjects

$$\{\text{MATH1510}, \text{MATH1110}, \text{SENG1110}, \text{INFO1010}\}.$$

The cartesian product of these two sets consists of the twelve possible ordered pairs which have a student as the first element and a subject as the second element.

Example 2.2 (continued)

A relation R can be defined as "is enrolled in" so that

$$\text{John } R \text{ MATH1510}$$

means John is enrolled in MATH1510.

The relation R is a subset of the cartesian product.

$$\begin{aligned} R = \{&(\text{John}, \text{MATH1510}), (\text{John}, \text{SENG1110}), (\text{John}, \text{INFO1010}), \\ &(\text{Sue}, \text{MATH1510}), (\text{Sue}, \text{MATH1110}), (\text{Sue}, \text{SENG1110}), \\ &(\text{Emily}, \text{MATH1510}), (\text{Emily}, \text{MATH1110})\} \end{aligned}$$

The Domain and Range of a Relation

Definition 2.3
The domain of a relation R from A to B is:

$$\{a \in A : (a, b) \in R \text{ for some } b \in B\}$$

Definition 2.4
The range of a relation R from A to B is:

$$\{b \in B : (a, b) \in R \text{ for some } a \in A\}$$

The Domain and Range of a Relation (continued)

Example 2.5
Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c\}$. Define a relation R from A to B by

$$R = \{(1, a), (1, b), (3, a), (4, b)\}$$

The domain of R is $\{1, 3, 4\}$ and the range is $\{a, b\}$.

The Inverse Relation

Definition 2.6
Let R be a relation from A to B. The inverse relation R^{-1} is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}.$$

Example 2.7
Let $A = \{1, 2, 3\}$. and let R be the relation "$\leq$" on A. That is $R = \{(a, b) \in A : a \leq b\}$. Explicitly we may write

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\},$$

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 2), (3, 2), (3, 3)\}.$$

The inverse of $\leq$ is $\geq$

Relations as Directed Graphs

A relation R, from A to B, can be represented by a digraph (directed graph), with the vertices representing the elements of $A \cup B$ and with an edge from vertex a to vertex b whenever $(a, b) \in R$.

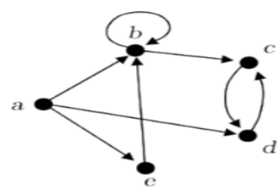
Example 2.8

Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$.
$R = \{(1, a), (2, b), (2, d), (3, a), (3, b), (3, d)\}$.

Example 2.9

Let $A = \{a, b, c, d, e\}$ and let R be a relation on A.

$$R = \{(a, b), (a, d), (a, e), (b, b), (b, c), (c, d), (d, c), (e, b)\}$$



Properties of Relations

We now consider several properties that allow us to classify relations into the following categories

- Reflexive
- Symmetric
- Antisymmetric
- Transitive
- Connected

We will discuss each in turn.

Reflexive Relations

Definition 2.10
A relation R on a set A is reflexive if

$$\text{for all } a \in A, \ (a, a) \in R.$$

That is, $a \ R \ a$ for all $a \in R$.

Examples 2.11

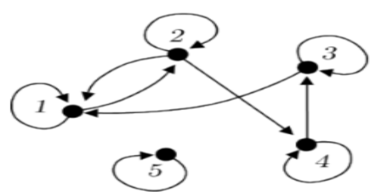
- $\leq$, $\geq$, $=$ and $\subseteq$ are all reflexive relations.
- "is the same colour as" is a reflexive relation.
- $<$, $>$, $\neq$ and $\subset$ are not reflexive relations.
- "is the parent of" is not a reflexive relation.

Example of a Reflexive Relation

Example 2.12
Consider the set $\{1, 2, 3, 4, 5\}$. The relation on this set given by

$$R = \{(1,1), (1,2), (2,1), (2,2), (2,4), (3,1), (3,3), (4,4), (4,3), (5,5)\}$$

is reflexive with associated digraph as follows.



Symmetric Relations

Definition 2.13
A relation R on a set A is symmetric if

$$\text{for all } a, b \in A, \ \text{if } (a, b) \in R \text{ then } (b, a) \in R.$$

That is, if $a \ R \ b$ then $b \ R \ a$.

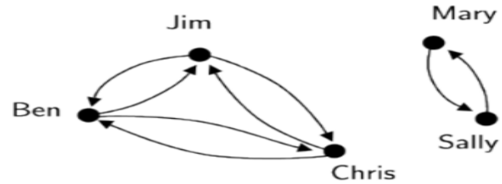
Examples 2.14

- $=$ and $\neq$ are symmetric relations.
- "is the same colour as" is a symmetric relation.
- "is a relative of" is a symmetric relation.
- $\subseteq$, $\leq$, $\geq$, $<$ and $<$ are not symmetric relations.
- "is the parent of" is not a symmetric relation.

Example of a Symmetric Relation

Example 2.15

Consider the set {Mary, Jim, Chris, Sally, Ben}. Let R be the relation "is the same gender as" represented in the following digraph.



This relation is symmetric.

Antisymmetric Relations

Definition 2.16

A relation R on a set A is antisymmetric if for all $a, b \in A$,

$$\text{if } (a,b) \in R \text{ and } a \neq b \text{ then } (b,a) \notin R.$$

That is, if $a \; R \; b$ and $a \neq b$ then $b \not R \; a$.

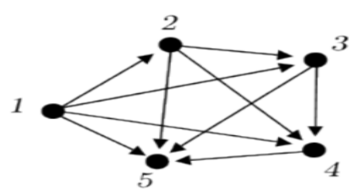
Examples 2.17

- $\leq$, $\geq$, $=$, $<$, $>$ and $\subseteq$ are all antisymmetric relations.
- "is the parent of" is an antisymmetric relation.
- $\neq$ is not an antisymmetric relation.
- "is a relative of" is not antisymmetric.

Example of an Antisymmetric Relation

Example 2.18

Consider the relation $<$ on the set { 1,2,3,4,5 }.



This relation is antisymmetric.

Transitive Relations

Definition 2.19
A relation R on a set A is transitive if for all $a, b, c \in A$

$$\text{if } (a, b) \in R \text{ and } (b, c) \in R \text{ then } (a, c) \in R.$$

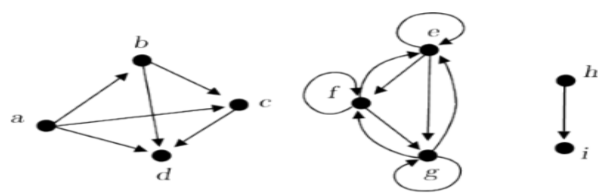
That is, if $a \; R \; b$ and $b \; R \; c$ then $a \; R \; c$.

Examples 2.20

- $\leq$, $\geq$, $=$, $<$, $>$ and $\subseteq$ are all transitive relations.
- "is the same colour as" is a transitive relation.
- $\neq$ is not a transitive relation.
- "is a parent of" is not transitive.

Example of a Transitive Relation

Example 2.21
Consider the set $\{a, b, c, d, e, f, g, h, i\}$ and the relation given by the following digraph.



This relation is transitive.