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Applications involving Fourier Transform

1) Heat equation on an infinite line

Consider an infinitely long rod or wire made of homogeneous material and uniform crossection whose lateral surface is impervious to heat.

Suppose that the ~~rodge~~ rod is kept along x-axis and the temp $u(x,t)$ is the same at any point in a perticular crossection but may change from crossection to crossection.

Let $f(x)$ be the initial temp distribution then the heat-conduction in the rod is given by the B.V.P.

$$\boxed{\text{Note: } \mathcal{F}^{-1}\left[e^{-a^2s^2} ; s\to x\right] = \frac{1}{a\sqrt{2}}\, e^{\frac{-x^2}{4a^2}}}$$

* Example-

$$\left.\begin{aligned} &u_{xx} = a^2 u_t \qquad -\infty < x < \infty,\ t > 0 \\ &\text{B.C. } u(x,t) \to 0,\quad u_x(x,t) \to 0 \text{ as } |x| \to \infty \\ &\text{I.C. } u(x,0) = f(x) \qquad -\infty < x < \infty \end{aligned}\right\} \cdots (1)$$

$\Rightarrow$ We define the fourier transform of the function $u(x,t)$ w.r.t. x by

$$\mathcal{F}[u(x,t) : x \to s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, u(x,t)\, dx = U(s,t)$$

with this definition we have fourier transform of

$$\mathcal{F}[u_x(x,t) : x \to s] = -is\, U(s,t)$$

$$\mathcal{F}[u_{xx}(x,t) : x \to s] = s^2\, U(s,t)$$

Also,

$$\mathcal{F}[u_t(x,t) : x \to s] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, u_t(x,t)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \int_{-\infty}^{\infty} e^{isx}\, u(x,t)\, dx$$

$$= U_t(s,t).$$

Denoting $\mathcal{F}[f(x)] = F(s)$.

and taking the fourier transform of eq^n ① it is to the problem.

$$-s^2 U(s,t) = a^2 U_t(s,t)$$

i.e. $U_t(s,t) + a^2 s^2 \cdot U(s,t) = 0$ --- ②

$\therefore$ I.C. $U(s,0) = F(s)$ --- ③

General solution of eq^n ② is

$$U(s,t) = A(s) \cdot e^{-a^2 s^2 t}$$

Using the I.C. $U(s,0) = F(s)$

$\Rightarrow F(s) = A(s) \cdot e^0$

$\Rightarrow F(s) = A(s)$.

$\cancel{\mathcal{F}[e^{-a^2x^2}; x \to s] = \frac{1}{a\sqrt{2}} e^{-\frac{s^2}{4a^2}}}$

$\cancel{\mathcal{F}^{-1}[e^{-\frac{s^2}{4a^2}}] = a\sqrt{2}\, e^{-a^2x^2}}$

Thus we have.

$$U(s,t) = F(s) \cdot e^{-a^2 s^2 t} \quad \text{---} ④$$

We know that

$$\mathcal{F}^{-1}[e^{-a^2 s^2 t}; s \to x] = \frac{1}{a\sqrt{2t}} e^{-\frac{x^2}{4a^2 t}} = g(x,t) \text{ say}$$

Taking inverse fourier transform of eq^n ④ and using the convolution theorem we have.

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) \cdot g(x-\xi, t)\, d\xi$$

$$= \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \cdot e^{-\frac{(x-\xi)^2}{4a^2 t}} \cdot d\xi$$

This is required solution.

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2. Heat equation on semi-infinite line.

example- consider B.V.P.

$$u_{xx} = a^{-2} u_t \quad , \quad 0<x<\infty \quad . \quad t>0 \quad \text{----} \; ①$$

B.C. $u_x(0,t) = 0$, $u(x,t) \to 0$, $u_x(x,t) \to 0$ as $x \to \infty$

I.C. $u(x,0) = f(x)$, $0<x<\infty$

⇒ since the interval is semiinfinite and in the B.C. it is given that

$$u_x(0,t) = 0$$

We use fourious cosine transform.

We define.

$$F_c[u(x,t) : x \to s] = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cdot \cos sx \, dx.$$

$$= U(s,t) \quad \text{say.}$$

with this definition we have

$$F_c[u_{xx}(x,t) : x \to s] = -s^2 U(s,t) - \sqrt{\frac{2}{\pi}} \cdot u_x(0,t)$$

$$= -s^2 \cdot U(s,t).$$

and $F_c[u_t(x,t) : x \to s] = U_t(s,t)$

Denoting $F_c[f(x) : x \to s] = F(s)$.

Taking fourious cosine transform of the eqⁿ. ①.

we get

$$-s^2 U(s,t) = -a^2 U_t(s,t).$$

$$U_t(s,t) + a^2 s^2 U(s,t) \quad \text{---} \; ②$$

I.C. $U(s,0) = F(s)$ ---- ③

A general solution of equation ② is

$$U(s,t) = A(s) \cdot e^{-a^2 s^2 t}$$

Using eqⁿ ③, we get

$$A(s) = F(s)$$

Hence.

$$U(s,t) = F(s).e^{-a^2s^2t}$$

taking the inverse fourier cosine transform we get the required solution as.

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(s).e^{-a^2s^2t} \cos sx \, ds.$$

H.P.

☆ Assignment ☆

Q.1. Find the inverse laplace transform of the following function.

a) $\frac{2}{(P+1)(P^2+1)}$

$\Rightarrow$ Consider $\frac{2}{(P+1)(P^2+1)} = \frac{A}{P+1} + \frac{BP+C}{P^2+1}$ --- ⊛

$$2 = A(P^2+1) + (BP+C)(P+1)$$

$$2 = AP^2 + A + BP^2 + BP + CP + C$$

$$2 = (A+B)P^2 + (B+C)P + A+C$$

$\Rightarrow$ $A+C = 2$ -- ①

$A+B = 0$ -- ②

$B+C = 0$ -- ③

solving above eqns.

$A+\cancel{B}-\cancel{B}-C = 0 \Rightarrow$ $A+C = 2$

$A-C = 0.$

$2A = 2$

$\boxed{A = 1}$ $\Rightarrow B = -1$

$C = 1$

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put the value in equation ①

$$L^{-1}\left[\frac{2}{(p+1)(p^2+1)}\right] = L^{-1}\left[\frac{1}{p+1} - \frac{p+1}{p^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{p+1} + \frac{1}{p^2+1} - \frac{p}{p^2+1}\right]$$

$$= e^{-t} + \sin t - \cos t$$

2] $\frac{1}{(p^4-1)}$

$\Rightarrow$ consider $\left[\frac{1}{(p^4-1)}\right] = \left[\frac{1}{(p^2-1)(p^2+1)}\right]$

$$= \frac{1}{2}\left[\frac{1}{p^2-1} - \frac{1}{p^2+1}\right]$$

taking inverse laplace transform.

$$L^{-1}\left[\frac{1}{p^4-1}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{p^2-1} - \frac{1}{p^2+1}\right]$$

$$= \frac{1}{2}[\sinh t - \sin t]$$

3) $\frac{p}{(p^2+a^2)(p^2+b^2)}$

$\Rightarrow \left[\frac{p}{(p^2+a^2)(p^2+b^2)}\right] = p\left[\frac{1}{p^2+a^2} - \frac{1}{p^2+b^2}\right]$

$$= p\left[\frac{p^2+b^2-p^2-a^2}{(p^2+a^2)(p^2+b^2)}\right]$$

$$= p\left[\frac{b^2-a^2}{(p^2+a^2)(p^2+b^2)}\right]$$

$$\Rightarrow \left[\frac{p}{(p^2+a^2)(p^2+b^2)}\right] = \frac{1}{(b^2-a^2)}\left[\frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\right]$$

taking inverse on both sides.

$$L^{-1}\left[\frac{p}{(p^2+a^2)(p^2+b^2)}\right] = \frac{1}{(b^2-a^2)} L^{-1}\left[\frac{p}{p^2+a^2} - \frac{p}{p^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}[\cos at - \cos bt]$$

$$= \frac{1}{b^2-a^2}(\cos at - \cos bt).$$

4). $\frac{4p^2-16}{p^3(p+2)^2}$

$\Rightarrow$ Here,

$$\frac{4p^2-16}{p^3(p+2)^2} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p^3} + \frac{D}{p+2} + \frac{E}{(p+2)^2} \quad \text{---} (*)$$

$$4p^2-16 = Ap^2(p+2)^2 + Bp(p+2)^2 + C\cdot(p+2)^2 + D(p+2)\cdot p^3 + E\cdot p^3$$

$$= Ap^2[p^2+4p+4] + Bp[p^2+4p+4] + C\cdot[p^2+4p+4] + D(p^4+2p^3) + Ep^3$$

$$= Ap^4 + A4p^3 + A4p^2 + Bp^3 + 4Bp^2 + 4Bp + Cp^2 + 4Cp + 4C + p^4D + 2Dp^3 + Ep^3$$

$$= (A+D)p^4 + (4A+B\ 2D+E)p^3 + (4A+4B+C)p^2 + (4B+4C)p + 4C.$$

comparing coefficient.

$A+D=0$. , $4B+4C=0$, $4A+B+2D+E=0$, $4C=-16$

$-D-2=0$, $4B=16$, $-8+4+4+E=0$ $\Rightarrow \boxed{C=-4}$

$\boxed{D=2}$, $\boxed{B=4}$, $\boxed{E=0}$

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$4A+4B+C=4$ $\qquad$ $4A+B+2D+E=0$

$4A+16-4=4$ $\qquad$ $-8+4+4+E=0$

$4A=4-12$ $\qquad$ $\Rightarrow E=0$

$4A=-8$

$\boxed{A=-2}$

Eq (*) becomes.

$$\frac{4p^2-16}{p^3(p+2)^2}=\frac{-2}{p}+\frac{4}{p^2}-\frac{4}{p^3}+\frac{2}{p+2}+\frac{0}{(p+2)^2}$$

taking inverse laplace transform.

$$L^{-1}\left[\frac{4p^2-16}{p^3(p+2)^2}\right]=L^{-1}\left[\frac{-2}{p}+\frac{4}{p^2}-\frac{4}{p^3}+\frac{2}{p+2}\right]$$

$$=-2+4t-4\frac{t^2}{2!}+2\cdot e^{-2t}$$

$$=2e^{-2t}+4t-2t^2-2.$$

5. $\dfrac{3p^2-6p+7}{(p^2-2p+5)^2}$

→ Given. $\dfrac{3p^2-6p+7}{(p^2-2p+5)}=\dfrac{3(p-1)^2+4}{[(p-1)^2+4]^2}$

taking inverse laplace transform.

$$L^{-1}\left[\frac{3p^2-6p+7}{(p^2-2p+5)^2}\right]=L^{-1}\left[\frac{3(p-1)^2+4}{[(p-1)^2+4]^2}\right]$$

$$=e^{t}L^{-1}\left[\frac{3p+4}{(p^2+4)^2}\right]$$

$$=e^{t}\cdot L^{-1}\left[\frac{3p}{(p^2+4)^2}+\frac{4}{(p^2+4)^2}\right]\quad ---*$$

using inverse laplace transform derivative.

If $L^{-1}\{F(p)\} = f(t)$. then

$$L^{-1}\left[\frac{d^n}{dp^n}(F(p)\right] = (-1)^n \cdot t^n \cdot L^{-1}[F(p)]$$

since $\frac{d}{dp}\left[\frac{3}{p^2+4}\right] = \frac{-3\cdot 2p}{(p^2+4)^2}$

$$= \frac{-6p}{(p^2+4)^2}$$

$$\therefore \frac{-6p}{(p^2+4)^2} = \frac{d}{dp}\left[\frac{3}{p^2+4}\right]$$

$$\frac{p}{(p^2+4)^2} = \frac{-1}{2}\frac{d}{dp}\left[\frac{1}{p^2+4}\right]$$

$$L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = -\frac{1}{2}\cdot L^{-1}\left[\frac{d}{dp}\cdot\left(\frac{1}{p^2+4}\right)\right]$$

$$= -\frac{1}{2}(-1)t\,L^{-1}\left[\frac{1}{p^2+4}\right]$$

$$= \frac{1}{2}\cdot t\cdot\frac{1}{2}\cdot\sin 2t$$

$$= \frac{1}{4}t\sin 2t \quad \text{---} \; (1).$$

since $L^{-1}\left[\frac{4}{(p^2+4)^2}\right] = L^{-1}[$

But We know

$$L^{-1}\left[\frac{p}{(p^2+4)^2}\right] = \frac{1}{4}\cdot t\sin t.$$

since by integration of inverse laplace transform,

$$L^{-1}\left[\frac{1}{(p^2+4)^2}\right] = \int_0^t \frac{1}{4}\cdot u\cdot\sin 2u\cdot du.$$

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$$= \frac{1}{4}\left[\frac{u.(-\cos 2u)}{2} + \frac{\sin 2u}{4}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{-t\cos 2t}{2} + \frac{\sin 2t}{4}\right] \quad \text{-- (2)}$$

$\therefore$ By (*) (1) & (2) we get.

$$L^{-1}\left[\frac{3p^2-6p+7}{(p^2-2p+5)^2}\right] = e^t.\left[\frac{3}{4}.t\sin 2t + \frac{\sin 2t}{4} - \frac{t.\cos 2t}{2}\right]$$

Q.2. Given that $L^{-1}[F(p)] = f(t)$ show that for constant a, b and k.

1) $L^{-1}[F(kp)] = \frac{1}{k}.F(t/k) \quad t>0$

2] $L^{-1}[F(ap+b)] = \frac{1}{a}.e^{-\frac{bt}{a}}.F\left(\frac{t}{a}\right) \quad a>0.$

$\Rightarrow$ Given that

$$L^{-1}[F(p)] = f(t).$$

$$F(p) = \int_0^\infty e^{-pt} f(t).dt$$

$$F(kp) = \int_0^\infty e^{-kpt}.f(t).dt \qquad \text{put } kt = x,\ t = \frac{x}{k}$$

$$dt = \frac{dx}{k}.$$

$$= \int_0^\infty e^{-xp} f\left(\frac{x}{k}\right).\frac{dx}{k}.$$

$$= \frac{1}{k}.\int_0^\infty e^{-pt}.f\left(\frac{t}{k}\right).dt.$$

$$~~L^{-1}[F(kp)]~~ = L\left[\frac{1}{k}\ f(t/k)\right]$$

$$L^{-1}[F(kp)] = \frac{1}{k}.f(t/k). \quad t>0$$

H.P

b). using change of scale property

If $L^{-1}[F(P)] = f(t)$.

$L^{-1}[F(P-a)] = e^{at} \cdot f(t)$

or $L[f(at)] = \frac{1}{a} \cdot F(P/a)$.

$a \cdot L[f(at)] = F(P/a)$.

$a \cdot [F(ap)] = L^{-1}[f(t/a)]$

$L^{-1}[f(t/a)] = a \cdot [F(ap)] = F_1(P)$.

using ~~b~~ shifting theorem.

$L\left[e^{-\frac{b}{a}t} \cdot f(t/a)\right] = a \cdot \left[F_1 \left[a \cdot (P + \frac{b}{a})\right]\right]$

$= a \cdot F\left[a \cdot \frac{(Pa+b)}{a}\right]$

$= a \cdot F(ap+b)$.

$\frac{1}{a} \cdot L\left[e^{-\frac{b}{a}t} \cdot f(t/a)\right] = F(ap+b)$

taking inverse

$L^{-1}[F(ap+b)] = \frac{1}{a} \cdot e^{-\frac{b}{a}t} \cdot f(t/a) \quad a>0$

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Q.5. convolution theorem

If $f(t)$ and $g(t)$ are piecewise continuous function on $t \geq 0$ and is $O(e^{\alpha t})$ and if $F(P)$ and $G(P)$ are laplace transform resp. of $f(t)$ & $g(t)$ then the inverse laplace transform of the product

$F(P) \cdot G(P)$ is given by the formula

$$L^{-1}[F(P) \cdot G(P)] = (f * g)t = \int_0^t f(t-u) \cdot g(u) \cdot du$$

$\Rightarrow$ Here we shall prove that

$$L\left[\int_0^t f(u) \cdot g(t-u) \cdot du\right] = F(P) \cdot G(P)$$

from which the required result follows immediately.

We have,

$$L\left[\int_0^t f(u) \cdot g(t-u) \cdot du\right]$$

$$= \int_0^{\infty} e^{-pt} \left\{ \int_0^t f(u) \cdot g(t-u) \cdot du \right\} \cdot dt$$

$$= \int_0^t \int_0^t e^{-pt} f(u) \cdot g(t-u) \cdot du \cdot dt$$

$$= \lim_{T \to t} \int_0^T \int_{u=0}^t e^{-pt} \cdot f(u) \cdot g(t-u) \cdot du \cdot dt$$

$$= \lim_{T \to \infty} I_T$$

$$\text{Where } I_T = \int_{t=0}^T \int_{u=0}^t e^{-pt} f(u) \cdot g(t-u) \cdot du \cdot dt$$

taking $t-u = y$ or $t = y+u$ we ~~say~~ see that the shaded Area A_{tu} of the tu-plane is transformed into area A_{uy} of the uy-plane as shown in fig

$\therefore$ By theorem on transformation of multiple integrals

We have.

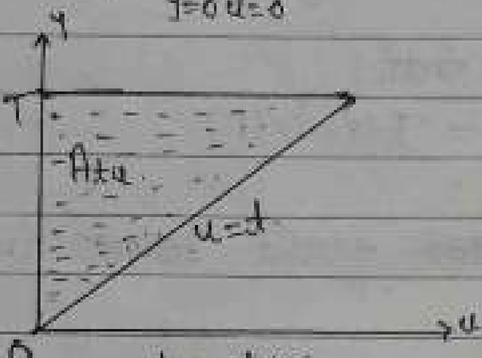
$$I_T = \iint_{A_{tu}} e^{-pt} f(u) \cdot g(t-u) \cdot du \, dt$$

$$I_T = \iint A_{uy} \cdot e^{-p(u+y)} \cdot f(u) \cdot g(y) \cdot \left| \frac{\partial(u,t)}{\partial(u,y)} \right| \cdot du\, dt$$

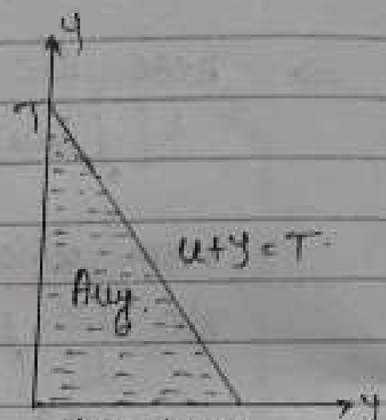
Were jacobian of transformation

$$J = \frac{\partial(u,t)}{\partial(u,y)} = \begin{vmatrix} \partial u/\partial u & \partial u/\partial y \\ \partial t/\partial u & \partial t/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$\therefore I_T = \int_{y=0}^{T} \int_{u=0}^{T} e^{-p(u+y)} \cdot f(u) \cdot g(u) \cdot dx\, dy$$



tu-plane

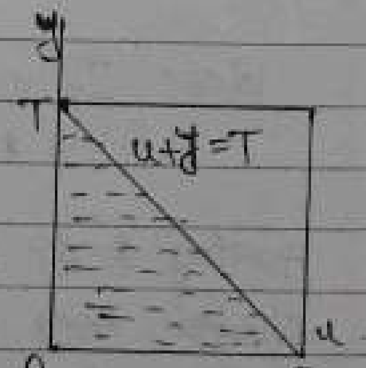


uy-plane

Now, Let us define function.

$$K(x,y) = \begin{cases} e^{-p(u+y)} \; f(u) \cdot g(y), & u+y \le T \\ 0 & u+y > T \end{cases}$$

Here the function $K(u,y)$ is defined over the square of side T such that integral I_T over the unshaded area shown in fig.



$$I_T = \int_{y=0}^{T} \int_{u=0}^{T} K(u,y) \cdot du\, dy.$$

Thus. $L\left[\int_0^t f(u) \cdot g(t-u) \cdot du \right]$

$$= \lim_{T \to \infty} I_T = \int_0^\infty \int_0^\infty K(u,y)\, du\, dy$$

$$= \int_0^\infty \int_0^\infty e^{-p(u+y)} \cdot f(u) \cdot g(y) \cdot du\, dy$$

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$$= \left[\int_0^\infty e^{-pu} \cdot f(u) \cdot du \right] \cdot \left[\int_0^\infty e^{-py} \cdot g(y) \cdot dy \right]$$

$$= L\{f(t)\}\ L[g(t)] \Rightarrow G(P) \cdot F(P).$$

Hence $\int_0^t f(u) \cdot g(t-u) \cdot du = L^{-1}[F(P) \cdot G(P)]$

$$= F * G.$$