Fourier Integral and Fourier Transform.

Fourier series representation of a function -

If f is a periodic function with period $2p$ defined on the interval $(-p, p)$ and is piecewise smooth. then it has the fourier series representation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\frac{n\pi x}{p} + b_n \cdot \sin\frac{n\pi x}{p} \right)$$

where $a_n = \frac{1}{p}\int_{-p}^{p} f(t) \cdot \cos\frac{n\pi t}{p} \, dt$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(t) \cdot \sin\frac{n\pi t}{p} \, dt.$$

Periodic functions defined on the entire real line have Fourier series representation. where as non periodic function define on the entire real line have the fourier integral representation which is given by the following theorem.

Theorem - Fourier Integral theorem

If f and f' are peicewise continuous function on every finite interval and if $\int_{-\infty}^{\infty} |f(x)| dx < \infty$ then

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) \cdot \cos[s(t-x)] \, dt \cdot dx \quad \text{--- (1)}$$

at points x. where f is cos

If x is a point of discontinuity the above integral converges to the average value $\frac{1}{2}[f(x)^{+} + f(x)^{-}]$ of the right and left hand limit.

Note - The cond^{n}s of in this theorem is only sufficient but not neccessary.

Recall - $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$

i.e. There exist function that have fourier integral representation but do not satisfy the conditions of the theorem.

We simplify the eqn ① as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t).[\cos st \cdot \cos sx + \sin st \cdot \sin sx \cdot]\,dt\,dx$$

$$= \int_0^{\infty}[A(s)\cdot \cos sx + B(s)\cdot \sin sx]\cdot ds \quad \text{---- ②}$$

Where $A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot \cos st \cdot dt$ --- ③

$$\therefore B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot \sin st \cdot dt \quad \text{---- ④}$$

Fourier cosine and sine Integral representation

If the function f is an even function. then it follows that

$$A(s) = \frac{2}{\pi}\int_0^{\infty} f(t)\cdot \cos st \cdot dt \text{ and } B(s)=0 \quad \text{---- ⑤}$$

so that f has fourier cosine integral representation. given by

$$f(x) = \int_0^{\infty} A(s)\cdot \cos sx \cdot ds \quad \text{--- ⑤}$$

Similarly,

If f is an odd function. then it has fourier sine integral representation given by

$$f(x) = \int_0^{\infty} B(s)\cdot \sin sx \cdot ds \quad \text{--- ⑥}$$

Where $B(s) = \frac{2}{\pi}\int_0^{\infty} f(t)\cdot \sin st \cdot dt$ --- ⑦

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★ Example-

Find an integeal eepeesentation of the function $f(x) = h(1-|x|)$ whee h is heavisides unit step function.

⇒ The fouiee integeal eepeesentation of integeal is

$$f(x) = \int_0^\infty (A(s)\cdot \cos sx + B(s)\cdot \sin sx)\cdot dx \quad \text{---} (1)$$

Wheee, $A(s) = \frac{1}{\pi}\cdot\int_{-\infty}^{\infty} f(t)\cdot \cos st\cdot dt.$

$$B(s) = \frac{1}{\pi}\cdot\int_{-\infty}^{\infty} f(t)\cdot \sin st\cdot dt.$$

We have, $A(s) = \frac{1}{\pi}\cdot\left[\int_{-1}^{1} 1\cdot \cos st\cdot ds\right.$

$$= \frac{2}{\pi}\cdot\int_0^1 \cos st\cdot ds$$

$$= \frac{2}{\pi s}\cdot [\sin st]_0^1$$

$$= \frac{2}{\pi s}\cdot [\sin s - 0]$$

$$A(s) = \frac{2\sin s}{\pi s}$$

Ny. $B(s) = \frac{1}{\pi}\int_{-1}^{1} 1\cdot \sin st\cdot dt = 0$

∴ The fouiee integeal eepeesentation is

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin s\cdot \cos sx}{s}\cdot ds.$$

Note -

Foe. $x=0$,

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin s}{s}\cdot ds$$

$$1 = \frac{2}{\pi}\int_0^\infty \frac{\sin s}{s} ds \Rightarrow \int_0^\infty \frac{\sin s}{s} ds = \frac{\pi}{2}.$$

Example - Find fourier cosine and fourier sine representation of the function $f(x) = \begin{cases} \cos x & 0 < x < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x \end{cases}$

⇒ The fourier cosine integral of the function is given by

$$f(x) = \int_0^\infty A(s) \cdot \cos sx \cdot dx \quad -- (1)$$

Where $A(s) = \int_{-\infty}^{\infty} f(t) \cdot \cos st \cdot dt \cdot \frac{1}{\pi}$.

$$= \frac{2}{\pi} \int_0^{\pi/2} \cos t \cdot \cos st \cdot dt.$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{2} \{ \cos t(1+s) + \cos t(1-s) \} dt$$

$$= \frac{1}{\pi} \int_0^{\pi/2} [\cos t(1+s) + \cos t(1-s)] \cdot dt$$

$$= \frac{1}{\pi} \left[\frac{\sin(1+s)t}{(1+s)} + \frac{\sin(1-s)t}{(1-s)} \right]_0^{\pi/2}$$

$$= \frac{1}{\pi} \left[\frac{\cos \frac{\pi}{2} s}{(1+s)} + \frac{\cos \frac{\pi}{2} s}{(1-s)} \right]$$

$$= \frac{\cos \frac{\pi}{2} s}{\pi} \left[\frac{1-s+1+s}{1-s^2} \right.$$

$$= \frac{2}{\pi} \cdot \frac{\cos(\pi/2 \cdot s)}{1-s^2}.$$

∴ The integral representation is

$$f(x) = \int_0^\infty \frac{2}{\pi} \left(\frac{\cos \frac{\pi s}{2}}{1-s^2} \right) \cdot \cos sx \cdot ds.$$

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Similarly, sine representation

$$f(x) = \int_0^\infty B(s) \cdot \sin sx \cdot dx \quad \text{---} (2)$$

Where,

$$B(s) = \frac{2}{\pi} \int_0^\infty f(t) \cdot \sin st \cdot dt.$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \cos t \cdot \sin st \cdot dt.$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \frac{1}{2} \{ \sin t(1+s) + \sin t(s-1) \} dt$$

$$= \frac{1}{\pi} \left[-\frac{\cos t(1+s)}{s+1} - \frac{\cos t(s-1)}{(s-1)} \right]_0^{\pi/2}$$

$$= \frac{1}{\pi} \left[\frac{-\sin \frac{\pi}{2} s}{1+s} + \frac{\sin \frac{\pi}{2} s}{s-1} + \frac{1}{s+1} + \frac{1}{s-1} \right]$$

$$\therefore B(s) = \frac{2}{\pi} \left[\frac{s - \sin \frac{\pi s}{2}}{s^2-1} \right]$$

$\therefore$ The required integral representation is.

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\frac{s - \sin \frac{\pi s}{2}}{s^2-1} \right) \cdot \sin sx \cdot dx.$$

1) Show that $\int_0^\infty \frac{\sin s \cdot \cos s}{s} \cdot ds = \frac{\pi}{4}$

$$\Rightarrow \int_0^\infty \frac{\sin s \cdot \cos s}{s} \cdot ds = \int_0^\infty \frac{2 \sin s \cos s}{s} \cdot ds = \int_0^\infty \frac{\sin 2s}{2s} \cdot ds.$$

put $2s = t \Rightarrow ds = dt/2$.

$$= \int_0^\infty \frac{\sin t}{t} \cdot \frac{dt}{2} = \frac{1}{2} \cdot \int_0^\infty \frac{\sin t}{t} \cdot dt.$$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

2) Show that $\int_0^\infty \frac{\sin ax}{x}\,dx = \frac{\pi}{2}$.

$\Rightarrow \int_0^\infty \frac{\sin ax}{x}\,dx = \int_0^\infty \frac{\sin ax}{ax}\cdot a\,dx$ put $ax = t$, $dx = \frac{dt}{a}$

$= \int_0^\infty \frac{\sin t}{t}\cdot a\cdot\frac{dt}{a}$

$= \int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

3) If $f(x) = \begin{cases} 0 & x<0 \\ e^x & x>0 \end{cases}$ then

a) Show that $f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos sx + s\sin x}{s^2+1}\,ds$.

b) Verify the directly that the above integral representation converges to the value $\frac{1}{2}$ at $x=0$

$\Rightarrow$ The fourier integral of function is given by

$f(x) = \int_0^\infty [A(s)\cdot\cos sx + B(s)\cdot\sin sx]\,dx$ ---- *

$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\cos st\cdot dt$

$= \frac{1}{\pi}\int_{-\infty}^{0} f(t)\cdot\cos st\cdot dt + \frac{1}{\pi}\int_0^\infty f(t)\cdot\cos st\,dt$

$= \frac{1}{\pi}\int_0^\infty e^{-t}\cos st\cdot dt$

$= \frac{1}{\pi}\left[\frac{e^{-t}}{1+s^2}\left[(-1)\cdot\cos st + s\cdot\sin st\right]\right]_0^\infty$

$= \frac{1}{\pi}\left(0 - \frac{1}{1+s^2}(-1)\right)$

$A(s) = \frac{1}{\pi}\left[\frac{1}{1+s^2}\right]$

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By. $B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot \sin st\cdot dt$

$= \frac{1}{\pi}\cdot\int_{0}^{\infty} e^{-t}\cdot \sin st\cdot dt$

$= \frac{1}{\pi}\left[\frac{e^{-t}}{1+s^2}\left[(-1)\cdot \sin st - s\cdot\cos st\right]\right]_0^{\infty}$

$= \frac{1}{\pi}\left[0 - \left(\frac{-s}{1+s^2}\right)\right]$

$B(s) = \frac{1}{\pi}\frac{s}{(1+s^2)}$

$\therefore$ Equation * becomes

$f(x) = \int_0^{\infty}[A(s)\cdot\cos sx + B(s)\cdot\sin sx]\,ds$

$= \int_0^{\infty}\left[\frac{1}{\pi(1+s^2)}\cdot\cos sx + \frac{s}{\pi(1+s^2)}\cdot\sin sx\right]ds.$

$f(x) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos sx + s\cdot\sin sx}{(1+s^2)}\,ds$

b) for $x=0$ ~~has~~ f is discontinuous at $x=0$

$f(0) = \frac{1}{\pi}\int_0^{\infty}\frac{\cos s(0) + s\cdot\sin s(0)}{1+s^2}\,ds.$

$= \frac{1}{\pi}\cdot\int_0^{\infty}\frac{1}{1+s^2}\,ds.$

$= \frac{1}{\pi}\left[\tan^{-1}(s)\right]_0^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2}-0\right] = \frac{1}{2} = f(0)$ - ①

$\therefore$ and directly. $f(0) = \frac{f(0^+) + f(0^-)}{2}$

$= \frac{0+1}{2} = \frac{1}{2} = f(0)$ - ②

$\therefore \Rightarrow$ The integral is convergeous to the value $\frac{1}{2}$

2) Find an integeal eepeesentation foe.

$$f(x) = \begin{cases} 1-x^2 & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

and deduced the value of integeal

$$I = \int_0^\infty \frac{\sin x - x\cdot\cos x}{x^3} \cdot \cos\frac{x}{2}\, dx.$$

$\Rightarrow$ The fouieie integeal eepeesention of the f is

$$f(x) = \int_{-\infty}^{\infty} (A(s)\cdot\cos sx + B(s)\cdot\sin sx)\cdot ds \quad \text{---(*)}$$

wheee: $A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\cos st\cdot dt$ and. ---①

$$B(s) = \frac{1}{\pi}\cdot\int_{-\infty}^{\infty} f(t)\cdot\sin st\cdot dt. \quad \text{---②}$$

∴ Equation ② becomes.

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\cos st\cdot dt$$

$$= \frac{1}{\pi}\int_{-1}^{1} (1-t^2)\cdot\cos st\cdot dt$$

$$= \frac{2}{\pi}\int_0^1 (\cos st - t^2\cdot\cos st)\cdot dt.$$

$$= \frac{2}{\pi}\left\{\left[\frac{\sin st}{s}\right]_0^1 - \left[t^2\cdot\frac{\sin st}{s} + 2t\,\frac{\cos st}{s^2} - \frac{2\sin st}{s^3}\right]_0^1\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\sin s}{s} - \left[\frac{\sin s}{s} + \frac{2\cos s}{s^2} + \frac{2\sin st}{s^3} - 0\right]\right\}$$

$$= \frac{4}{\pi}\left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3}\right]$$

$$A(s) = \frac{4}{\pi}\left[\frac{\sin s - s\cdot\cos s}{s^3}\right]$$

and. oleaely $B(s) = 0$

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∴ The required integral representation is.

$$f(x) = \frac{4}{\pi}\int_0^\infty \left[\frac{\sin s - s\cos s}{s^3}\right]\cos xs\, ds$$

deduced integral at $x = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = \frac{4}{\pi}\int_0^\infty \left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)\cdot ds.$$

We have

$$f\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

$$\Rightarrow \frac{3}{4} = \frac{4}{\pi}\int_0^\infty \frac{\sin x - x\cdot\cos x}{x^3}\cdot\cos\left(\frac{x}{2}\right)\cdot dx \quad [\because s = x \Rightarrow ds = dx]$$

$$\Rightarrow \frac{3}{4}\cdot\frac{\pi}{4} = \int_0^\infty \frac{\sin x - x\cdot\cos x}{x^3}\cdot\cos\frac{x}{2}\cdot dx$$

$$\Rightarrow \int_0^\infty \frac{\sin x - x\cos x}{x^3}\cdot\cos\frac{x}{2}\cdot dx = \frac{3\pi}{16}$$

* Example - obtain the fourier integral representation of the following function.

1) $f(x) = \begin{cases} \sin x & |x| < \pi \\ 0 & |x| > \pi \end{cases}$

2) $f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$

3) $f(x) = \begin{cases} \cos x & , 0 < x < \pi \\ 0 & , \text{otherwise} \end{cases}$

To prove the fourier integral theorem we prove the following 2 results

* Lemma - Riemann - Lebesgue lemma

If f is peicewise continuous and absolutely integrable on the entire real axis then

$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} f(t)\cdot\cos\lambda t\cdot dt = 0$ and $\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} f(t)\cdot\sin\lambda t\cdot dt = 0$ are

equivalently $\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\cdot dt = 0$

$\Rightarrow$ We will prove the result by the assumption that f is continuous and f' is bounded on the entire real axis.

Using integration by part.

$$\int_{-P}^{P} e^{i\lambda t}\cdot f(t)\cdot dt = \left[f(t)\cdot \frac{e^{i\lambda t}}{i\lambda} \right]_{-P}^{P} - \frac{1}{i\lambda}\int_{-P}^{P} f'(t)\cdot e^{i\lambda t}\cdot dt \quad \text{--- (1)}$$

As $f(t)\cdot e^{i\lambda t}$ is bdd on all finite interval of any value of λ we have.

$$\lim_{\lambda\to\infty} \left[f(t)\cdot \frac{e^{i\lambda t}}{i\lambda} \right]_{-P}^{P} = 0 \quad \text{also}$$

as f' is bounded we have

$$\left| \int_{-P}^{P} f'(t)\cdot e^{i\lambda t}\cdot dt \right| \le \int_{-P}^{P} |f'(t)|\cdot dt < \infty \quad \forall \lambda.$$

Hence taking limit as $\lambda \longrightarrow \infty$ in eqn (1) we get $\lim_{\lambda\to\infty} \int_{-P}^{P} e^{i\lambda t} f(t)\, dt = 0$

Extending the limit's of integration we get

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} e^{i\lambda t} f(t)\cdot dt = 0.$$

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★ lemma-2.

If f is peciewise smooth and absolutely indegeable on the entiee eeal axis and if x is a point of continuity of f then

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t)\cdot \frac{\sin \lambda t}{t}\, dt = f(x)$$

⇒ Fiest we will peove the eesult.

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-p}^{p} \frac{\sin \lambda t}{t}\, dt = \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\lambda p}^{\lambda p} \frac{\sin t}{t}\, dt \quad (\text{put } t = \lambda t$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t}\, dt = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin t}{t}\, dt$$

$$= \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$$

$$\therefore \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-p}^{p} \frac{\sin \lambda t}{t}\, dt = 1$$

In oedeeed to peove lemma we will show that

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-p}^{p} \left(\frac{f(x+t) - f(x)}{t}\right) \cdot \sin t \cdot dt = 0$$

The function $\frac{f(x+t) - f(x)}{t}$ is peciewise continuous and absolutely indegeable foe $t \neq 0$. and.

$\lim_{t\to 0} \frac{f(x+t) - f(x)}{t} = f'(x)$ exists since f is peicewise smooth. moeeovee $\frac{f(x+t) - f(x)}{t}$ is absolutly intgeable hence by ~~R-Integeable~~ Riemann lemma.

We have

$$\lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-p}^{p} \left(\frac{f(x+t) - f(x)}{t}\right) \sin \lambda t\, dt = 0$$

so taking limit of $p\to\infty$

$$\therefore \lim_{\lambda\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t)\cdot \frac{\sin \lambda t}{t}\, dt = f(x)$$

Note - Now we will prove fourier integral theorem for point x where f is continuous

$$\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cdot\cos[s(t-x)]\,dt\,ds = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\int_0^{\infty}\cos[s(t-x)]\,ds\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\frac{\sin(\lambda(t-x))}{t-x}\,dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\,\frac{\sin\lambda t}{t}\,dt \quad [t-x=t]$$

$$= f(x) \quad [\text{By lemma } (2)]$$

Ex Prove that a) $\lim_{\lambda\to\infty}\int_0^{p}\frac{\sin\lambda t}{t}\,dt = \frac{\pi}{2}$

b) $\lim_{\lambda\to\infty}\int_{-p}^{0}\frac{\sin\lambda t}{t}\,dt = \frac{\pi}{2}$

$\Rightarrow$ Given $\lim_{\lambda\to\infty}\int_0^{p}\frac{\sin\lambda t}{t}\,dt$

If f is piecewise smooth and absolutely integrable on the entire real line and if x is a point of continuity of f then we have

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-p}^{p} f(t+x)\cdot\frac{\sin\lambda t}{t}\,dt = f(x).$$

taking $f(x) = 1$

$$\lim_{\lambda\to\infty}\int_{-p}^{p}\frac{\sin\lambda t}{t}\,dt = \pi \quad \cdots (1)$$

$$\lim_{\lambda\to\infty}\cdot 2\int_0^{p}\frac{\sin\lambda t}{t}\,dt = \pi.$$

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$$\Rightarrow \lim_{\lambda\to\infty} \int_0^p \frac{\sin\lambda t}{t}\,dt = \frac{\pi}{2}$$

By using eq^n (1)

$$\lim_{\lambda\to\infty} \int_{-p}^{p} \frac{\sin\lambda t}{t}\,dt = \pi$$

$$\lim_{\lambda\to\infty} \left[\int_{-p}^{0} \frac{\sin\lambda t}{t}\,dt + \int_0^p \frac{\sin\lambda t}{t}\,dt \right] = \pi$$

$$\lim_{\lambda\to\infty} \int_{-p}^{0} \frac{\sin\lambda t}{t}\,dt + \frac{\pi}{2} = \pi$$

$$\lim_{\lambda\to\infty} \int_{-p}^{0} \frac{\sin\lambda t}{t}\,dt = \frac{\pi}{2}.$$

Fourier Transform -

let $f(t)$ be a function defined on R. f be peicewise continuous on each finite interval on R also suppose that f is absolutely integeable on R then the integeal $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\,dt$ is called as the fourier transform of the function $f(t)$. and we denoting by $\mathcal{F}(f(t))$, $\mathcal{F}$

$$\therefore \mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f(t)\,dt = F(s).$$

In this case the function $f(t)$ is called as the inverse fourier transform of $F(s)$ is given by.

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} \cdot F(s)\,ds. = \mathcal{F}^{-1}[F(s)]$$

Fourier sine and cosine transforms.

The fourier sine transform of a function and it's inverse sine transform are define by

$$F_s[f(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \sin st \, dt = F_s(s) \quad s>0.$$

$$F_s^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \cdot \sin st \cdot ds = f(t). \quad t>0$$

Ally. The fourier cosine transform of a function $f(t)$ and it's inverse cosine are defined by

$$F_c[f(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \cos st \cdot dt = F_c(s) \quad s>0.$$

$$F_c[F_c(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cdot \cos st \, ds = f(t) \quad t>0$$

Note - If f is even then.

1) $F(f(t)) = F_c[f(t)] \quad t>0$

If f is odd then.

2) $F[f(t)] = i \cdot F_s[f(t)] \quad t>0.$

Ex. Find the fourier transform of $e^{-a|t|}$ $a>0$.

let $f(t) = e^{-a|t|}$

since f is an even function it's fourier transform is given by.

$$F[e^{-a|t|}] = F_c[e^{-at}]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos st \, dt \quad -- (1)$$

We know that

$$L[\cos at] = \int_0^\infty e^{-pt} \cos at \cdot dt = \frac{p}{p^2+a^2}.$$

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$$\Rightarrow \int_0^\infty e^{-at} \cdot \cos st \cdot dt = \frac{a}{a^2+s^2}$$

$\therefore$ from equation (1)

$$F_c[e^{-a|t|}] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2+a^2} \checkmark$$

Note:

$$F_c^{-1}\left[\frac{1}{s^2+a^2}\right] = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{a} \cdot e^{-a|t|}, \quad a>0$$

Ex. Find the fourier sine and cosine transforms of the function $t \cdot e^{-at}$, $a>0$

$\Rightarrow$ Let $f(t) = t \cdot e^{-at}$, $a>0$

The Fourier sine transform of $f(t)$ is.

$$F_s[f(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cdot \sin st \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty t \cdot e^{-at} \cdot \sin st \cdot dt \quad \text{--- (1)}$$

We know that.

$$L[\sin at] = \int_0^\infty e^{-pt} \cdot \sin at \cdot dt = \frac{a}{p^2+a^2}$$

$$\therefore \int_0^\infty e^{-at} \cdot \sin st \cdot = \frac{s}{a^2+s^2} \quad \text{---- (2)}$$

diff. w.r.t. a on both sides.

$$\therefore \int_0^\infty -t \cdot e^{-at} \sin st \cdot dt = \frac{-2as}{(a^2+s^2)^2}.$$

$$\therefore \int_0^\infty e^{-at} \cdot t \cdot \sin st \cdot dt = \frac{2as}{(s^2+a^2)^2}.$$

$\therefore$ Equation (1) becomes.

$$F_s[f(t)] = \sqrt{\frac{2}{\pi}} \cdot \frac{2as}{(s^2+a^2)^2}.$$

The Fourier cosine transform of $f(t)$ is

$$F_c[f(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty t \cdot e^{at} \cos st \cdot dt \quad \text{---- (3)}$$

diff. equation (2) w.r.t. s.

$$\int_0^\infty e^{at} \cdot t \cdot \cos st \, dt = \frac{s^2 + a^2 - s.2s}{(s^2+a^2)^2}$$

$$= \frac{a^2 - s^2}{(s^2+a^2)^2}$$

put in eqn (3) we get

$$F_c[f(t)] = \sqrt{\frac{2}{\pi}} \cdot \frac{a^2 - s^2}{(s^2+a^2)^2}$$

Ex. $f(t) = e^{-a^2t^2}$; $a > 0$

$$\Rightarrow F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f(t) \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot e^{-a^2t^2} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2t^2 - ist)} \cdot dt \quad \text{---- (1)}$$

$$\Rightarrow \left(at - \frac{is}{2a}\right)^2 = a^2t^2 - 2at \cdot \frac{is}{2a} + \frac{-s^2}{4a^2} + \frac{s^2}{4a^2}$$

$$\Rightarrow \left(at - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} = (a^2t^2) - ist$$

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$\therefore$ Equation (1) becomes

$$\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(at - \frac{is}{2a})^2 - \frac{s^2}{4a^2}} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(at - \frac{is}{2a})^2} \cdot e^{\frac{-s^2}{4a^2}} \cdot dt$$

$$= \frac{e^{\frac{-s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(at - \frac{is}{2a})^2} \cdot dt$$

put $x = at - \frac{is}{2a}$

$$\frac{dx}{a} = dt$$

$$\therefore \mathcal{F}[\delta(t)] = \frac{e^{\frac{-s^2}{4a^2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot \frac{dx}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{-s^2/4a^2} \cdot 2 \int_{0}^{\infty} e^{-x^2} \cdot dx$$

But we know that

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$2 \int_0^{\infty} e^{-x^2} \cdot dx = \Gamma\frac{1}{2} = \sqrt{\pi}$$

$$\mathcal{F}[\delta(t)] = \frac{1}{a\sqrt{2\pi}} \cdot e^{\frac{-s^2}{4a^2}} \cdot \sqrt{\pi}$$

$$= \frac{1}{a\cdot\sqrt{2}} \cdot e^{\frac{-s^2}{4a^2}}$$

Note - Take $a = 1/\sqrt{2}$ in above example.

$$\mathcal{F}\left[e^{-\frac{t^2}{2}}\right] = \frac{1}{\frac{\sqrt{2}}{\sqrt{2}}} \cdot e^{-\frac{s^2}{2}} = e^{-s^2/2}$$

Thus $e^{-\frac{t^2}{2}}$ is self-reciprocal i.e. it is its own transform.

Home-Work:

Ex: 3, 4, 5, 6, 12, 14, 16, 18 Page - 57 - 58.

Ex. Determine the fourier transform of the following fun^n.

1). $f(t) = \begin{cases} 1 & , |t| < b \\ 0 & , \text{otherwise.} \end{cases}$

2) $f(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \text{otherwise.} \end{cases}$

3) $f(t) = \frac{1}{5t^2+1}$.

4) $f(t) = \frac{t}{t^2+7}$

5) $f(t) = \frac{t}{(t^2+4)^2}$.

6) $f(t) = \frac{1-t^2}{(t^2+4)^2}$.

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Properties of Fourier transform.

Theorem - 2 Linearity properties.

If $F(s)$ and $G[s]$ are the Fourier transforms resp. of $f(t)$ and $g(t)$, then for any constants c_1 and c_2 it follows that

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = c_1 F(s) + c_2 G(s)$$

similarly

$$\mathcal{F}^{-1}[c_1 F(s) + c_2 G(s)] = c_1 f(t) + c_2 g(t).$$

$\Rightarrow$ By definition of Fourier transform

$$\mathcal{F}[c_1 f(t) + c_2 g(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [c_1 f(t) + c_2 g(t)] \cdot e^{ist} \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_1 e^{ist} f(t) \cdot dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_2 e^{ist} g(t) \cdot dt$$

$$= \frac{c_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t) \, dt + \frac{c_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} g(t) \cdot dt$$

Also, By definition.

$$\mathcal{F}^{-1}[c_1 F(s) + c_2 G(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [c_1 \cdot F(s) + c_2 \cdot G(s)] \cdot e^{-ist} \cdot ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_1 \cdot F(s) \cdot e^{-ist} ds + \frac{c_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(s) \cdot e^{-ist} \cdot ds$$

$$= c_1 \cdot f(t) + c_2 \cdot g(t)$$

Theorem - 3. Change of Scale property.

If $\mathcal{F}[f(t)] = F(s)$. then $\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$.

$\Rightarrow$ (i) let $a > 0$. By debination of Fourier transform.

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f(at) \cdot dt.$$

put $at = x \Rightarrow dt = \frac{dx}{a}$.

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{is\frac{x}{a}} f(x) \cdot \frac{1}{a} \cdot dx$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\frac{x}{a}} \cdot f(x) \cdot dx.$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\frac{s}{a}t} \cdot f(t) \cdot dt$$

$$= \frac{1}{a} \cdot F\left(\frac{s}{a}\right) \quad \text{---} \quad (1) \qquad a > 0.$$

(ii) When $a < 0$, let $a = -k$. where $k > 0$.

Then.

$$\mathcal{F}[f(at)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f(at) \cdot dt \qquad \text{where } -kt = x, \quad dt = -\frac{dx}{k}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{-\infty} e^{is\left(-\frac{x}{k}\right)} \cdot f(x) \cdot \cdot \left(\frac{dx}{-k}\right).$$

$$= +\frac{1}{k\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is\left(-\frac{x}{k}\right)} \cdot f(x) \cdot dx$$

$$= +\frac{1}{k} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(-\frac{s}{k}\right)t} \cdot f(t) \cdot dt$$

$$= \frac{1}{-a} \cdot F\left(-\frac{s}{k}\right).$$

$$= \frac{-1}{a} F\left(\frac{s}{a}\right) \quad \text{---} \quad (2)$$

From (1) and (2)

$$\mathcal{F}[f(at)] = \frac{1}{|a|} \cdot F\left(\frac{s}{a}\right).$$

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Theorem - 4 Shifting Property

If $f(t)$ and $F(s)$ are fourier transform pairs then

a] $\mathcal{F}[e^{iat} \cdot f(t)] = F(s+a)$.

b) $\mathcal{F}[f(t-a)] = e^{ias} \cdot F(s)$.

⇒ By definition of fourier transform.

a). $\mathcal{F}[e^{iat} f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot e^{iat} f(t) \cdot dt$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s+a)t} f(t) \cdot dt$.

$= F(s+a)$.

b) Also, $\mathcal{F}[f(t-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t-a)\, dt$

put $t-a = x$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{is(a+x)} \cdot f(x) \cdot dx$.

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ias} \cdot e^{isx} f(x) \cdot dx$

$= \frac{e^{ias}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f(t) \cdot dt$.

$= e^{ias} \cdot F(s)$.

OR $\mathcal{F}^{-1}[e^{ias} \cdot F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} \cdot e^{ias} \cdot F(s) \cdot ds$.

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is(t-a)} \cdot F(s) \cdot ds$.

$= f(t-a)$.

Ex. Find the fourier inverse transform of $\frac{1}{s^2+ias+b}$, $b>0$.

$\Rightarrow$ We have

$$\mathcal{F}^{-1}\left[\frac{1}{s^2+ias+b}\right] = \mathcal{F}^{-1}\left[\frac{1}{(s+\frac{ia}{2})^2+\frac{a^2}{4}+b}\right]$$

$$= e^{-\frac{ia}{2}t}\,\mathcal{F}^{-1}\left[\frac{1}{s^2+\frac{a^2+4b}{4}}\right]$$

We know that

$$\mathcal{F}\left[e^{-a|t|}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{a}{s^2+a^2}.$$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{s^2+a^2}\right] = \sqrt{\frac{\pi}{2}}\cdot\frac{1}{a}\cdot e^{-a|t|}$$

$$\boxed{e^{-a|t|} = \mathcal{F}^{-1}\left[\sqrt{\frac{2}{\pi}}\cdot\frac{a}{s^2+a^2}\right]}$$

$$\therefore \mathcal{F}^{-1}\left[\frac{1}{s^2+ias+b}\right] = e^{-\frac{at}{2}}\cdot\sqrt{\frac{\pi}{2}}\cdot\frac{2}{\sqrt{a^2+4b}}\cdot e^{-\frac{\sqrt{a^2+4b}\cdot|t|}{2}}.$$

$$= \sqrt{\frac{2\pi}{a^2+4b}}\cdot\exp\left[-\frac{1}{2}\left(at+\sqrt{a^2+4b}\cdot|t|\right)\right]$$

Ex. Find the Fourier sine transform of $\frac{e^{-at}}{t}$ $a>0$.

$\Rightarrow$ $f(t) = \frac{e^{-at}}{t}$

Then fourier sine transform of $f(t)$ is

$$\mathcal{F}_s[f(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cdot\sin st\cdot dt.$$

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$$H_s\left[\frac{e^{-at}}{t}\right] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-at}}{t} \sin st \cdot dt \quad \text{---} \quad (1)$$

We know that

$$L[\sin at] = \int_0^\infty e^{-pt} \cdot \sin at \cdot dt = \frac{a}{p^2+a^2}$$

$$\Rightarrow \int_0^\infty e^{-at} \cdot \sin at \cdot dt = \frac{s}{a^2+s^2} \quad \text{---} \quad (2)$$

Integrating eqn (2) w.r.t. a from a to ∞.

$$\int_a^\infty \left\{ \int_0^\infty e^{-at} \sin st \cdot dt \right\} \cdot da = \int_a^\infty \frac{s}{a^2+s^2} \cdot da$$

$$\int_0^\infty \sin st \cdot \left\{ \int_a^\infty e^{-at} \cdot da \right\} \cdot dt = s \cdot \left[\frac{1}{s} \cdot \tan^{-1}\left(\frac{a}{s}\right) \right]_a^\infty$$

$$\int_0^\infty \sin st \cdot \left[\frac{e^{-at}}{-t} \right]_a^\infty \cdot dt = \tan^{-1}(\infty) - \tan\left(\frac{a}{s}\right)$$

$$\int_0^\infty \sin st \cdot \frac{e^{-at}}{t} \cdot dt = \frac{\pi}{2} - \tan^{-1}\left(\frac{a}{s}\right)$$

$$\int_0^\infty \frac{\sin st}{t} \cdot e^{-at} \cdot dt = \cot^{-1}\left(\frac{a}{s}\right) = \tan^{-1}\left(\frac{s}{a}\right)$$

putting in equation (1)

$$H_s\left[\frac{e^{-at}}{t}\right] = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{s}{a}\right)$$

Theorem - 5 Differentiation Property

If $f, f', \ldots, f^{(n-1)}$ are cts every-where and absolutely integrable, $f^{(n)}$ is piecewise smooth and absolutely integrable, and

$$\lim_{|t|\to\infty} f(t) = \lim_{|t|\to\infty} f'(t) = \cdots = \lim_{|t|\to\infty} f^{(n-1)}(t) = 0$$

Then $\mathcal{F}[f^{(n)}(t)] = (is)^n \cdot F(s)$, $n = 1, 2, 3 \ldots$

Where $F(s)$ is the fourier transform of $f(t)$.

$\Rightarrow$ First we will prove the theorem $n = 1$

We have to prove that

$$\mathcal{F}[f'(t)] = -is \cdot F(s)$$

$\therefore$ By definition of fourier transform

$$\mathcal{F}[f'(t)] = -is \cdot F(s)$$

$\therefore$ By definition of fourier transform

$$\mathcal{F}[f'(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot f'(t) \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[e^{ist} f(t) \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} is \cdot e^{ist} f(t) \cdot dt$$

since $\lim_{|t|\to\infty} f(t) = 0$

$$\Rightarrow \mathcal{F}[f'(t)] = 0 - \frac{(is)}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{ist} f(t) \cdot dt$$

$$= (-is) \cdot F(s)$$

$$\mathcal{F}[f'(t)] = (-is) \cdot \mathcal{F}[f(t)] \quad \text{---} \quad (1)$$

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Using the equation (1) for $f''(t)$

We have.

$$\mathcal{F}[f'(t)] = (-is)\cdot \mathcal{F}[f'(t)]$$
$$= (-is)(-is)\,\mathcal{F}[f(t)]$$
$$= (-is)^2 F(s)$$

By repited the application of equation (1) we get

$$\mathcal{F}[f^{(n)}(t)] = (-is)^n \cdot F(s) \qquad n = 1, 2 \text{ ----}$$

Ex If f and f' are continuous every where and absolutely integrable. f' is absolutely smooth and integrable and $\lim_{t\to\infty} f(t) = 0$ then prove that

1) $\mathcal{F}_c[f'(t)] = s\cdot F_s(s) - \sqrt{\frac{2}{\pi}}\cdot f(0)$

2) $\mathcal{F}_s[f'(t)] = -s\cdot F_c(s)$.

⇒ 1) By definition of fourier cosine transform.

$$\mathcal{F}_c[f(t)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cdot \cos st\cdot dt.$$

$$\mathcal{F}_c[f'(t)] = \sqrt{\frac{2}{\pi}}\cdot\int_0^\infty f'(t)\cdot \cos st\cdot dt$$

$$= \sqrt{\frac{2}{\pi}}\left[\cos st\cdot f(t)\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty (-s)\sin st\cdot f(t)\cdot dt$$

$$= \sqrt{\frac{2}{\pi}}[0 - f(0)] + s\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cdot \sin st\ dt.$$

$$= s\cdot \mathcal{F}_s[f(t)] - \sqrt{\frac{2}{\pi}}\cdot f(0).$$

$$= s\cdot F_s(s) - \sqrt{\frac{2}{\pi}}\cdot f(0).$$

$e^{at}\cos st$

2) dly By definition of fourier transform.

$$\mathcal{F}_s[f'(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(t) \cdot \sin st \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin st\, f(t)\right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty s \cdot \cos st \cdot f(t) \cdot dt.$$

$$= \sqrt{\frac{2}{\pi}} [0-0] - s\sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(t) \cdot \cos st \cdot dt$$

$$= -s \cdot F_c(s).$$

If $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = 0$,

Ex. Deeive the transform. relation.

1) $\mathcal{F}_c[f''(t)] = -s^2 F_c(s) - \sqrt{\frac{2}{\pi}} f'(0)$

2) $\mathcal{F}_s[f''(t)] = -s^2 F_s(s) + \sqrt{\frac{2}{\pi}} \cdot s \cdot f(0)$.

⇒ 1) By definition of fourier transform.

$$\mathcal{F}_c[f''(t)] = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f''(t) \cdot \cos st \cdot dt.$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\cos st \cdot f'(t)\right]_0^\infty - \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sin st \cdot f'(t) \cdot dt.$$

$$= \sqrt{\frac{2}{\pi}} [0 - f'(0)] + s \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \sin st \cdot f'(t) \cdot dt.$$

since $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} f'(t) = 0$

$$\mathcal{F}_c[f''(t)] = -\sqrt{\frac{2}{\pi}} \cdot f'(0) + s \cdot \mathcal{F}_s[f'(t)]$$

$$= -\sqrt{\frac{2}{\pi}} \cdot f'(0) + s \cdot [-s \cdot F_c(s)]$$

$$= -s^2 F_c(s) - \sqrt{\frac{2}{\pi}} \cdot f'(0).$$

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2) Ny By using definition of fourier transform.

$$F_s[f''(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f''(t) \cdot \sin st \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} \left[\sin st \, f'(t) \right]_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty s \cdot \cos st \cdot f'(t) \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} (0 - 0) - s \cdot F_c[f'(t)]$$

$$= -s \left[(s) \cdot F_s(s) \right] - \sqrt{\frac{2}{\pi}} \cdot f(0) \Big]$$

$$= -s^2 F_s(s) + \sqrt{\frac{2}{\pi}} \cdot s \, f(0)$$

Note - while solving

From this examples. it follows. that ~~in any~~. problem. in which $f(0)$ is known but $f'(0)$ is unknown we should. use fourier sine transform. while if $f'(0)$ is known and $f(0)$ is unknown we should. use fourier cosine transform.

2nd order i.v.p/b.v.p.

Theorem - 6 Fourier transform of multiplication by powers of t

If f is. absolutly integrable. and picewise smooth and if $t^n \cdot f(t)$ has a fourier transform. then.

$$F[t^n [f(t)] = (-i)^n \cdot F^{(n)}(s),$$

where $F(s)$ is fourier transform of $f(t)$

$\Rightarrow$ By definition of fourier transform

$$F[f(t)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t) \cdot dt \quad -- (1)$$

diff Both sides w.r.t. s.

We get.

$$F'(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} it\ e^{ist} \cdot f(t) \cdot dt$$

$$= \frac{1}{\sqrt{2\pi}}\, i \int_{-\infty}^{\infty} e^{ist} \cdot t \cdot f(t) \cdot dt$$

Multiplying both sides by $(-i)$.

$$(-i) \cdot F'(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{ist} \cdot t \cdot f(t) \cdot dt.$$

$$= \mathcal{F}[t \cdot f(t)]$$

$(-i)F''(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} it^2 e^{ist} f(t)\,dt$

$\Rightarrow (-i)^2 F''(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} \cdot t^2 f(t)\,dt$

$= \mathcal{F}[t^2 f(t)]$.

Diff eqn (1) sucssesively w.r.t. s.

we get

$$\mathcal{F}[t^n \cdot f(t)] = (-i)^n \cdot F^{(n)}(s).$$

Ex. Find the fourier transform $t \cdot e^{t - \frac{t^2}{2}}$

$\Rightarrow$ let $f(t) = t \cdot e^{t - \frac{t^2}{2}}$

We have. $\mathcal{F}[e^{\frac{t^2}{2}}] = e^{-\frac{s^2}{2}} = F_1(s)$. say.

$$\mathcal{F}[t \cdot e^{\frac{t^2}{2}}] = (-i)^1 \cdot \frac{d}{ds} F_1(s).$$

$$= (-i) \cdot [-s \cdot e^{\frac{-s^2}{2}}] = is \cdot e^{\frac{-s^2}{2}} = F(s).$$

We know that if

$\mathcal{F}[f(t)] = F(s)$ then $\mathcal{F}[e^{iat} \cdot f(t)] = F(s+a)$.

$$\mathcal{F}[t \cdot e^{t - \frac{t^2}{2}}] = \mathcal{F}[e^{i(-i)t} \cdot t\, e^{\frac{-t^2}{2}}] = i(s-i) \cdot e^{\frac{-(s-i)^2}{2}}$$

$$= (is+1)\ e^{is} \cdot e^{\frac{1-s^2}{2}}.$$

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★ Convolution integeals of fouriour.

The convolution of two integeable function f and g is given by

$$(f*g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(t-u) \cdot du.$$

#. Theoeem - Fouriour convolution theoeem.

If F(s) and G(s) denote the fouriour teansform of the function f(t) and g(t) resp. then fouriour inveese of

$$\mathcal{F}^{-1}[F(s) \cdot G(s)] = (f*g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) \cdot g(t-u) \cdot du.$$

⇒ The fouriour teansfoem of the convolution f*g is given by

$$\mathcal{F}[(f*g)(t)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{ist} \cdot (f*g)(t) \cdot dt.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{ist} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(u) \cdot g(t-u) \cdot du \cdot dt.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ist} \cdot f(u) \cdot g(t-u) \cdot du\, dt.$$

By changing the oedee of integeation then making the change of vaeiables $t = u+x$ we get

$$\mathcal{F}[(f*g)(t)] = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{is(u+x)} \cdot f(u) \cdot g(x) \cdot dx\, du.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{isu} \cdot f(u)\, du \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{isx} \cdot g(x)\, dx.$$

$$= F(s) \cdot G(s)$$

$$\Rightarrow (f*g)(t) = \mathcal{F}^{-1}[F(s) \cdot G(s)]$$