The Laplace Transform.

Integral transform :-

Let $f(t)$ be a function of variable t define for $t \in (-\infty, \infty)$ and $K(p,t)$ be a function of two independent variable p and t ~~$k(p,t)$~~ then the integral

$$\int_{-\infty}^{\infty} K(p,t) \cdot f(t) \cdot dt = F(p)$$ is called as the integral transform of the function $f(t)$ which is a function of variable t it is denoted by.

$$I[f(t)] = \int_{-\infty}^{\infty} K(p,t) \cdot f(t) \cdot dt = F(p). \text{ (say)} \quad \text{--- (1)}$$

provided the integral is convergent. depending upon the kernal K. and the function $f(t)$, we have manly five types of integral transforms. namely

1] Laplace. Transform
2] fourier Transform
3] Z - transform.
4] Hankel. transform.
5] Melian transform

If the kernal. K is. defined. as. $K(p,t) = e^{-pt}$ and $f(t)$ is define for $t \geq 0$ then the integral. in (1) i.e.

$$\int_{0}^{\infty} e^{-pt} \cdot f(t) \cdot dt$$

Is called as the laplace transform of $f(t)$ denoted. by $L[f(t)]$ or $F[p]$ provided the integral exist

$$\Rightarrow L[f(t)] = \int_{0}^{\infty} e^{-pt} \cdot f(t) \cdot dt.$$

Laplace transform of some elimentary functions

1] $f(t) = e^{at} = \frac{1}{p-a}$

$\Rightarrow$ let $f(t) = e^{at}$

By defination.

$$L[e^{at}] = \int_0^\infty e^{-pt} \cdot e^{at} \cdot dt$$

$$= \int_0^\infty e^{-t(p-a)} \cdot dt$$

$$= \left[\frac{e^{-t(p-a)}}{p-a} \right]_0^\infty$$

$$= 0 + \frac{e^0}{p-a} \qquad p > a.$$

$$L[e^{at}] = \frac{1}{p-a}$$

2] $f(t) = 1$.

$$L[f(t)] = \int_0^\infty e^{-pt} \cdot f(t) \cdot dt$$

$$= \int_0^\infty e^{-pt} \cdot 1 \cdot dt$$

$$= \left[\frac{e^{-pt}}{-p} \right]_0^\infty$$

$$= 0 - \frac{1}{-p}$$

$$\Rightarrow L[1] = 1/p.$$

3] $f(t) = \cos at$

$\rightarrow$ let $f(t) = \cos at$

$$L\{f(t)\} = L[\cos at] = \int_0^{\infty} e^{-pt} \cdot \cos at \cdot dt$$

Recall. $\int e^{ax} \cdot \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cdot \cos bx + b \cdot \sin bx)$.

$\int e^{ax} \cdot \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \cdot \sin bx - b \cdot \cos bx)$.

$$\Rightarrow \int_0^{\infty} e^{-pt} \cdot \cos at \, dt = \left[\frac{e^{-pt}}{p^2+a^2}(-p \cdot \cos at + a \cdot \sin at)\right]_0^{\infty}$$

$$= 0 - \left[\frac{e^0}{p^2+a^2}(-p)\right]$$

$$\Rightarrow L[\cos at] = \frac{p}{p^2+a^2}$$

4] $f(t) = \sin at$

$$L[\sin at] = \int_0^{\infty} e^{-pt} \cdot \underset{\sin at}{\cos at} \, dt$$

$$= \left[\frac{e^{-pt}}{p^2+a^2}(p \sin at - a \cos at)\right]_0^{\infty}$$

$$= 0 - \left[\frac{e^0}{p^2+a^2}(0-a)\right]$$

$$= \frac{a}{p^2+a^2}$$

$$\Rightarrow L[\sin at] = \frac{a}{p^2+a^2}$$

6. $f(t) = t^n$ $\qquad n > -1.$

$$\Rightarrow L[f(t)] = L[t^n] = \int_0^\infty e^{-pt} \cdot t^n \cdot f(t).$$

put $pt = x$

$pdt = dx.$

$$\Rightarrow L[t^n] = \int_0^\infty \left(\frac{x}{p}\right)^n \cdot e^{-x} \cdot \frac{1}{p} dx.$$

$$= \frac{1}{p^{n+1}} \int_0^\infty x \cdot e^{-x} dx$$

$$= \frac{\Gamma(n+1)}{p^{n+1}}, \quad Re(p) > 0, \ \& \ n > -1.$$

In particular if n is a +ve integer.

$$L[t^n] = n!/p^{n+1}.$$

7. $f(t) = \cosh at$

$$\Rightarrow L[\cosh at] = \int_0^\infty e^{-pt} \cdot \cosh at \cdot dt.$$

$$= \int_0^\infty e^{-pt} \left(\frac{e^{at} + e^{-at}}{2} \right) \cdot dt.$$

$$= \frac{1}{2} \int_0^\infty e^{-pt} \cdot (e^{at} + e^{-at}) \cdot dt.$$

$$= \frac{1}{2} \left\{ \int_0^\infty e^{-t(p-a)} dt + \int_0^\infty e^{-t(p+a)} \cdot dt \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{-t(p-a)}}{-(p-a)} + \frac{e^{-t(p+a)}}{-(p+a)} \right\}_0^\infty$$

$$L[\cosh at] = \frac{1}{2}\left\{(0+0) + \frac{1}{p-a} + \frac{1}{p+a}\right\}$$

$$= \frac{1}{2}\,\frac{p+a+p-a}{p^2-a^2}$$

$$\Rightarrow L[\cosh at] = \frac{p}{p^2-a^2}.$$

8. $f(t) = \sinh at$

$$L[\sinh at] = L\left[\frac{e^{at} - e^{-at}}{2}\right]$$

$$= \left\{L[e^{at}] - L[e^{-at}]\right\}\frac{1}{2}.$$

$$= \frac{1}{2}\left\{\frac{1}{p-a} - \frac{1}{p+a}\right\}$$

$$= \frac{1}{2}\left[\frac{p+a-p+a}{p^2-a^2}\right]$$

$$= \frac{1}{2}\cdot\frac{2a}{p^2-a^2}.$$

$$= \frac{a}{p^2-a^2}.$$

$$\Rightarrow L[\sinh at] = \frac{a}{p^2-a^2}.$$

OR.

$$L[\sinh at] = \int_0^\infty e^{-pt}\cdot\sinh at\cdot dt = \int_0^\infty e^{-pt}\left(\frac{e^{at} - e^{-at}}{2}\right)\cdot dt$$

$$= \frac{a}{p^2-a^2}.$$

Evaluate the laplace transform of the following by using defination.

1] $f(t) = t^2$.

$\Rightarrow$ $L[f(t)] = L[t^2] = \int_0^\infty e^{-pt} \cdot t^2 \cdot dt$.

$$= \left[t^2 \cdot \frac{e^{-pt}}{-p} - 2t \cdot \frac{e^{-pt}}{p^2} + 2\frac{e^{-pt}}{-p^3} \right]_0^\infty$$

$$= \left[0 - 0 + \frac{2}{p^3} \right]$$

$$L[t^2] = \frac{2}{p^3}.$$

Note.:- $\int u.v.dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ----$

2] $f(t) = t \cdot e^{2t}$.

$\Rightarrow$ $L[f(t)] = L[t \cdot e^{2t}] = \int_0^\infty e^{-pt} \cdot t \cdot e^{2t} \cdot dt$

$$= \int_0^\infty t \cdot e^{-t(p-2)} \cdot dt.$$

$$= \int_0^\infty t \cdot e^{-t(p-2)} \cdot dt.$$

$$= \left[t \cdot \frac{e^{-t(p-2)}}{p-2} - 1 \cdot \frac{e^{-t(p-2)}}{(p-2)^2} \right]_0^\infty$$

$$= \left[0 + \frac{1}{(p-2)^2} \right]$$

$$L[t \cdot e^{2t}] = 1/(p-2)^2.$$

3] $f(t) = t \cdot \sin kt = \frac{2pk}{(p^2+k^2)^2}$.

$\Rightarrow \quad L\{f(t)\} = L[t \cdot \sin kt] \quad = \quad L\{\sin kt\} = \frac{k}{p^2+k^2}$

$$\therefore L\{t \cdot \sin kt\} = \frac{2pk}{(p^2+k^2)^2}$$

\# $f(t) = \cos^2 kt$

$\Rightarrow \quad L\{f(t)\} = L[\cos^2 kt]$

$$= L\left[\tfrac{1}{2}(1+\cos 2kt)\right] \qquad \left[\because \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)\right]$$

$$= L\left[\tfrac{1}{2}\right] + \tfrac{1}{2} \cdot L[\cos 2kt]$$

$$= \frac{1}{2p} + \frac{1}{2} \cdot \frac{p}{p^2+4k^2}$$

$$= \frac{1}{2}\left(\frac{1}{p} + \frac{p}{p^2+4k^2}\right)$$

<u>Note</u> \# $f(t) = h(t-a)$, $a>0$ where $h(t-a)$ is called as Heaviside's unit step function defined as

$$h(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$$

Q. $L[f(t)] = L[\sin at \cdot \sin bt]$

$\Rightarrow \quad L[\sin at \cdot \sin bt] = L\left[\tfrac{1}{2}\cdot\cos(a-b)t - \tfrac{1}{2}\cos(a+b)t\right]$

$$= \frac{1}{2}\int_0^\infty e^{-pt}\cos(a-b)t\,dt - \frac{1}{2}\int_0^\infty e^{-pt}\cos(a+b)t \cdot dt$$

$$= \frac{1}{2}\left[\frac{e^{-pt}}{p^2+(a-b)^2}\{-p\cdot\cos(a-b)t + (a-b)\sin(a-b)t\}\right]_0^\infty$$

$$- \frac{1}{2}\cdot\left\{\frac{e^{-pt}}{p^2+(a+b)^2}(-p\cdot\sin(a+b)t + (a+b)\cdot\cos(a+b)t)\right\}_0^\infty$$

$$= \frac{1}{2}\left[0 - \frac{(-p)}{p^2+(a-b)^2}\right] - \frac{1}{2}\left[0 - \frac{(-p)}{p^2+(a+b)^2}\right]$$

$$= \frac{1}{2}\left[\frac{p}{(p^2+a^2+b^2)-2ab}\right] - \frac{1}{2}\left[\frac{p}{p^2+a^2+b^2+2ab}\right]$$

$$= \frac{1}{2} \cdot \frac{p(p^2+a^2+b^2)+2ab \cdot p - p(p^2+a^2+b^2)+2abp}{(p^2+a^2+b^2)^2 - (2ab)^2}$$

$$= \frac{4abp}{2\left[(p^2+a^2+b^2)^2 - (2ab)^2\right]}$$

$$\Rightarrow L[\sin at \cdot \sin bt] = \frac{2abp}{(p^2+a^2+b^2)^2 - (2ab)^2}$$

2] $f(t) = e^{at} \cdot \cosh kt$

$\Rightarrow L[f(t)] = L[e^{at} \cdot \cosh kt]$

$$\Rightarrow L[\cosh kt] = \frac{p}{p^2-k^2}$$

By first shifting property replace $p = (p-a)$

$$\Rightarrow L[e^{at} \cdot \cosh kt] = \frac{p-a}{(p-a)^2 - k^2}$$

Basic operational properties of Laplace transform

Th^m. 1 Linearity property.

If $F(P)$ and $G(P)$ are the laplace transforms resp. of $f(t)$ and $g(t)$ then for any constant c_1 and c_2

$$L[c_1 \cdot f(t) + c_2 \cdot g(t)] = c_1 F(P) + c_2 \cdot G(P)$$

$\Rightarrow$ By defination of Laplace transform.

$$L[c_1 f(t) + c_2 g(t)] = \int_0^\infty e^{-pt} [c_1 f(t) + c_2 \cdot g(t)] \, dt$$

$$= \int_0^\infty e^{-pt} \cdot c_1 \cdot f(t) \cdot dt + \int_0^\infty e^{-pt} \cdot c_2 g(t) \, dt$$

$$= c_1 \int_0^\infty e^{-pt} \cdot f(t) \cdot dt + c_2 \int_0^\infty e^{-pt} \cdot g(t) \cdot dt$$

$$= c_1 [F[p]] + c_2 \cdot G[P]$$

$$\Rightarrow L[c_1 \cdot f(t) + c_2 \cdot g(t)] = c_1 \cdot F[P] + c_2 \cdot G[P]$$

Th^m-2. First translation or shifting theorem.

If $F(P)$ is the laplace transform of $f(t)$ then.

$$L[e^{at} \cdot f(t)] = F(P-a).$$

$\Rightarrow$ By defination of laplace transform.

$$L[e^{at} \cdot f(t)] = \int_0^\infty e^{-pt} \cdot e^{at} \cdot f(t) \cdot dt$$

$$= \int_0^\infty e^{-t(P-a)} \cdot f(t) \cdot dt$$

$$= F(P-a).$$

$$\Rightarrow L[e^{at} \cdot f(t)] = F(P-a).$$

Th^m-3. Second shifting theorem

If $F(p)$ is the Laplace transform of $f(t)$, then $L[f(t-a) \cdot h(t-a)] = e^{-ap} \cdot F(p)$, $a > 0$

$\Rightarrow$ Heviside's unit step function $h(t-a)$ is defined as,

$$h(t-a) = \begin{cases} 0 & , t < a \\ 1 & t \geq a \end{cases}$$

$$L[f(t-a) \cdot h(t-a)] = \int_0^\infty e^{-pt} \cdot [f(t-a) \cdot h(t-a) \cdot dt$$

$$= \int_0^a f(t-a) \cdot 0 \cdot dt + \int_a^\infty e^{-pt} \cdot f(t-a) \cdot 1 \cdot dt$$

where $(t-a) = x \Rightarrow dt = dx$

$$= \int_0^\infty e^{-p(a+x)} \cdot f(x) \cdot dx$$

$$= e^{-pa} \cdot \int_0^\infty e^{-px} \cdot f(x) \cdot dx$$

$$= e^{-pa} \cdot \int_0^\infty e^{-pt} \cdot f(t) \cdot dx$$

$$= e^{-pa} L[f(t)]$$

$$\Rightarrow L[f(t-a) \cdot h(t-a)] = e^{-pa} \cdot F(p)$$

Note. This property can also be stated as follows

If $L[f(t)] = F(p)$, then $L[G(t)] = e^{-ap} \cdot F(p)$.

Where $G(t) = \begin{cases} f(t-a) & , t > a \\ 0 & , t < a \end{cases}$

Change of scale of Property

If $L[f(t)] = F(p)$ then $L[f(at)] = \frac{1}{a} F(p/a)$.

$\Rightarrow$ By definition

$$L[f(t)] = F(p) = \int_0^\infty e^{-pt} \cdot f(t) \cdot dt$$

$$L[f(at)] = \int_0^\infty e^{-pt} f(at) \cdot dt$$

Where $at = u$.

$a \cdot dt = du$. $\Rightarrow dt = \frac{du}{a}$

$$= \int_0^\infty e^{-\frac{pu}{a}} \cdot f(u) \cdot \frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\frac{p u}{a}} \cdot f(u) \cdot du$$

$$= \frac{1}{a} \cdot F\left[\frac{p}{a}\right]$$

$$L[f(at)] = \frac{1}{a} \cdot F\left[\frac{p}{a}\right]$$

Th^m-5 Laplace transform of derivative.

If $f, f', f'' \text{------} f^{(n-1)}$ are all cts. function on $t \geq 0$, $f^{(n)}$ is piecewise cts. on $t \geq 0$ and if all are of exponential order c_0, then for $n = 1, 2 \cdots$

$$L[f^{(n)}(t)] = p^{n} F(p) - p^{(n-1)} f(0) - p^{(n-2)} \cdot f'(0) - \cdots f^{(n-1)}(0)$$

$\Rightarrow$ By definition of Laplace transform.

$$L[f'(t)] = \int_0^\infty e^{-pt} \cdot f'(t) \cdot dt$$

$$= \left[e^{-pt} \cdot f(t) \right]_0^\infty - \int_0^\infty -p \cdot e^{-pt} \cdot f(t) \cdot dt$$

$$= [0 - f(0)] + p \cdot \int_0^\infty e^{-pt} \cdot f(t) \cdot dt$$

$$= p \cdot F(p) - f(0).$$

$L[f'(t)] = p.F(p) - f(0).$

Thus the statement is true for $n=1$.

since f and f' are cts on $t \geq 0$ f'' is piecewise cts on $t \geq 0$ and all are of exponential order c_0,

Applying the statement A for $f^{(n)}$.

We get,

$$L[f''(t)] = p.L[f'(t).] - f'(0).$$

$$= p.[p.F(p) - f(0)]$$

$$= p^2.F(p) - pf(0) - f'(0) \quad \text{--- (B)}$$

with repited the applications of statements A and B We get.

$$L[f^{(n)}(t)] = p^n.F(p) - p^{n-1}f(0) - p^{n-2}f'(0) ---- -f^{(n-1)}(0).$$

Note :- If f is discontinuous at $t=0$ then eq^n A becomes

$$L[f'(t)] = p.F(p). - f(0^-)$$

Def^n Piecewise/ sectionally / continuous function :-

" A function $f(t)$ is said to be piecewise/ sectionally cts function on a closed interval. $0 \leq t \leq b$ if it is define on that the interval can be subdivided into a finite number of subintervals in each of which $f(t)$ is cts and has finite right and left hand limits

2) Function of Exponential order:-

A piecewise cts. function f is said to be of exponential order so if there exist a real number c_0 s.t.

$$\lim_{t\to\infty} |f(t)|\, e^{-ct} = \begin{cases} 0 & c > c_0 \\ \text{no limit} & c < c_0 \end{cases}$$ At the point $c = c_0$

the above limit may or may not exist.

- A function of exponential order is denoted by $O(e^{ct})$

Ex. 1) $f(t) = e^{at}$ is of exponential order a.

since $\lim_{t\to\infty} e^{at} \cdot e^{-ct}$

$= \lim_{t\to\infty} e^{-t(c-a)} = 0$ If $c > a$

⇒ All bunded function $f(t)$ are of exponentional order $\underline{0}$.

Note. Note that an unbdd. function can also be an exponential order Ex: t^n

Ex The function $f(t) = e^{t^2}$ is not of exponential order

⇒ since $\lim_{t\to\infty} e^{t^2} \cdot e^{-ct} = \lim_{t\to\infty} e^{t^2 - ct} = \infty$

Function of class A :-

"A function which is piecewise cts on every finite inteeval in the eange $t \geq 0$ is of exponential order is known as a function of class A"

Existance theorem of laplace transform.

If f is piecewise cts. on $t \geq 0$ and is $o(e^{c_0 t})$ then f has the laplace transform $F(p)$ in the half plane $Re(p) > c_0$ moreover the laplace transform of integral converges both absolutely and uniformly for $Re(p) \geq c_2 > c_0$

⇒ To prove that laplace transform $F(p)$ of the function $f(t)$ exists

i.e. we have to prove that laplace transform integral

$$F(p) = \int_0^\infty e^{-pt} \cdot f(t) \cdot dt.$$

$$|F(p)| = \left| \int_0^\infty e^{-pt} \cdot f(t) \cdot dt \right|$$

$$\leq \int_0^\infty |e^{-pt} \cdot f(t) \cdot dt|.$$

$$\leq \int_0^\infty e^{-ct} |f(t)| \cdot dt \quad \text{---- } ①.$$

($\because c = \text{Real-}P$)

since $f(t)$ is exponential order c_0 and f is the piecewise cts function on $t \geq 0$.

for $\epsilon > 0$ we have to find a number c_1 with $c_0 < c_1 < c$ s.t.

$|f(t)| \cdot e^{-c_1 t} < \epsilon$ --- * for some $t > t_0$

from eq^n ①

$$|F(p)|. = \int_0^{t_0} e^{-ct} |f(t)| \cdot dt + \int_{t_0}^\infty e^{-ct} \cdot |f(t)| \cdot dt.$$

here the first integral in finite limit exist since f is piecewise cts function.
Also second integral satisfied.

$$\int_{t_0}^{\infty} e^{-ct} |f(t)|\, dt = \int_{t_0}^{\infty} e^{-ct}\, e^{-c_1 t}\, e^{c_1 t} |f(t)| \cdot dt.$$

$$= \int_{0}^{\infty} e^{-(c-c_1)t} \cdot e^{-c_1 t} |f(t)| \cdot dt.$$

$$< \epsilon \int_{0}^{\infty} e^{-(c-c_1)t} \cdot dt.$$

which is exist for $|c > c_1|$.

$$\therefore |F(p)| = \left| \int_{0}^{\infty} e^{-pt} f(t) \cdot dt \right| < \infty$$

i.e. $|F(p)| < \infty$

This prove that laplace transform integral convergeoves absolutely in the half plane $Re(p) > c_0$

lly

We can show that the integral convergeous uniformally for $Re(p) \geq c_2 < c_0$.

Note: This only sufficent condition but not for existance of laplace transform neccsseary.

Ex) $L[t^{-1/2}] = 1/\sqrt{t}$ $\left[\because t^n = \frac{\sqrt{n+1}}{p^{n+1}} \ \forall n > -1\right]$

$$L[t^{-1/2}] = \forall \frac{\Gamma(-\frac{1}{2}+1)}{p^{-1/2+1}}$$

$$= \forall \frac{\Gamma 1/2}{p^{1/2}} \quad \left\{\because \Gamma 1/2 = \sqrt{\pi}\right.$$

$$= \frac{\sqrt{\pi}}{\sqrt{p}} = \sqrt{\frac{\pi}{p}}$$

Th^m - 7 Laplace transform of integration

If f is picewise cts on $t \geq 0$ and is $O(e^{at})$ then Laplace transform of

$$L\left[\int_0^t f(u)\cdot du\right] = \frac{F(p)}{p}$$ where $F(p)$ is the L.t of $f(t)$.

$\Rightarrow$ Define $g(t) = \int_0^t f(u)\cdot du$.

Then as f is picewise cts.

If follows that g is cts with

$g'(t) = f(t)$. and

$g(0) = 0$

$\therefore$ Hence, By theorem of L.t. of derivative.

$$L[f(t)] = L[g'(t)] = p.L[g(t)] - g(0)$$

$$F[p] = p.L\left[\int_0^t f(u)\cdot du\right] - 0$$

$$\Rightarrow F[p] = p.L.\left[\int_0^t f(u)\, du\right]$$

$$\text{i.e. } L\left[\int_0^t f(u)\cdot du\right] = \frac{F(p)}{p} \quad p > 0.$$

Note From Theorem 5 and 7 we observed that the processes of diff and integration of a function in the t domain are transformed into the algebric operation of multiplication & division in the p- domain this is an important features. of Laplace transform

Th^m-8 Laplace transform of multiplication by powers of t

If f is picewise cts on $t \geq 0$ and is $o(e^{ct})$, then laplace of

$$L[t^n \cdot f(t)] = (-1)^n \cdot F^{(n)}(p) \quad . n = 1, 2 \ldots$$

Where $F(p)$ is L.T. of $f(t)$.

⇒ we will prove the theorem by mathematical induction.

By defination of Laplace transform.

$$F(p) = \int_0^\infty e^{-pt} \cdot f(t) \cdot dt.$$

Diff w.r.t p

$$F^{(1)}(p) = \frac{d}{dp} \int_0^\infty e^{-pt} \cdot f(t) \cdot dt.$$

$$= \int_0^\infty \frac{\partial}{\partial p} (e^{-pt} \cdot f(t) \cdot dt \quad [\because \text{ by leibintz rule}]$$

$$= \int_0^\infty -t \cdot e^{-pt} \cdot f(t) \cdot dt.$$

$$= (-1) \cdot \int_0^\infty e^{-pt} \cdot t \cdot f(t) \cdot dt.$$

$$F'(p) \cdot (-1) = L[t \cdot f(t)]$$

Thus the statement is true for $n = 1$.

Assume that the statement is true for $n = m$ so

Then $L[t^m \cdot f(t)] = (-1)^m \cdot F^{(m)}(p)$

Diff w.r.t. p.

$$\frac{d}{dp} \left[\int_0^\infty e^{-pt} \cdot t^m \cdot f(t) \right. = (-1)^m \cdot F^{(m+1)}(p)$$

$$\int_0^\infty \frac{\partial}{\partial p}\left[e^{-pt} \cdot t^m f(t)\right] dt = (-1)^m F^{(m+1)}(p)$$

$$\int_0^\infty -t\, e^{-pt} \cdot t^m f(t)\, dt = (-1)^m \cdot F^{(m+1)}(p)$$

$$(-1) \cdot \int_0^\infty e^{-pt} \cdot t^{m+1} f(t)\, dt = (-1)^m \cdot F^{(m+1)}(p)$$

$$L\left[t^{m+1} \cdot f(t)\right] = (-1)^{m+1} \cdot F^{(m+1)}(p).$$

Thus the statement is true for. $n = m+1$

Hence by principal of mathematical induction it is true for $n = 1, 2, 3.$

Th^m. 9. Laplace transform of division by t.

If f is picewise cts on $t \geq 0$ and is $O(e^{at})$ and. Laplace transform of $\frac{f(t)}{t}$ is.

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(u) \cdot du.$$

$\Rightarrow$ By defination of L.T.

$$F(p) = \int_0^\infty e^{-pt} \cdot f(t)\, dt$$

By replacing p by u and integrating we get

$$\int_p^\infty F(u) \cdot du = \int_p^\infty \left[\int_0^\infty e^{-ut} \cdot f(t) \cdot dt\right] du$$

$$= \int_0^\infty \left[\int_p^\infty e^{-ut} \cdot du\right] f(t)\, dt$$

$$= \int_0^\infty e^{-pt} \cdot \frac{f(t)}{t} \cdot dt.$$

K. satish

$$\left[\because \int_p^\infty e^{-ut}\,du = \left[\frac{e^{-ut}}{-t}\right] = 0 + \frac{e^{-pt}}{t}\right]$$

$$\therefore \int_0^\infty F(u)\cdot du = \int_0^\infty e^{-pt}\cdot \frac{f(t)}{t}\cdot dt$$

$$\Rightarrow L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(u)\cdot du$$

Th^n-10 Laplace transform of periodic Function

let f be piecewise cts on $t \geq 0$ and of $o(e^{at})$

If f is periodic with period T then

$$L[f(t)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt}\cdot f(t)\cdot dt.$$

$\Rightarrow$ By definition of Laplace transform.

$$L[f(t)] = \int_0^\infty e^{-pt}\cdot f(t)\cdot dt.$$

$$= \int_0^T e^{-pt}\cdot f(t)\cdot dt + \int_T^\infty e^{-pt}\cdot f(t)\cdot dt.$$

$$= \int_0^T e^{-pt}\cdot f(t)\cdot dt + \int_0^\infty e^{-p(T+u)}\cdot f(T+u)\cdot du.$$

where $t = T+u$.

$$= \int_0^T e^{-pt}\cdot f(t)\cdot dt + e^{-pT}\cdot \int_0^\infty e^{-pu}\cdot f(u)\cdot du.$$

$$L[f(t)] = \int_0^T e^{-pt}\cdot f(t)\,dt + e^{-pT}\, L[f(t)]$$

$$L\{f(t)\} - e^{-pT} \cdot L\{f(t)\} = \int_0^T e^{-pt} \cdot f(t) \cdot dt$$

$$L\{f(t)\}\left[1 - e^{-pT}\right] = \int_0^T e^{-pt} \cdot f(t) \cdot dt$$

$$\Rightarrow L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} \cdot f(t) \cdot dt.$$

Ex. Find the Laplace transform of the half wave rectified sinusoidal. $f(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$

where $f[(t)+2\pi] = f(t)$.

$\Rightarrow$ Given,

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

since f is periodic with period $T = 2\pi$.

$$L\{f(t)\} = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} \cdot f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2\pi p}} \int_0^{2\pi} e^{-pt} \cdot f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2\pi p}} \left\{ \int_0^{\pi} e^{-pt} \cdot \sin t \, dt + \int_{\pi}^{2\pi} 0 \cdot dt \right\}$$

$$= \frac{1}{1-e^{-2\pi p}} \left\{ \frac{e^{-pt}}{p^2+1} \left(-p \sin t - 1 \cdot \cos t\right) \right\}_0^{\pi}$$

$$= \frac{1}{1-e^{-2\pi p}} \left\{ \frac{e^{-\pi p}}{p^2+1} (1) - \frac{1}{p^2+1} (-1) \right]$$

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$$= \frac{1}{1-e^{-2\pi p}} \left\{ \frac{1+e^{-\pi p}}{p^2+1} \right\}.$$

$$\Rightarrow L[f(t)] = \frac{1+e^{-\pi p}}{(1-e^{-2\pi p})(p^2+1)} = \frac{1}{(1-e^{-\pi p})(p^2+1)}$$

Q. Evaluate the laplace transform of the following functions.

1) $f(t) = e^{4t}.\cosh 5t$.

$\Rightarrow$ $L[f(t)] = L[e^{4t}.\cosh 5t]$

We have.

$$L[\cosh 5t] = \frac{p}{p^2-25} = F[p]$$

By first shifting property.

$$L[e^{at}.\cosh 5t] = F(p-a).$$

$$= \frac{p-4}{(p-4)^2-25}$$

$$= \frac{p-4}{p^2-8p+16-25}$$

$$= \frac{p-4}{p^2-8p-9}$$

2) $f(t) = e^{-t}(3\cos 6t - 5\sin 6t)$.

$\Rightarrow$ Given $f(t) = e^{-t}(3\cos 6t - 5\sin 6t)$.

$$L[3\cos 6t - 5\sin 6t] = \frac{3p}{p^2+36} - \frac{5 \cdot 6}{p^2+36}.$$

$$= \frac{3p-30}{p^2+36}.$$

By using first shifting property

$$L[e^{-at} \cdot f(t)] = F(p-a).$$

$$L[e^{-t}(3\cos 6t - 5\sin 6t)] = \frac{3(p+1) - 30}{(p+1)^2 + 36}$$

$$= \frac{3p-27}{p^2+2p+37}.$$

3]. $f(t) = 3 \cdot t^4 \cdot e^{5t}$

$\Rightarrow$ $L(3 \cdot t^4) = 3 \cdot L[t^4]$

$$= \frac{3 \cdot 4!}{p^5}.$$

$$= \frac{72}{p^5} = F(p).$$

By using first shifting property.

$$L[3t^4 e^{5t}] = F(p-5) = \frac{72}{(p-5)^5}$$

Note: 1] $\frac{d^n}{dx^n}\left[\frac{1}{ax+b}\right] = \frac{(-1)^n \cdot n! \, a^n}{(ax+b)^{n+1}}.$

2] $2\cos^2\theta = 1 + \cos 2\theta$

3. $2\sin^2\theta = 1 - \cancel{\sin 2\theta}$ $\cos 2\theta$

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4] $f(t) = 2e^{-2t} \cdot \cos^2 3t$

$\Rightarrow$ $f(t) = 2e^{-2t} \cos^2 3t$

we have

$$L[2\cos^2 3t] = L[1 + \cos 6t]$$

$$= \frac{1}{p} + \frac{p}{p^2+36}$$

By using first shifting property.

$$L[e^{-2t} \cdot 2\cos^2 3t] = F(p+2)$$

$$= \frac{1}{p+2} + \frac{p+2}{(p+2)^2+36}$$

$$= \frac{1}{p+2} + \frac{p+2}{p^2+4p+40}$$

$$= \frac{p^2+4p+40+p^2+4p+4}{(p+2)(p^2+4p+40)}$$

$$= \frac{p^2+8p+44}{(p+2)(p^2+4p+40)}$$

Ex. $f(t) = t^2 \cdot \sin kt$

$\Rightarrow$ $f(t) = t^2 \cdot \sin kt$

we have

$$L[\sin kt] = \frac{k}{p^2+k^2}$$

By multiplication of power's of t.

$$\therefore L[t^n \cdot f(t)] = (-1)^n \cdot \frac{d^n}{dp^n} [F(p)]$$

$$\therefore L[t^2 \cdot f(t)] = (-1)^2 \cdot \frac{d^2}{dp^2} \left[\frac{k}{p^2+k^2}\right]$$

$$= \frac{d}{dp} \left[\frac{-2pk}{(p^2+k^2)^2}\right]$$

$$= \frac{(p^2+k^2)^2(-2k) + 2pk \cdot 2(p^2+k^2)2p}{(p^2+k^2)^4}$$

$$= \frac{-2k(p^2+k^2)^2 + 8p^2k \cdot (p^2+k^2)}{(p^2+k^2)^4}$$

$$= \frac{-2kp^2 - 2k^3 + 8p^2 k}{(p^2+k^2)^3}$$

$$= \frac{6p^2k - 2k^3}{(p^2+k^2)^3}$$

Ex. $f(t) = 5t \cdot e^{3t} \cdot \sin^2 t$

→ Given.

$L[5t \cdot e^{3t} \sin^2 t]$

Now,

$L[5t \sin^2 t]$

$$L[\sin^2 t] = L\left[\frac{1}{2} - \frac{\cos 2t}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right] = F(p)$$

$L[5t \sin^2 t] = 5\, L[t \cdot \sin^2 t$

$$= 5 \cdot (-1) \cdot \frac{d}{dp} \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+4}\right]$$

$$= -5\left[\frac{-1}{2p^2} - \frac{1}{2} \frac{p^2+4 - p \cdot 2p}{(p^2+4)^2}\right]$$

$$= -5\left[\frac{-1}{2p^2} - \frac{1}{2} \frac{4-p^2}{(p^2+4)^2}\right]$$

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$$= \frac{5}{2}\left[\frac{1}{p^2} + \frac{4-p^2}{(p^2+4)^2}\right] = F_1(p) \text{ say.}$$

$\therefore L[5t.\, e^{3t}.\sin 2t] = F_1(p-3)$

By first shifting property.

$$= \frac{5}{2}\left[\frac{1}{(p-3)^2} + \frac{4-(p-3)^2}{[(p-3)^2+4]^2}\right]$$

$$= \frac{5}{2}\left[\frac{1}{p^2-6p+9} + \frac{4-p^2+6p-9}{(p^2-6p+9+4)^2}\right]$$

$$= \frac{5}{2}\left[\frac{1}{p^2-6p+9} + \frac{6p-p^2-5}{(p^2-6p+13)^2}\right]$$

Ex. $f(t) = t.\, e^{-2t}.\cos t$.

$\Rightarrow$ $L[t.\cos t] = L[\cos t] = \frac{p}{p^2+1} = F(p)$ say

By multiplication of t

$$L[t.\cos 2t] = (-1)^1.\frac{d}{dp}\left[\frac{p}{p^2+1}\right].$$

$$= -\left[\frac{p^2+1-2p^2}{(p^2+1)^2}\right]$$

$$= \frac{p^2-1}{(p^2+1)^2} = F_1(p)$$

By. first shifting property.

$$L[t.\, e^{-2t}\cos t] = F_1(p+2) = \frac{p^2+4p+3}{(p^2+4p+5)^2}.$$

Ex. $f(t) = \frac{\sinh t}{t}$.

$\Rightarrow \quad L\left[\frac{\sinh t}{t}\right] = L[\sinh t] = \frac{1}{p^2-1} = F(u)$

By using division of t.

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(u)\cdot du$$

$$L\left[\frac{\sinh t}{t}\right] = \int_p^\infty \frac{1}{u^2-1}\cdot du \quad --- (*)$$

$$\frac{A}{u-1} + \frac{B}{u+1} = \frac{A(u+1)+B(u-1)}{(u-1)(u+1)}$$

$$1 = A(u+1) + B(u-1).$$

put $u=1 \Rightarrow A = 1/2$

$u=-1 \Rightarrow B = -1/2$

$\therefore$ eq^n $*$ becomes.

$$\int_p^\infty \left[\frac{1/2}{(u-1)} + \frac{(-1/2)}{(u+1)}\right]\cdot du$$

$$= \frac{1}{2}\int_p^\infty \left[\frac{1}{u-1} - \frac{1}{u+1}\right]\cdot du$$

$$= \frac{1}{2}\left[\log(u-1) - \log(u+1)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{u-1}{u+1}\right)\right]_p^\infty$$

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$$= \frac{1}{2}\left\{\log 1 - \log \frac{p-1}{p+1}\right]$$

$$= \frac{1}{2}\cdot \log \frac{p+1}{p-1}$$

Ex. $f(t) = \dfrac{e^{-at} - e^{-bt}}{t}$

$\Rightarrow$ $L\{e^{-at} - e^{-bt}\} = \left[\dfrac{1}{p+a} - \dfrac{1}{p+b}\right] = F(a)$

By using division of t.

$$L\left[\frac{f(t)}{t}\right] = \int_p^\infty F(u)\cdot du$$

$$L\left[\frac{e^{-at} - e^{-bt}}{t}\right] = \int_p^\infty \left[\frac{1}{u+a} - \frac{1}{u+b}\right] du$$

$$= \left[\log(u+a) - \log(u+b)\right]_p^\infty$$

$$= \left[\log\left(\frac{u+a}{u+b}\right)\right]_p^\infty$$

$$= 0 - \log\left(\frac{p+a}{p+b}\right)$$

$$= \log\left(\frac{p+b}{p+a}\right).$$

Ex. $f(t) = \dfrac{\sin t}{t}$

$\rightarrow$ $L\{\sin t\} = \dfrac{1}{p^2+1}$

$$\therefore L\left[\frac{\sin t}{t}\right] = \int_p^\infty \frac{1}{u^2+1}\,du$$

$$= \left[\tan^{-1}\left(\frac{u}{1}\right)\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(p)$$

$$= \cot^{-1}(p) = \tan^{-1}\left(\frac{1}{p}\right)$$

Ex. $f(t) = \dfrac{e^{3t}\cdot \sin 5t}{t}$

$$\Rightarrow L\{\sin 5t\} = \frac{5}{p^2+25} = F(u).$$

$\therefore$ By using multiplication of $\frac{1}{t}$

$$L\left[\frac{\sin 5t}{t}\right] = \int_p^\infty \frac{5}{u^2+25}\,du$$

$$= \left[\tan^{-1}(u/5)\right]_p^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{p}{5}\right) = F_1(p) \text{ say}$$

$$\therefore L\left[\frac{e^{3t}\cdot \sin 5t}{t}\right] = F_1(p-3)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{p-3}{5}\right)$$

$$= \cot^{-1}\left(\frac{p-3}{5}\right)$$

$$= \tan^{-1}\left(\frac{5}{p-3}\right)$$

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Ex $f(t) = \begin{cases} 2 & 0 < t < 1 \\ t & t > 1 \end{cases}$

$$\Rightarrow L\{f(t)\} = \int_0^\infty e^{-pt} f(t)\cdot dt$$

$$= \int_0^1 e^{-pt} 2\,dt + \int_1^\infty e^{-pt}\, t\cdot dt$$

$$= 2\left[\frac{e^{-pt}}{-p}\right]_0^1 + \left[t\cdot\frac{e^{-pt}}{-p} - \frac{e^{-pt}}{p^2}\right]_1^\infty$$

$$= \frac{2}{-p}\left[e^{-p}-1\right] + \left[0 - \left[\frac{1\cdot e^{-p}}{-p} - \frac{e^{-p}}{p^2}\right]\right]$$

$$= -\frac{2}{p}\left(e^{-p}-1\right) + \frac{e^{-p}}{p} + \frac{e^{-p}}{p^2}$$

$$= -\frac{2}{p}\left[e^{-p}-1\right] + \frac{e^{-p}}{p} + \frac{e^{-p}}{p^2}$$

$$\Rightarrow e^{-p}\left\{-\frac{2}{p} + \frac{1}{p} + \frac{1}{p^2}\right\} + \frac{2}{p}$$

Ex $f(t) = \begin{cases} t^2 & 0 < t < 3 \\ e^{-t} & t > 0 \end{cases}$

$$\Rightarrow L\{f(t)\} = \int_0^3 e^{-pt}\cdot t^2\cdot dt + \int_3^\infty e^{-t}\cdot e^{-pt}\cdot dt$$

$$= \left[t^2\cdot\frac{e^{-pt}}{-p} - 2t\frac{e^{-pt}}{p^2} + \frac{2\cdot e^{-pt}}{p^3}\right]_0^3 + \int_3^\infty e^{-t(p+1)}\cdot dt$$

$$= \left[t^2 \frac{e^{-pt}}{-p} - 2t \cdot \frac{e^{-pt}}{p^2} - \frac{2e^{-pt}}{p^3} \right]_0^3 + \left[\frac{e^{-t(p+1)}}{-(p+1)} \right]_3^\infty$$

$$= \left[3^2 \frac{e^{-3p}}{-p} - 2\times 3 \frac{e^{-3p}}{p^2} - \frac{2e^{-3p}}{p^3} + \frac{2}{p^3} \right] + \left[0 + \frac{e^{-3(p+1)}}{p+1} \right]$$

$$= -e^{-3p} \left(\frac{9}{p} + \frac{6}{p^2} + \frac{2}{p^3} \right) + \frac{2}{p^3} + \frac{e^{-3p} e^{-3}}{p+1}$$

$$= \frac{2}{p^3} - e^{-3p} \left[\frac{9}{p} + \frac{6}{p^2} + \frac{2}{p^3} + \frac{e^{-3}}{p+1} \right]$$

Ex. $f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0. & t > \pi \end{cases}$

$$\Rightarrow \quad L[f(t)] = \int_0^\pi e^{-pt} \sin t \, dt + \int_\pi^\infty e^{pt} \cdot 0 \cdot dt$$

$$= \left[\frac{e^{-pt}}{p^2+1} \left(-p \sin t - \cos t \right) \right]_0^\pi$$

$$= \frac{1}{p^2+1} \left[e^{-\pi p} (0+1) - (1)\cdot(-1) \right]$$

$$= \frac{1}{p^2+1} \left[e^{-\pi p} + 1 \right]$$

4] $f(t) = \begin{cases} \cos t & 0 < t < \pi \\ \sin t & t > \pi \end{cases}$

$$\Rightarrow \quad L[f(t)] = \int_0^\pi e^{-pt} \cos t \, dt + \int_\pi^\infty e^{-pt} \sin t \, dt$$

$$= \left[\frac{e^{-pt}}{p^2+1} (-p \cos t - \sin t) \right]_0^\pi + \left[\frac{e^{-pt}}{p^2+1} (-p \sin t - \cos t) \right]_\pi^\infty$$

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$$= \left[\frac{e^{-p\pi}}{p^2+1}(p) - \frac{1}{p^2+1}(\cdot p)\right] + \left[0 - \frac{e^{-p\pi}}{p^2+1}(0+1)\right]$$

$$= \frac{e^{-\pi p}}{p^2+1}(p) + \frac{p}{p^2+1} - \frac{e^{-p\pi}}{p^2+1}$$

$$= \left(\frac{p}{p^2+1} - \frac{1}{p^2+1}\right) e^{-\pi p} + \frac{p}{p^2+1}$$

$$= \frac{e^{-\pi p}}{p^2+1}(p-1) + \frac{p}{p^2+1}.$$

Ex. If $F(P)$ denotes the laplace transform of $f(t)$. show that laplace transform of $e^{-bt/a} \cdot f(t/a) = a \cdot F(ap+b)$ $a>0$

$\therefore$ $F(P) = L[f(t)]$ To prove that

$$L\left[e^{-\frac{bt}{a}} \cdot f(t/a)\right] = a \cdot F(ap+b).$$

$\Rightarrow$ As $L[f(t)] = F(P)$

By change of scale property

$$L[f(t/a)] = a \cdot F(ap) = F_1(P) \text{ say}$$

By first shifting property.

$$L\left[e^{-\frac{bt}{a}} \cdot f(t/a)\right] = F_1\left[p + \frac{b}{a}\right]$$

$$= a \cdot [F(a(p+b/a))$$

$$= a \cdot F\left[\left(\frac{ap+b}{a}\right) \cdot a\right.$$

$$= a \cdot F(ap+b)$$

$$\Rightarrow L\left[e^{-\frac{b}{a}t} f[t/a]\right] = a \cdot F[ap+b]$$

Ex. Given that laplace transform of $L\left[\frac{\sin t}{t}\right] = \tan^{-1}\frac{1}{p}$ find $L\left[\frac{\sin at}{t}\right]$ where $a>0$.

$\Rightarrow$ $L\left[\frac{\sin t}{t}\right] = \tan^{-1}(1/p) = F(p)$ (say)

$$\therefore L\left[\frac{\sin at}{at}\cdot a\right] = a\cdot L\left[\frac{\sin at}{at}\right]$$

$$= a\cdot\frac{1}{a}\cdot F\left[\frac{p}{a}\right]$$

$$= \tan^{-1}\left(\frac{1}{p/a}\right)$$

$$\Rightarrow L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{p}\right)$$

Ex. Find laplace transform of $f(t) = \int_0^t (u^2-u-e^{-u})\cdot du$

$\Rightarrow$ Given $f(t) = \int_0^t (u^2-u-e^{-u})\cdot du$.

$$\Rightarrow f_1(t) = t^2-t+e^{-t}$$

$$\therefore L[f_1(t)] = L[t^2-t+e^{-t}]$$

$$= \frac{2!}{p^3} - \frac{1!}{p^2} + \frac{1}{p+1} = F(p) \text{ say}$$

$$\therefore L[f(t)] = L\left[\int_0^t (u^2-u-e^{-u})du\right] = \frac{F(p)}{p}$$

$$= \frac{2}{p^4} - \frac{1}{p^3} + \frac{1}{p(p+1)}$$

Ex. find laplace transform of $f(t)$ is $\int_0^t e^{4u}\cosh 5u\cdot du$

$\Rightarrow$ Given $f(t) = \int_0^t (e^{4u}\cosh 5u)\, du$

$$\Rightarrow f_1(t) = e^{4t}\cosh 5t$$

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$$L[\cosh 5t] = \frac{p}{p^2-25}$$

$$L[e^{4t}\cdot \cosh 5t] = \frac{p-4}{(p-4)^2-25}$$

$$= \frac{p-4}{p^2-8p-9} = F(p)$$

$$\therefore L\left[\int_0^t (e^{4u}\cdot \cosh 5u)\cdot du\right] = \frac{1}{p}\cdot F(p)$$

$$= \frac{p-4}{p(p^2-8p-9)}.$$

Ex. If $L[f(t)] = \frac{1}{p}e^{-1/p}$ find the $L[e^{-2t}\cdot f(3t)]$

$\Rightarrow$ Given $L[f(t)] = \frac{1}{p}\cdot e^{-1/p} = F(p)$.

$$L[f(3t)] = \frac{1}{3}\cdot F\left(\frac{1}{3}p\right).$$

$$= \frac{1}{3}\cdot\frac{3}{p}\cdot e^{-3/p} = F_1(p).$$

By first shifting property.

$$\therefore L[e^{-2t}\cdot f(3t)] = F_1(p+2)$$

$$= \frac{1}{(p+2)}\cdot e^{-3/(p+2)}$$

Q) Find the laplace transform of sine integral. denoted by $S_i(t) = \int_0^t \frac{\sin u}{u} \, du$

$\Rightarrow$ We have to find.

$$L[S_i(t)] = L\left[\int_0^t \frac{\sin u}{u} \, du\right] \quad \text{---} \; ①$$

$$\Rightarrow L\left[\frac{\sin t}{t}\right] = \tan^{-1}\frac{1}{p}$$

$$L[\sin t] = \frac{1}{p^2+1}$$

$$\therefore L\left[\frac{\sin t}{t}\right] = \int_p^{\infty} \frac{1}{u^2+1} \, du$$

$$= [\tan^{-1} u]_p^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1}(p)$$

$$= \tan^{-1}(1/p) \quad F(p)$$

$$\therefore L[S_i(t)] = \frac{F(p)}{p} = \frac{1}{p} \cdot \tan^{-1}\left(\frac{1}{p}\right)$$

F(p) is the laplace transform of f(t) then prove that $L\left[\int_0^t \frac{f(u)}{u} \, du\right] = \frac{1}{p} \cdot \int_0^{\infty} F(s) \, ds$.

$\Rightarrow$ Given $L[f(t)] = F(P)$

$$\therefore L\left[\frac{f(t)}{t}\right] = \int_p^{\infty} F(s) \, ds = F_1(P)$$

$$\therefore L\left[\int_0^t \frac{f(u)}{u} \, du\right] = \frac{1}{p} \cdot F_1(P) = \frac{1}{2} \cdot \int_p^{\infty} F(s) \, ds$$

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Ex. Given that laplace of $L[t^{-1/2}] = \sqrt{\pi/p}$ show that $L\left\{\left[\frac{1+2at}{\sqrt{t}}\right]\cdot e^{at}\right\} = \ldots\ldots$

$\Rightarrow$ $$L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{p}}$$

By first shifting property

$$L\left[\frac{e^{at}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{p-a}}$$

$$\therefore L\left(\frac{t\,e^{at}}{\sqrt{t}}\right) = (-1)\frac{d}{p}\left(\sqrt{\frac{\pi}{p-a}}\right)$$

$$= (-1)\sqrt{\pi}\frac{d}{dp}\left((p-a)^{-1/2}\right)$$

$$= -\sqrt{\pi}\left[\frac{-1}{2}(p-a)^{-3/2}\right]$$

$$= \frac{\sqrt{\pi}}{2}\frac{1}{(p-a)^{3/2}}$$

$$= \sqrt{\frac{\pi}{(p-a)}}\cdot\frac{1}{2(p-a)}.$$

$$\therefore L\left[\left(\frac{1+2at}{\sqrt{t}}\right)e^{at}\right] = L\left[\frac{e^{at}}{\sqrt{t}}\right] + 2a\cdot L\left[\frac{t\cdot e^{at}}{\sqrt{t}}\right]$$

$$= \sqrt{\frac{\pi}{p-a}} + \frac{2a}{2(p-a)}\sqrt{\frac{\pi}{(p-a)}}$$

$$= \sqrt{\frac{\pi}{p-a}}\left(1+\frac{a}{p-a}\right)$$

$$\Rightarrow \sqrt{\frac{\pi}{p-a}}\left(\frac{p}{p-a}\right).$$

Ex. Find the Laplace transform of the following periodic functions.

1] $f(t) = \begin{cases} 1 & 0<t<c. \\ -1 & c<t<2c. \end{cases}$ where $f(t+2c) = f(t)$

$\Rightarrow$ $$L[f(t)] = \frac{1}{1-e^{-pT}} \int_0^T e^{-pt} \cdot f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2pc}} \int_0^{2c} f(t) \cdot dt$$

$$= \frac{1}{1-e^{-2pc}} \left[\int_0^c e^{-pt} \cdot f(t) \cdot dt + \int_c^{2c} e^{-pt} \cdot f(t) \cdot dt \right]$$

$$= \frac{1}{1-e^{-2pc}} \left\{ \left[\frac{e^{-pt}}{-p} \right]_0^c - \left[\frac{e^{-pt}}{-p} \right]_c^{2c} \right\}$$

$$= \frac{1}{1-e^{-2pc}} \left\{ \left[\frac{e^{-pc}}{-p} + \frac{1}{p} \right] - \left[\frac{e^{-2pc}}{-p} + \frac{e^{-pc}}{p} \right] \right.$$

$$= \frac{1}{1-e^{-2pc}} \left[-\frac{2e^{-pc}}{p} + \frac{e^{-2pc}}{p} + \frac{1}{p} \right]$$

2] $f(t) = \begin{cases} 3t & 0<t<2. \\ 6. & 2<t<4 \end{cases}$ $f(t+4) = f(t)$

$\Rightarrow$ $$L\{f(t)\} = \frac{1}{1-e^{-4p}} \left[\int_0^2 e^{-pt} \cdot 3t \, dt + \int_2^4 e^{-pt} \cdot 6 \cdot dt \right]$$

$$= \frac{1}{1-e^{-4p}} \left\{ \left[3t \frac{e^{-pt}}{-p} - \frac{3e^{-pt}}{p^2} \right]_0^2 + \left[\frac{6 \cdot e^{-pt}}{-p} \right]_2^4 \right\}$$

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$$= \frac{1}{1-e^{-4p}}\left\{ -6\cdot\frac{e^{-2p}}{p} - 3\frac{e^{-2p}}{p} + \frac{3}{p^2} - \frac{6\cdot e^{-4p}}{p} + 6\frac{e^{-2p}}{p} \right\}$$

$$= \frac{1}{1-e^{-4p}}\left(\frac{3}{p^2} - 9\frac{e^{-2p}}{p} - 6\frac{e^{-4p}}{p} \right)$$

Inverse Laplace Transform.

If the laplace transform of a function $f(t)$ is $F(p)$ then we define the function $f(t)$ as the inverse laplace transform of the function $F(p)$ and defined by.

$$L^{-1}[F(p)] = f(t).$$

Theorem -

If f is a piecewise continuous function for $t \geq 0$ and is $o(e^{\alpha t})$ and $F(p)$ is the the laplace transform of $f(t)$ then

$\lim\limits_{|p| \to \infty} F(p) = 0$ From this theorem it follows that If $\lim\limits_{|p| \to \infty} F(p) \neq 0$, then $F(p)$ is not laplace transform of any piecewise continuous function of exponential order.

Thus, The function's like polynomial in p, e^p, $\sin p$, $\cos p$ etc are not the laplace transform of any transform.

If $f(t)$ and $g(t)$ are two functions which are identical except for a finite number of points. then their integral's are same and hence, they have the same laplace transform say $F(p)$.

Thus we conclude that The inverse laplace transform of $F(p)$ is either $f(t)$ or $g(t)$

The relation between the functions f and g is given by the following theorem.

Lerche's Theorem -

If $f(t)$ and $g(t)$ have the same laplace transform $F(p)$ then $f(t) - g(t) = N(t)$.

Where $N(t)$ is called as the null function defined as $\int_0^t N(t)\,dt = 0 \;\forall\; t > 0$. , Thus we conclude that, the inverse laplace transform of a function is unique upto the addition of a null functions.

Note – 1) If $\lim_{|p|\to\infty} F(p) \neq 0$ then $F(p)$ is not L.T. of $f(t)$.

2) Laplace transform is unique but L^{-1} is not unique.

The Inverse Laplace transform of some standard functions.

$F(p)$		$f(t) = L^{-1}[F(p)]$
1) $\frac{1}{p}$		1
2) $\frac{1}{p^n}$	n – +ve integer	$\frac{t^{n-1}}{(n-1)!}$
3) $\frac{1}{p^{n+1}}$	$n > -1$.	$\frac{t^n}{\Gamma(n+1)}$
4) $\frac{1}{p-a}$		e^{at}
5) $\frac{p}{p^2+a^2}$		$\cos at$
6) $\frac{1}{p^2+a^2}$		$\frac{1}{a}\sin at$
7) $\frac{p}{p^2-a^2}$		$\cosh at$
8) $\frac{1}{p^2-a^2}$.		$\frac{1}{a}\sinh at$

Properties of Inverse laplace transform

1) Linearity Property –

If $F_1(p)$ and $F_2(p)$ are laplace transform of function's $f_1(t)$ and $f_2(t)$ resp and c_1 and c_2 are any constant then

$$L^{-1}[c_1 F_1(p) + c_2 F_2(p)] = c_1 L^{-1}[F(p)] + c_2 \cdot L^{-1}[F(p)]$$
$$= c_1 \cdot f_1(t) + c_2 f_2(t)$$

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2) First shifting theorem -

If $L^{-1}[F(p)] = f(t)$ then

$$L^{-1}[F(p-a)] = e^{at} \cdot L^{-1}[F(p)]$$
$$= e^{at} \cdot f(t).$$

3) Second shifting theorem -

If $L^{-1}[F(p)] = f(t)$ then

$$L^{-1}[e^{-ap} F(p)] = G(t).$$

where $G(t) = \begin{cases} f(t-a) & , t>a \\ 0 & t<a \end{cases}$

i.e. $G(t) = f(t-a) \cdot h(t-a)$

4) change of scale property -

If $L^{-1}[F(p)] = f(t)$ Then

$$L^{-1}[F(ap)] = \frac{1}{a} \cdot f(t/a).$$

5) Inverse laplace transform of derivative

If $L^{-1}[F(p)] = f(t)$. then

$$L^{-1}\left[\frac{d^n}{dp^n} F(p)\right] = (-1)^n \cdot t^n L^{-1}[F(p)]$$
$$= (-1)^n \cdot t^n \cdot f(t).$$

6) Inverse laplace transform of integral.

If $L^{-1}[F(p)] = f(t)$. Then

$$L^{-1}\left[\int_p^\infty F(u)\,du\right] = \frac{1}{t} \cdot L^{-1}[F(p)] = \frac{f(t)}{t}$$

7) Inverse laplace transform of multiplication by power of p

If $L^{-1}[F(p)] = f(t)$ and $f(0) = 0$ then.

$$L^{-1}[p \cdot F(p)] = f'(t).$$

8) Inverse laplace transform of division of powers of p.

If $f(t)$ is piecewise continuous and of exponential order such that $\lim_{t\to 0} \frac{f(t)}{t}$ exist's then $L^{-1}\left[\frac{F(p)}{p}\right] = \int_0^t f(u)\,du.$

Q. Obtain the inverse laplace transform of following functions

1) $F(P) = \dfrac{1}{2P+3}$

$\Rightarrow L^{-1}[F(P)] = L^{-1}\left[\dfrac{1}{2P+3}\right] = \dfrac{1}{2} L^{-1}\left[\dfrac{1}{P+\frac{3}{2}}\right]$

$= \dfrac{1}{2} e^{-\frac{3}{2}t}.$

2) $F(P) = \dfrac{P-5}{P^2+6P+13}$

$\Rightarrow L^{-1}\left[\dfrac{P-5}{P^2+6P+13}\right] = L^{-1}\left[\dfrac{P-5+3-3}{P^2+6P+9-9+13}\right]$

$= L^{-1}\left[\dfrac{(P+3)-8}{(P+3)^2+4}\right]$

$= \left\{L^{-1}\left[\dfrac{P-8}{P^2+4}\right]\right\} e^{-3t}$

$= e^{-3t}\left\{L^{-1}\left[\dfrac{P}{P^2+4}\right] - 8 \cdot L^{-1}\left[\dfrac{1}{P^2+4}\right]\right.$

$= e^{-3t}\left(\cos 2t - 8 \cdot \dfrac{\sin 2t}{2}\right)$

$= e^{-3t}(\cos 2t - 4\sin 2t).$

3) $F(P) = \dfrac{3P+7}{P^2+5}$

$L^{-1}\left[\dfrac{3P+7}{P^2+5}\right] = 3 \cdot L^{-1}\left[\dfrac{P}{P^2+5}\right] + 7\, L^{-1}\left[\dfrac{1}{P^2+5}\right]$

$= 3 \cdot \cos\sqrt{5}t + \dfrac{7}{\sqrt{5}} \cdot \sin\sqrt{5}\, t.$

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4) $\dfrac{2p}{(p-3)^5}$

$\Rightarrow \quad L^{-1}\left[\dfrac{2p}{(p-3)^5}\right] = L^{-1}\left[\dfrac{2(p-3)+6)}{(p-3)^5}\right]$

$= e^{3t} L^{-1}\left[\dfrac{2p+6}{p^5}\right]$

$= e^{3t} L^{-1}\left\{\dfrac{2}{p^4} + \dfrac{6}{p^5}\right\}$

$= e^{3t}\left[\dfrac{2t^3}{3!} + \dfrac{6t^4}{4!}\right]$

$= e^{3t}\left[\dfrac{t^3}{3} + \dfrac{t^4}{4}\right]$

5) $\dfrac{p}{p^2-6p+13}$.

$\Rightarrow \quad L^{-1}\left[\dfrac{p}{p^2-6p+13}\right] = L^{-1}\left[\dfrac{(p-3)+3}{(p-3)^2+4}\right]$

$= e^{3t} \cdot L^{-1}\left[\dfrac{p+3}{p^2+4}\right]$

$= e^{3t}(\cos 2t + 3/2 \sin 2t)$

6) $L^{-1}\left[\dfrac{5p-2}{3p^2+4p+8}\right]$

$\Rightarrow \quad L^{-1}\left[\dfrac{5p-2}{3p^2+4p+8}\right] = \dfrac{1}{3} L^{-1}\left[\dfrac{5p-2}{p^2+\frac{4p}{3}+\frac{8}{3}}\right]$

$= \dfrac{1}{3} L^{-1}\left[\dfrac{5(p+\frac{2}{3}) - 2 - 10/3}{(p+\frac{2}{3})^2 + \frac{8}{3} - 4/9}\right]$

$= \dfrac{1}{3} e^{-2/3 t} L^{-1}\left[\dfrac{5p - 16/3}{p^2 + 20/9}\right]$

$= \dfrac{1}{3} e^{-\frac{2}{3}t}\left\{ L^{-1}\left[\dfrac{5p}{p^2+20/9}\right] - \dfrac{16}{3} L^{-1}\left[\dfrac{1}{p^2+20/9}\right]\right\}$

$= \dfrac{1}{3} e^{-2/3 t}\left[5\cos\left(\dfrac{2\sqrt{5}}{3}t\right) - \dfrac{16}{3} \cdot \dfrac{3}{2\sqrt{5}} \sin\left(\dfrac{2\sqrt{5}}{3}t\right)\right]$

7) $\dfrac{2p+3}{4p^2+4p+5}$

$$\Rightarrow L^{-1}\left[\frac{2p+3}{4p^2+4p+5}\right] = L^{-1}\left[\frac{2p+3}{p^2+p+\frac{5}{4}}\right]\frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{2(p+1/2)-1+3}{(p+\frac{1}{2})^2-\frac{1}{4}+\frac{5}{4}}\right]$$

$$= \frac{1}{4}e^{-\frac{1}{2}t}\,L^{-1}\left[\frac{2p+2}{p^2+1}\right]$$

$$= \frac{1}{4}e^{-\frac{1}{2}t}(2\cos t+2\sin t)$$

$$= \frac{1}{2}e^{-\frac{1}{2}t}(\cos t+\sin t)$$

8) $\dfrac{p^2}{(p+2)^4}$

$$\Rightarrow L^{-1}\left[\frac{p^2}{(p+2)^4}\right] \Rightarrow L^{-1}\left[\frac{[(p+2)-2]^2}{(p+2)^4}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{(p-2)^2}{p^4}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{p^2-4p+4}{p^4}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{1}{p^2}-\frac{4}{p^3}+\frac{4}{p^4}\right]$$

$$= e^{-2t}\left[t-4\frac{t^2}{2!}+4\frac{t^3}{3!}\right]$$

$$= e^{-2t}\left(t-2t^2+\frac{2}{3}t^3\right)$$

9) $\dfrac{1}{p^2-6p+10}$

$$\Rightarrow L^{-1}\left[\frac{1}{p^2-6p+10}\right] = L^{-1}\left[\frac{1}{p^2-6p+9+1}\right]$$

$$= L^{-1}\left[\frac{1}{(p-3)^2+1}\right] = e^{3t}L^{-1}\left[\frac{1}{p^2+1}\right]$$

$$= e^{3t}\sin t$$

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10) $\dfrac{e^{-5p}}{(p-3)^4}$

$$\Rightarrow L^{-1}\left[\frac{1}{(p-3)^4}\right] = e^{3t}\left[\frac{1}{p^4}\right]$$

$$= e^{3t}\cdot\frac{t^3}{3!} = b(t).\ \text{say.}$$

$$L^{-1}\left[\frac{e^{-5p}}{(p-3)^4}\right] = e^{3(t-5)}\ \frac{(t-5)^3}{3!}$$

$$= e^{3(t-5)}\ \frac{(t-5)^3}{6}$$

11). $F(p) = \dfrac{(p+1)\,e^{-\pi p}}{p^2+p+1}$

$$\Rightarrow L^{-1}\left[\frac{p+1}{p^2+p+1}\right] = L^{-1}\left[\frac{(p+1/2)+1/2}{(p+1/2)^2+3/4}\right]$$

$$= e^{-t/2}\ L^{-1}\left[\frac{p+1/2}{p^2+3/4}\right]$$

$$= e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right)$$

$$= e^{-t/2}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right) = b(t)\ \text{say}$$

$$\therefore L^{-1}\left[e^{-\pi p}F(p)\right] = b(t-\pi).\ h(t-\pi)$$

$$= e^{-\frac{(t-\pi)}{2}}\left[\cos\frac{\sqrt{3}}{2}(t-\pi) + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}(t-\pi)\right]h(t-\pi).$$

Where $h(t)$ is heaviside unit step function.

12) $\dfrac{1}{p(p+1)}$

$$\Rightarrow L^{-1}\left[\frac{1}{p(p+1)}\right]$$

Here $\dfrac{1}{p(p+1)} = \dfrac{A}{p} + \dfrac{B}{p+1} = \dfrac{A(p+1)+B(p)}{p(p+1)}$

$$\Rightarrow 1 = A(p+1) + B(p).$$

$$1 = A(P+1) + B(P)$$

put $P=-1 \Rightarrow B=-1$.

put $P=0 \Rightarrow A=1$.

$$\therefore L^{-1}\left[\frac{1}{P(P+1)}\right] = L^{-1}\left[\frac{1}{P} - \frac{1}{P+1}\right]$$

$$= L^{-1}\left[\frac{1}{P}\right] - L^{-1}\left[\frac{1}{P+1}\right]$$

$$= 1 - e^{-t}.$$

13) $\dfrac{1}{(P-1)(P+2)(P+4)}$

$\Rightarrow$ Here $\dfrac{1}{(P-1)(P+2)(P+4)} = \dfrac{A}{(P-1)} + \dfrac{B}{(P+2)} + \dfrac{C}{(P+4)}$.

$$1 = A(P+2)(P+4) + B(P-1)(P+4) + C(P-1)(P+2)$$

put $P=-2 \Rightarrow 1 = (-3)(2)B$.

$\Rightarrow B = -\dfrac{1}{6}$.

put $P=1 \Rightarrow 1 = 3 \cdot 5 A$.

$\Rightarrow A = \dfrac{1}{15}$

& $P=-4 \Rightarrow 1 = (-5)(-2)C$

$\Rightarrow C = \dfrac{1}{10}$

$$L^{-1}\left[\frac{1}{(P-1)(P+2)(P+4)}\right] = L^{-1}\left[\frac{1}{15(P-1)} - \frac{1}{6(P+2)} + \frac{1}{10(P+4)}\right]$$

$$= \frac{1}{15}e^{t} - \frac{1}{6}e^{-2t} + \frac{1}{10}e^{-4t}.$$

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14) $\dfrac{3p-2}{p^3(p^2+4)}$

Here, $\dfrac{3p-2}{p^3(p^2+4)} = \dfrac{A}{p} + \dfrac{B}{p^2} + \dfrac{C}{p^3} + \dfrac{Dp+E}{p^2+4}$

$$3p-2 = p^2(p^2+4)A + Bp(p^2+4) + C.(p^2+4) + (Dp+E)p^3.$$
$$= p^4(A+D) + p^3(B+E). + p^2(4A+C) + p(4B) + 4C$$

comparing the coefficient of like power's of p.

$$A+D=0 \quad , \quad B+E=0, \quad 4A+C=0 \ . \quad 4B=3 \quad 4C=-2.$$
$$\Rightarrow B=\frac{3}{4} \quad C=\frac{-1}{2}.$$

$$4A-\frac{1}{2}=0.$$

$$\Rightarrow 4A=\frac{1}{2} \Rightarrow A=\frac{1}{8}$$

$$B+E=0 \Rightarrow \frac{3}{4}+E=0 \Rightarrow E=-3/4$$

$$A+D=0 \Rightarrow D=-1/8.$$

$$L^{-1}\left[\frac{3p-2}{p^3(p^2+4)}\right] = L^{-1}\left[\frac{1}{8}\frac{1}{p} + \frac{3}{4}\frac{1}{p^2} - \frac{1}{2}\frac{1}{p^3}\right] - \frac{-3/4}{p^2+4}\Big]$$

$$= \frac{1}{8}.1 + \frac{3}{4}t - \frac{1}{2}\frac{t^2}{2} - \frac{1}{8}.\cos 2t - \frac{3}{4}\frac{1}{2}\sin 2t$$

$$\Rightarrow L^{-1}\left[\frac{3p-2}{p^3(p^2+4)}\right] = \frac{1}{8} + \frac{3t}{4} - \frac{t^2}{4} - \frac{1}{8}\cos 2t - \frac{3}{8}\sin 2t.$$

Assignment No. - 3.

Q. Find the inverse laplace transform of the following functions.

1) $\dfrac{2}{(p+1)(p^2+1)}$

2) $\dfrac{p+1}{(p^2-4p)(p+5)^2}$

3) $\dfrac{1}{p^4-1}$

4) $\dfrac{p}{(p^2+a^2)(p^2+b^2)}$

5) $\dfrac{4p^2-16}{p^3(p+2)^2}$

6) $\dfrac{p^2-3}{(p+2)(p-3)(p^2+2p+5)}$

7) $\dfrac{3p^2-6p+7}{(p^2-2p+5)^2}$

Q. Given that $L^{-1}[F(p)] = f(t)$, show that for the constant's a, b, and k that.

a) $L^{-1}[F(kp)] = \frac{1}{k} \cdot f(t/k) \quad k>0$

b) $L^{-1}[F(ap+b)] = \frac{1}{a} \cdot e^{-\frac{bt}{a}} f(t/a) \quad a>0$

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Thm. Convolution theorem.

If $f(t)$ and $g(t)$ are piece-wise continuous function's on $t \geq 0$ and are $o(e^{ct})$ and if $F(p)$ and $G(p)$ are the laplace transformations resp. of $f(t)$ and $g(t)$ then the inverse laplace transform of the product $F(p) \cdot G(p)$ is given by the formula.

$$L^{-1}[F(p) \cdot G(p)] = (f * g)(t)$$
$$= \int_0^t f(t-u)\, g(u) \cdot du.$$

Note-

The integral $\int_0^t f(t-u) \cdot g(u) \cdot du$ is called as the convolution / faulting of f and g denoted by $(f * g)(t)$

This convolution is commutative.

i.e.
$$(f*g)(t) = \int_0^t f(t-u)\, g(u) \cdot du$$
$$= \int_0^t f(u) \cdot g(t-u) \cdot du$$
$$= (g * f)(t).$$

Find the inverse laplace transform of the following functions.

1) $\dfrac{2p}{(p^2+1)^2}$.

$\rightarrow$ $F(p) = \dfrac{-1}{p^2+1}$. then

$L^{-1}[F(p)] = -\sin t = f(t)$ say.

Now, $\frac{2p}{(p^2+1)^2} = \frac{d}{dp}\left[\frac{-1}{p^2+1}\right]$

$= \frac{d}{dp}[F(p)]$

$= L^{-1}\left[\frac{d}{dp}[F(p)]\right]$

$= (-1)^1 \cdot t^1 \cdot f(t)$

$= t \cdot \sin t$

2) $\frac{1}{(p^2+1)^2}$

$\Rightarrow$ we have $L^{-1}\left[\frac{2p}{(p^2+1)^2}\right] = t \cdot \sin t$

$\Rightarrow L^{-1}\left[\frac{p}{(p^2+1)^2}\right] = \frac{1}{2} \cdot t \cdot \sin t = f(t)$ say.

i.e. $L^{-1}\left[\frac{1}{(p^2+1)^2}\right] = \int_0^t \frac{1}{2} \cdot u \cdot \sin u \, du$

$= \frac{1}{2}\left[u \cdot (-\cos u) - 1(-\sin u)\right]_0^t$

$= \frac{1}{2}\left[-t \cos t + \sin t\right]$

3) $\log\left(1 + \frac{1}{p^2}\right)$

$\Rightarrow$ let $F(p) = \log\left(1 + \frac{1}{p^2}\right)$

$= \log(1 + p^2) - 2 \log p$

$\therefore F'(p) = \frac{2p}{p^2+1} - \frac{2}{p}$

$\therefore L^{-1}[F'(p)] = 2 \cdot \cos t - 2$

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$$(-1)^1 \cdot t \cdot L^{-1}[F(P)] = 2\cos t - 2$$

$$L^{-1}[F(P)] = \frac{2(1-\cos t)}{t}$$

4) $\log(1 + 1/p)$

$\Rightarrow$ let

$$F(P) = \log\left(1 + \frac{1}{P}\right)$$

$$= \log(1+P) - \log P$$

$$F'(P) = \frac{1}{1+P} - \frac{1}{P}$$

$$\therefore L^{-1}[F'(P)] = e^{-t} - 1$$

$$(-1)^1 \cdot t \cdot L^{-1}[F(P)] = e^{-t} - 1$$

$$L^{-1}[F(P)] = \frac{1 - e^{-t}}{t}$$

5) $\log\left(\frac{p+1}{p-1}\right)$

$\Rightarrow$ let $F(P) = \log\left(\frac{p+1}{p-1}\right)$

$$= \log(p+1) - \log(p-1)$$

$$F'(P) = \frac{1}{p+1} - \frac{1}{p-1}$$

$$L^{-1}[F'(P)] = e^{-t} - e^{t}$$

$$(-1) \cdot t \cdot L^{-1}[F(P)] = e^{-t} - e^{t} \Rightarrow L^{-1}[F(P)] = \frac{e^{t} - e^{-t}}{t}$$

7). $\dfrac{1}{p^2(p^2+k^2)}$ $\Rightarrow L^{-1}\left[\dfrac{1}{p^2}\right] = t = f(t)$ --- (say)

$\Rightarrow L^{-1}\left[\dfrac{1}{p^2+k^2}\right] = \dfrac{1}{k}\cdot\sin kt = g(t)$ --- (say)

By convolution theoeem.

$$L^{-1}\left[\frac{1}{p^2(p^2+k^2)}\right] = \int_0^t f(t-u)\cdot g(u)\cdot du$$

$$= \int_0^t (t-u)\cdot\frac{1}{k}\cdot\sin ku\cdot du$$

$$= \frac{1}{k}\cdot\int_0^t (t-u)\cdot(\sin ku)\cdot du$$

$$= \frac{1}{k}\left\{\int_0^t t\cdot\sin ku\,du - \int_0^t u\cdot\sin ku\cdot du\right\}$$

$$= \frac{1}{k}\left\{\left[t\cdot\frac{(-\cos ku)}{k}\right]_0^t - \left[u\cdot\left(\frac{-\cos ku}{k}\right) - 1\cdot\left(-\frac{\sin ku}{k^2}\right)\right]_0^t\right\}$$

$$= \frac{1}{k}\left\{-\frac{t\cdot\cancel{\cos kt}}{\cancel{k}} + \frac{t}{k} + \frac{t\cdot\cancel{\cos kt}}{\cancel{k}} - \frac{\sin kt}{k^2} + 0\right\}$$

$$= \frac{1}{k}\left[\frac{t}{k} - \frac{\sin kt}{k^2}\right]$$

$$= \frac{1}{k^2}\left(t - \frac{\sin kt}{k}\right).$$

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7) $\dfrac{2}{(p+1)(p^2+1)}$

$\Rightarrow \quad L^{-1}\left[\dfrac{2}{p+1}\right] = 2 \cdot e^{-t}$ ---- $f(t)$ (say.)

and $L^{-1}\left[\dfrac{1}{p^2+1}\right] = \sin t$ ... $g(t)$ (say)

$\therefore$ By convolution theorem.

$$L^{-1}\left[\frac{2}{(p+1)(p^2+1)}\right] = \int_0^t f(t-u)\, g(u) \cdot du$$

$$= \int_0^t 2\, e^{-(t-u)} \cdot \sin u \cdot du$$

$$= 2\int_0^t e^{-(t-u)} \cdot \sin u \cdot du$$

$$= 2\int_0^t e^{-t} \cdot e^{u} \cdot \sin u \cdot du$$

$$= 2 e^{-t} \int_0^t e^{u} \sin u \cdot du$$

$$= 2 e^{-t}\left[\frac{e^{u}}{2}(\sin u - \cos u)\right]_0^t$$

$$= 2 e^{-t}\left[\frac{e^{t}}{2}(\sin t - \cos t) - \frac{1}{2}(0-1)\right]$$

$$= 2 e^{-t}\left[\frac{e^{t}}{2} \cdot (\sin t - \cos t) + \frac{1}{2}\right]$$

$$= \sin t - \cos t + e^{-t}$$

Note - 1) If f is piece-wise smooth and absolutely integrable on the entire real axis and if x is a point of continuity of f then

$$\lim_{\lambda \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t+x) \cdot \frac{\sin \lambda t}{t}\, dt = f(x).$$

2) $F(p)$ is $O(p^k)$ mean's then $\exists\ M > 0$ such that $|p^k \cdot F(p)| < M$. for $|p|$ sufficiently large.

★ Complex Inversion Formula.

If $F(P)$ is an analytic function of the complex variable p in the half plane. $Re(P) > c$. and further. $F(P)$ is $o(p^k)$.

Where k is real and $k > 1$ then the inversion integral.

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \cdot F(P) \cdot dp.$$

converge's to the real function. $f(t)$ which is independent of c and whose laplace transform is $F(P)$ for $Re(P) > c$.

Also the function $f(t)$ is $o(e^{c_0 t})$. and is continuous everywhere and $f(t) = 0$ when $t \leq 0$.

⇒ Suppose that. $f(t)$ and $f'(t)$ are continuous function and $f(t)$ is $o(e^{c_0 t})$ then the laplace transform of integral.

$$F(P) = \int_0^\infty e^{-pu} \cdot f(u) \cdot du$$ converge's absolutely and uniformally in the half plane in $Re(P) \geq c_2 \geq c_0$ and that $F(P)$ is analytic for

multiplying by e^{pt} and integrating with respect to p over the limit $(c - i\lambda)$ to $(c + i\lambda)$ we get

$$\int_{c-i\lambda}^{c+i\lambda} e^{pt} \cdot F(P) \cdot dp = \int_{c-i\lambda}^{c+i\lambda} e^{pt} \cdot \int_0^\infty e^{-pu} \cdot f(u) \cdot du \cdot dp$$

$$= \int_0^\infty f(u) \cdot \int_{c-i\lambda}^{c+i\lambda} e^{p(t-u)} \cdot dp \cdot du.$$

$$= \int_0^\infty f(u) \cdot \left\{ \frac{e^{p(t-u)}}{t-u} \right\}_{c-i\lambda}^{c+i\lambda} \cdot du.$$

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$$\Rightarrow \int_{c-i\lambda}^{c+i\lambda} e^{pt} \cdot F(p)\,dp = \int_0^\infty \frac{f(u)}{t-u} \left[e^{(c+i\lambda)(t-u)} - e^{(c-i\lambda)(t-u)} \right] du$$

$$= \int_0^\infty f(u) \cdot \frac{1}{t-u} \cdot e^{c\cdot(t-u)} \left[2i \sin\lambda(t-u) \right] \cdot du$$

$$= 2i \cdot \int_0^\infty f(u) \cdot e^{c\cdot(t-u)} \cdot \frac{\sin\lambda(t-u)}{t-u}\, du$$

$$= 2i \int_{-t}^\infty f(t+x) \cdot e^{-cx} \cdot \frac{\sin\lambda x}{x} \cdot dx$$

$$\frac{1}{2\pi i} \int_{c-i\lambda}^{c+i\lambda} e^{pt} \cdot F(p)\,dp = \frac{1}{\pi} e^{ct} \int_{-t}^\infty f(t+x) \cdot e^{-c(x+t)} \frac{\sin\lambda x}{x} \cdot dx$$

taking limit as $\lambda \to \infty$ we get.

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \cdot F(p) \cdot dp = e^{ct} \cdot f(t) \cdot e^{-ct}$$

$$= f(t) \qquad \text{where } t > 0.$$

$$\Rightarrow f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{pt} \cdot F(p)\,dp.$$

Heaviside's Expansion Theorem / Formula.

let $F(P)$ and $G(P)$ be two polynomial's in p where $F(P)$ has degree less than that of $G(P)$ if $G(P)$ has n distinct zero's α_r, $r = 1, 2, \ldots n$ i.e.

$G(P) = (p-\alpha_1)(p-\alpha_2) \cdots (p-\alpha_n)$ then

$$L^{-1}\left[\frac{F(P)}{G(P)}\right] = \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t}$$

$\Rightarrow$ Since, $F(P)$ is a polynomial of degree less than that of $G(P)$, and $G(P)$ has n distinct zero's α_r, $r = 1, 2 \cdots n$. we can write,

$$\frac{F(P)}{G(P)} = \frac{F(P)}{(P-\alpha_1)(P-\alpha_2)\cdots\cdots(P-\alpha_n)}$$

$$= \frac{A_1}{(P-\alpha_1)} + \frac{A_2}{(P-\alpha_2)} + \cdots\cdots + \frac{A_r}{(P-\alpha_r)} + \cdots + \frac{A_n}{(P-\alpha_n)} \quad \text{(A)}$$

multiplying by $(P-\alpha_r)$ and taking limit as $P-\alpha_r$. we get.

$$A_r = \lim_{P\to\alpha_r} \frac{F(P)}{G(P)}(P-\alpha_r).$$

$$= \lim_{P\to\alpha_r} \frac{F(P) + (P-\alpha_r)F'(P)}{G'(P)}. \quad \{\because \text{L' Hospital rule}$$

$$A_r = \frac{F(\alpha_r)}{G'(\alpha_r)}.$$

From (A)

$$\frac{F(P)}{G(P)} = \frac{F(\alpha_1)}{G'(\alpha_1)(P-\alpha_1)} + \frac{F(\alpha_2)}{G'(\alpha_2)(P-\alpha_2)} + \cdots + \frac{F(\alpha_r)}{G'(\alpha_r)(P-\alpha_r)} + \cdots + \frac{F(\alpha_n)}{G'(\alpha_n)(P-\alpha_n)}$$

taking the inverse laplace transform, we have.

$$L^{-1}\left[\frac{F(P)}{G(P)}\right] = \frac{F(\alpha_1)}{G'(\alpha_1)}e^{\alpha_1 t} + \frac{F(\alpha_2)}{G'(\alpha_2)}e^{\alpha_2 t} + \cdots\cdots + \frac{F(\alpha_r)}{G'(\alpha_r)}e^{\alpha_r t} + \cdots + \frac{F(\alpha_n)}{G'(\alpha_n)}e^{\alpha_n t}.$$

$$= \sum_{r=1}^{n} \frac{F(\alpha_r)}{G'(\alpha_r)} e^{\alpha_r t} \qquad \text{Where } r = 1, 2 \cdots n.$$

Note - 1) If f is piecewise continuous on $t>0$ and is $O(e^{at})$ and. $L[f(t)] = F(P)$ then.

$$\lim_{|P|\to\infty} F(P) = 0$$

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2) The behaviour of the function $f(t)$ in the nbhd of origin is reflected in the behaviour of it's laplace transform $F(p)$ as $|p| \to \infty$.

Ex. Let. $f(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$ then.

$$\lim_{t \to 0} f(t) = a_0$$

and $L[f(t)] = F(p) = \frac{a_0}{p} + \frac{a_1}{p^2} + \frac{a_2 2!}{p^3} + \dots \frac{a_n n!}{p^{n+1}}$

$$\lim_{|p| \to \infty} p \cdot F(p) = a_0 = \lim_{t \to 0} f(t)$$

Initial value Theorem.

If $f(t)$ is piecewise continuous on $t > 0$ and is $o(e^{\alpha t})$ then it's laplace transform $F(p)$ satisfie's $\lim_{|p| \to \infty} p \cdot F(p) = \lim_{t \to 0} f(t) = f(0)$

⇒ We will prove the theorem for a stronger case when $f(t)$ is continuous. for $t \geq 0$

We know that

$$L[f'(t)] = \int_0^\infty e^{-pt} f'(t)\, dt = p \cdot F(p) - f(0)$$

If $f'(t)$ is piecewise continuous and is of exponential order then it's laplace transform satisfie's the condition.

$$\lim_{|p| \to \infty} \int_0^\infty e^{-pt} f'(t) \cdot dt = 0 \qquad [\because \text{by above th note}]$$

i.e. $\lim_{|p| \to \infty} \{p \cdot F(p) - f(0)\} = 0$

i.e. $\lim_{|p| \to \infty} p \cdot F(p) = f(0) = \lim_{t \to 0} f(t)$.

Final value theorem-

If $f(t)$ is piecewise continuous on $t \geq 0$ and is $o(e^{\alpha t})$ and if $\int_0^\infty f'(t)\,dt$ exist, then the transform $F(p)$ satisfies

$$\lim_{p \to 0} p.F(p) = \lim_{t \to \infty} f(t) = f(\infty).$$

$\Rightarrow$ We will prove the theorem for the stronger case when $f(t)$ is continuous on $t \geq 0$

We know that,

$$L[f'(t)] = \int_0^\infty e^{-pt}.f'(t)\,dt$$

$$= p.F(p) - f(0).$$

taking limit as $p \to 0$.

$$\lim_{p \to 0} \int_0^\infty e^{-pt}.f'(t)\,dt = \lim_{p \to 0} [p.F(p) - f(0)]$$

$$\therefore \int_0^\infty f'(t)\,dt = \lim_{p \to 0} p.F(p) - f(0)$$

$$f(\infty) - f(0) = \lim_{p \to 0} p.F(p) - f(0)$$

$$\Rightarrow \lim_{p \to 0} p.F(p) = f(\infty) = \lim_{t \to \infty} f(t).$$

$$\Rightarrow \lim_{p \to 0} p.F(p) = \lim_{t \to \infty} f(t).$$

★ Example - Find the laplace transform of the co-sine integral $C_i(t)$ defined by $C_i(t) = \int_\infty^t \frac{\cos u}{u}\,du$, $t > 0$

$\Rightarrow$ let $f(t) = C_i(t) = \int_\infty^t \frac{\cos u}{u}\,du.$

$$= -\int_t^\infty \frac{\cos u}{u}\,du$$

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$$f'(t) = \frac{\cos t}{t}$$

taking laplace transform on both sides and on integrating we get

$$p.F(p) - f(0) = \int \frac{p}{p^2+1}.dp + K'$$

$$p.F(p) = \frac{1}{2}.\log(p^2+1) + K' + f(0).$$

$$= \frac{1}{2} \log(p^2+1) + K.$$

$$F(p) = \frac{1}{2p} \log(p^2+1) + \frac{K}{p}$$

By final value theorem

$$\lim_{p\to 0} p.F(p) = \lim_{t\to\infty} f(t) = 0$$

i.e. $\lim_{p\to 0}\left[\frac{1}{2}\log(p^2+1)+K\right] = 0 \Rightarrow K=0.$

Hence.

$$L[C_i(t)] = F(p) = \frac{1}{2p}.\log(p^2+1).$$

2) verify the initial value theorem for the following functions

1) $f(t) = 5 + 4\cos t$

$\Rightarrow$ Here. $\lim_{t\to 0} f(t) = \lim_{t\to 0}(5+4\cos 2t)$

$$= 5+4 = 9 \quad \text{-- (i)}$$

Now $F(p) = L\{f(t)\} = \frac{5}{p} + \frac{4p}{p^2+4}$

$$\lim_{|p|\to\infty} p.F(p) = \lim_{|p|\to\infty}\left[5 + \frac{4p^2}{p^2+4}\right]$$

$$\Rightarrow \lim_{|p|\to\infty} [p \cdot F(p)] = \lim_{|p|\to\infty}\left[5 + \frac{4}{1+\frac{4}{p^2}}\right]$$

$$= 5 + \frac{4}{1+0}$$

$$= 9 \quad \text{----} (2)$$

From eq^n (1) and (2) we get

$$\lim_{|p|\to\infty} p \cdot F(p) = \lim_{t\to 0} f(t)$$

Intial value theorem is verified.

2) $f(t) = (3t-2)^3$

$$\Rightarrow \lim_{t\to 0} f(t) = \lim_{t\to 0} (3t-2)^3$$

$$= (0-2)^3 = -8 \quad \text{---} (1)$$

Also.

$$L[f(t)] = L[27t^3 - 54t^2 + 36t - 8]$$

$$F(p) = \left[\frac{27 \cdot 3!}{p^4} + \frac{(-54) \cdot 2!}{p^3} + 36\frac{1}{p^2} - \frac{8}{p}\right]$$

$$\lim_{|p|\to\infty} p \cdot F(p) = \lim_{|p|\to\infty}\left[\frac{27 \cdot 3!}{p^3} - \frac{54 \cdot 2!}{p^2} + \frac{36}{p} - 8\right]$$

$$= 0 - 0 + 0 - 8 = -8 \quad \text{--} (2)$$

From eq^n (1) and (2)

$$\lim_{|p|\to\infty} p \cdot F(p) = \lim_{t\to 0} f(t)$$

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3) Verify the final value theorem for the function

$$f(t) = 3 + e^{-2t}(\cos t + \sin t)$$

$\Rightarrow$ $\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty} [3 + e^{-2t}(\cos t + \sin t)]$

$= 3+0 = 3 \ldots (1)$

$L[f(t)] = F(p) = \lim\limits_{p\to0} \left\{ \frac{3}{p} + \left[\frac{p+2}{(p+2)^2+1} + \frac{1}{(p+2)^2+1} \right] \right\}$

$\lim\limits_{p\to0} p \cdot F(p) = \lim\limits_{p\to0} \left[3 + \frac{p^2+2p+p}{(p+2)^2+1} \right]$

$= \lim\limits_{p\to0} \left[3 + \frac{p^2+2p+p}{p^2+4p+5} \right]$

$= 3+0 = 3 \ldots (2)$

from (1) and (2)

$\lim\limits_{p\to0} F(p) = \lim\limits_{t\to\infty} f(t)$

1) Find the inverse laplace transform of the following

1) $F(p) = \frac{1}{(p+1)(p-3)^2}$

2.) $F(p) = \frac{p}{(p+1)^3(p-1)^3}$.

3) $F(p) = \frac{p^2}{(p^2+4)^2}$.

4) $F(p) = \frac{p^2}{p^4 4}$

Application's Involving Laplace Inverse.

laplace transform can be convinently use to solve.

1) Integral's of the type's $\int_0^{\infty} f(t) \cdot dt$

2) Ordinary diff^n equation's with initial ~~val~~ conditions.

3) Partial diff^n equations.

4) Integral equation of convolation type.

Application to Evaluate integral's.

1) $\int_0^{\infty} \frac{\sin t}{t} \cdot dt$.

$\Rightarrow L\left[\frac{\sin t}{t}\right] \Rightarrow L[\sin t] = \frac{1}{p^2+1} = f(p)$

$\therefore L\left[\frac{\sin t}{t}\right] = \int_p^{\infty} f(u) \cdot du$

$= \int_p^{\infty} \frac{1}{u^2+1} \cdot du$

$= [\tan^{-1} u]_p^{\infty}$

$\int_0^{\infty} e^{-pt} \cdot \frac{\sin t}{t} \cdot dt = \frac{\pi}{2} - \tan^{-1} p$

put $p=0$ we get,

$\int_0^{\infty} \frac{\sin t}{t} \cdot dt = \pi/2$.

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2) $\int_0^\infty e^{-t} \cdot \frac{\sin t}{t} \, dt$

$\Rightarrow$ We have. $L\left[\frac{\sin t}{t}\right] = \frac{\pi}{2} - \tan^{-1} p$

i.e. $\int_0^\infty e^{-pt} \cdot \frac{\sin t}{t} \cdot dt = \frac{\pi}{2} - \tan^{-1} p$

put $p = 1$

$$\int_0^\infty e^{-t} \cdot \frac{\sin t}{t} \, dt = \frac{\pi}{2} - \tan^{-1}(1).$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

3) $\int_0^\infty t \cdot e^{-2t} \cos t \cdot dt$

$\Rightarrow$ We have,

$$L\{\cos t\} = \frac{p}{p^2+1}$$

$$L\{t \cdot \cos t\} = -\frac{d}{dp}\left[\frac{p}{p^2+1}\right]$$

$$= -\frac{(p^2+1) - 2p \cdot p}{(p^2+1)^2}$$

$$= \frac{-p^2 - 1 + 2p^2}{(p^2+1)^2}$$

i.e. $\int_0^\infty e^{-pt} \cdot t \cdot \cos t \, dt = \frac{p^2-1}{(p^2+1)^2}$

put $p = 2$. we get

$$\int_0^\infty t \cdot e^{-2t} \cos t \cdot dt = \frac{4-1}{(4+1)^2} = \frac{3}{25}$$

4) $\int_0^\infty t^2 \cdot e^{-3t} \cdot \sin t \cdot dt$

$\rightarrow$ We have. $L[\sin t] = \frac{1}{p^2+1}$

$$L[t^2 \cdot \sin t] = \frac{d^2}{dp^2}\left(\frac{1}{p^2+1}\right)$$

$$= \frac{d}{dp}\left[\frac{(p^2+1)0 - 1(2p)}{(p^2+1)^2}\right]$$

$$= \frac{d}{dp}\left[\frac{-2p}{(p^2+1)^2}\right]$$

$$= \left[\frac{(p^2+1)^2(-2) - (-2p)2(p^2+1)(2p)}{(p^2+1)^4}\right]$$

$$= \frac{(p^2+1)(-2) - [(-2p)(4p)]}{(p^2+1)^3}$$

$$= \frac{6p^2-2}{(p^2+1)^3}$$

$$\int_0^\infty e^{-pt} \cdot t^2 \cdot \sin t \cdot dt = \frac{6p^2-2}{(p^2+1)^3}$$

put $p = 3$.

$$\int_0^\infty t^2 e^{-3t} \cdot \sin t \cdot dt = \frac{54-2}{(10)^3} = \frac{52}{100}.$$

5) $\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t} dt$

$\Rightarrow$ we have $L[e^{-3t} - e^{-6t}] = \frac{1}{p+3} - \frac{1}{p+6}$ $F(p)$

$$L\left[\frac{e^{-3t} - e^{-6t}}{t}\right] = \int_p^\infty \left[\frac{1}{u+3} - \frac{1}{u+6}\right] du$$

$$= \left[\log\frac{(u+3)}{(u+6)}\right]_p^\infty = 0 - \log\frac{(p+3)}{(p+6)}$$

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$$\rightarrow \int_0^\infty e^{-pt}\left[\frac{e^{-3t}-e^{-6t}}{t}\right]dt = \log\frac{p+6}{p+3}$$

put $p=0$ $\quad = \log 2.$

6) $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt.$

$\Rightarrow$ We have

$$L[\cos 6t - \cos 4t] = \frac{p}{p^2+36} - \frac{p}{p^2+16}$$

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_p^\infty \left[\frac{u}{u^2+36} - \frac{u}{u^2+16}\right] du$$

$$= \frac{1}{2}\int_p^\infty \frac{2u}{u^2+36}\,du - \frac{1}{2}\int_p^\infty \frac{2u}{u^2+16}\,du$$

$$= \left[\frac{1}{2}\log(u^2+36) - \frac{1}{2}\log(u^2+16)\right]_p^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{u^2+36}{u^2+16}\right)\right]_p^\infty$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{p^2+36}{p^2+16}\right)\right]$$

$$\int_0^\infty e^{-pt}\left[\frac{\cos 6t - \cos 4t}{t}\right]dt = \frac{1}{2}\log\left(\frac{p^2+16}{p^2+36}\right)$$

put $p=0$

$$= \frac{1}{2}\log\left(\frac{16}{36}\right)$$

$$= \frac{1}{2}\log\left(\frac{4}{9}\right).$$

$$= \log\left(\frac{4}{9}\right)^{1/2}$$

$$= \log\left(\frac{2}{3}\right).$$

4) $\int_0^\infty \frac{\cos tx}{x^2+1} \cdot dx \quad t>0$

$\rightarrow$ let $f(t) = \int_0^\infty \frac{\cos t \cdot x}{x^2+1} \cdot dx$

taking laplace transform we get

$$L[f(t)] = F(P) = \int_0^\infty e^{-Pt}\left(\int_0^\infty \frac{\cos tx}{x^2+1} \cdot dx\right) \cdot dt$$

$$= \int_0^\infty \frac{1}{x^2+1} \cdot \left(\int_0^\infty e^{-Pt} \cdot \cos tx \cdot dt\right) dx$$

$$= \int_0^\infty \frac{1}{x^2+1} \cdot \frac{P}{P^2+x^2} \cdot dx \quad \text{--- (1)}$$

let $\frac{P}{(x^2+1)(P^2+x^2)} = \frac{A}{(x^2+1)} + \frac{B}{(x^2+P^2)}$

$$= \frac{A(x^2+P^2) + B(x^2+1)}{(x^2+1)(x^2+P^2)}$$

$\Rightarrow P = A(x^2+P^2) + B(x^2+1) \quad \text{--- (*)}$

put $x^2 = -P^2 \Rightarrow P = B(1-P^2) \quad \therefore B = \frac{-P}{P^2-1}$

~~put~~

put $x^2 = -1 \Rightarrow P = A(P^2-1) \quad \therefore A = \frac{P}{P^2-1}$

$\therefore$ equation (1) becomes.

$$L[f(t)] = F(P) = \int_0^\infty \frac{P}{P^2-1} \frac{1}{x^2+1} - \frac{P}{P^2-1} \frac{1}{x^2+P^2} \cdot dx$$

$$= \frac{P}{P^2-1}\left[\tan^{-1} x - \frac{1}{P} \cdot \tan^{-1} \frac{x}{P}\right]_0^\infty$$

$$= \frac{P}{P^2-1}\left[\frac{\pi}{2} - \frac{1}{P} \cdot \frac{\pi}{2}\right]$$

$$= \frac{P}{(P+1)(P-1)} \cdot \frac{\pi(P-1)}{2P} = \frac{\pi}{2(P+1)}$$

K. satish...

taking inverse laplace transform

$$f(t) = \frac{\pi}{2} e^{-t}$$

i.e. $$\int_0^\infty \left(\frac{\cos tx}{x^2+1}\right) dx = \frac{\pi}{2} \cdot e^{-t}$$

H.W solve the integral

1) $\int_0^\infty e^{-tx^2} \cdot dx$ 2) $\int_0^\infty \frac{x \cdot \sin tx}{x^2+1}$ 3) $\int_0^\infty \frac{\sin tx}{x} \cdot dx$

4) $\int_0^\infty \cos tx^2 \cdot dx$ 5) $\int_0^\infty \frac{\sinh t}{t} \cdot dt$ 6) $\int_0^\infty \exp\left(-x^2 - \frac{t^2}{x^2}\right) dx$

7) $\int_0^\infty \frac{\sin tx}{\sqrt{x}} \cdot dx$

Ans.

1) $\frac{1}{2}\sqrt{\pi/t}$ 2) $\frac{\pi}{2} e^{-t}$ 3) $\frac{\pi}{2}$ 4) $\frac{1}{2}\sqrt{\frac{\pi}{2t}}$

Solution of Ordinary differential Equation.

Bessel's function of order n :-

It is defined as $J_n(t) = \frac{t^n}{2^n \Gamma(n+1)} \left\{ 1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot4(2n+2)(2n+4)} - \cdots \right\}$

Some function's	laplace transform.
1) $J_0(at)$	$\frac{1}{\sqrt{p^2+a^2}}$
2) $J_n(at)$	$\frac{(\sqrt{p^2+a^2}-p)^n}{a^n(\sqrt{p^2+a^2})}$
3) $\sin\sqrt{t}$	$\frac{\sqrt{\pi}}{2\cdot p^{3/2}} e^{-1/4p}$
4) $\frac{\cos\sqrt{t}}{\sqrt{t}}$	$\sqrt{\frac{\pi}{p}}\, e^{-1/4p}$
5) $S_i(t) = \int_0^t \frac{\sin u}{u}\, du$	$\frac{1}{p} \tan^{-1}\frac{1}{p}$
6) $C_i(t) = \int_t^\infty \frac{\cos u}{u}\, du$	$\frac{\log(p^2+1)}{2p}$
7) $E_i(t) = \int_t^\infty \frac{e^{-u}}{u}\, du$	$\frac{\log(p+1)}{p}$
8) $h(t-a)$	$\frac{e^{-ap}}{p}$

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9) $\delta(t-a)$ $\qquad\qquad$ e^{-ap}

10) $\delta(t)$ $\qquad\qquad$ 1

Ex. Solve the following initial value problem.

1) $y'' - 6y' + 9y = t^2 \cdot e^{3t}$. $y(0) = 2$, $y'(0) = 6$

$\Rightarrow$ Taking the laplace transform of the given equations and denoting $L[y(t)] = Y(p)$ we get

$$p^2 Y(p) - p \cdot y(0) - y'(0) - 6[pY(p) - y(0)] + 9Y(p) = \frac{2}{(p-3)^3}$$

$$Y(p)[p^2 - 6p + 9] = \frac{2}{(p-3)^3} + 2p + 6 - 12$$

$$Y(p)(p-3)^2 = \frac{2}{(p-3)^3} + 2(p-3)$$

$$Y(p) = \frac{2}{(p-3)^5} + \frac{2(p-3)}{(p-3)^2}$$

$$= \frac{2}{(p-3)^5} + \frac{2}{(p-3)}$$

Taking inverse laplace transform we get

$$y(t) = L^{-1}\left[\frac{2}{(p-3)^5} + \frac{2}{p-3}\right]$$

$$= e^{3t} L^{-1}\left[\frac{2}{p^5} + \frac{2}{p}\right]$$

$$= e^{3t}\left(2 \cdot \frac{t^4}{4!} + 2 \cdot 1\right)$$

$$= 2e^{3t}\left(\frac{t^4}{4!} + 1\right)$$

2) $y'' + 2y' + y = 3t \cdot e^{-t}$ $y(0) = 4$, $y'(0) = 2$

$\Rightarrow$ Taking the laplace transform of the given equation and denoting $L[y(t)] = Y(p)$

$$p^2 \cdot Y(p) - p \cdot 4 - 2 + 2 \cdot pY(p) - 2 \cdot 4 + Y(p) = \frac{3}{(p+1)^2}$$

$$Y(p)[p^2 + 2p + 1] = \frac{3}{(p+1)^2} + 4p + 10$$

$$Y(p) = \frac{3}{(p+1)^4} + \frac{4(p+1) + 6}{(p+1)^2}$$

Taking inverse laplace transform.

$$y(t) = e^{-t} L^{-1}\left[\frac{3}{p^4} + \frac{4p+6}{p^2}\right]$$

$$= e^{-t}\left[3 \cdot \frac{t^3}{3!} + 4 \cdot 1 + 6t\right]$$

$$= e^{-t}\left(\frac{t^3}{2} + 6t + 4\right)$$

3) $y'' + 4y = f(t)$, $y(0) = 1$, $y'(0) = 0$

where $f(t) = \begin{cases} 4t & 0 \le t \le 1 \\ 4 & t > 1 \end{cases}$

$\Rightarrow$ $p^2 Y(p) - p \cdot 1 - 0 + 4Y(p) = L[f(t)]$ --- ①

Here,

$$L[f(t)] = \int_0^\infty e^{-pt} f(t) \cdot dt$$

$$= \int_0^1 e^{-pt} \cdot 4t \, dt + \int_1^\infty 4 \cdot e^{-pt} \cdot dt$$

$$= \left[4t \cdot \frac{e^{-pt}}{-p} - 4\frac{e^{-pt}}{p^2}\right]_0^1 + 4\left[\frac{e^{-pt}}{-p}\right]_1^\infty$$

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$\Rightarrow \quad L[f(t)] = \frac{-4\bar{e}^{p}}{p} - \frac{4\bar{e}^{p}}{p^2} + \frac{4}{p^2} + 4\left(\frac{\bar{e}^{p}}{p}\right)$

$= \frac{-4\bar{e}^{p}}{p^2} + \frac{4}{p^2}$

put in eq^n ②

$(p^2+4)Y(p) = \frac{-4\bar{e}^{p}}{p^2} + \frac{4}{p^2} + p$

$Y(p) = \frac{-4\bar{e}^{p}}{p^2(p^2+4)} + \frac{4}{p^2(p^2+4)} + \frac{p}{(p^2+4)} \quad -(*)$

let $\frac{4}{p^2(p^2+4)} = \frac{A}{p^2} + \frac{B}{p^2+4} \quad ---③$

$4 = A(p^2+4) + Bp^2$

put $p^2 = -4 \Rightarrow 4 = -4B \Rightarrow B = -1$

then $p^2 = 0 \Rightarrow 4 = 4A \Rightarrow A = 1$

$L^{-1}\left[\frac{4}{p^2(p^2+4)}\right] = L^{-1}\left[\frac{1}{p^2} - \frac{1}{p^2+4}\right] = t - \frac{1}{2}\sin 2t$

taking inverse laplace transform of equation ②

$y(t) = \left[\frac{1}{2}\sin 2(t-1) - (t-1)\right]h(t-1) + t - \frac{1}{2}\sin 2t + \cos 2t$

4) $\quad y'' - y = e^{t}\cos t \qquad y(0) = 0, \ y'(0) = 0$

$\Rightarrow \quad p^2 Y(p) - p.0 - 0 - Y(p) = \frac{p-1}{(p-1)^2+1}$

$(p^2-1)Y(p) = \frac{(p-1)}{[p-1]^2+1}$

$Y(p) = \cancel{(p-1)} \Big/ \cancel{(p-1)}(p+1)\cdot[(p-1)^2+1]$

$$Y(p) = \frac{1}{[(p-1)^2+1][(p-1)+2]}$$

taking inverse laplace transform. we have.

$$y(t) = e^{t} \cdot L^{-1}\left\{\frac{1}{(p^2+1)(p+2)}\right\} \quad --- (1)$$

$$L^{-1}\left\{\frac{1}{p+2}\right\} = e^{-2t} = f(t) \text{ (say)}$$

$$L^{-1}\left\{\frac{1}{p^2+1}\right\} = \sin t = g(t) \text{ (say)}$$

$\therefore$ By convolution Theorem

$$L^{-1}\left[\frac{1}{(p^2+1)(p+2)}\right] = \int_0^t f(u) \cdot g(t-u) \cdot du$$

$$= \int_0^t e^{-2(t-u)} \cdot \sin u \cdot du$$

$$= e^{-2t} \int_0^t e^{2u} \cdot \sin u \cdot du$$

$$= e^{-2t}\left[\frac{e^{2u}}{2^2+1}(2 \sin u - \cos u)\right]_0^t$$

$$= e^{-2t}\left[\frac{e^{2t}}{5}(2\sin t - \cos t) - \frac{1}{5}(0-1)\right]$$

$$\frac{2\sin t - \cos t}{5} + \frac{e^{-2t}}{5}$$

$$\Rightarrow y(t) = \frac{e^{t}}{5}\left[2\sin t - \cos t + e^{-2t}\right]$$

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5) $y'' - 4y' + 4y = t$, $y(0) = 1$, $y'(0) = 0$

$\Rightarrow [p^2 Y(p) - p(1) - 0] - 4[p \cdot Y(p) - 1] + 4 \cdot Y(p) = \frac{1}{p^2}$

$(p^2 - 4p + 4) Y(p) = \frac{1}{p^2} + p - 4$

$(p-2)^2 \cdot Y(p) = \frac{1}{p^2} + p - 4$

$Y(p) = \frac{1}{p^2(p-2)^2} + \frac{(p-2)-2}{(p-2)^2}$

taking inverse laplace transform

$y(t) = e^{2t} L^{-1}\left[\frac{1}{p^2 \cdot p^2} + \frac{p-2}{p^2}\right]$

$= e^{2t}\left[\frac{t^3}{3!} + 1 - 2t\right]$

$y(t) = e^{2t}\left(\frac{t^3}{3!} - 2t + 1\right)$

6) $y'' - 3y' + 2y = 4e^{2t}$ $y(0) = -3$ $y'(0) = 5$

$\Rightarrow$

$p^2 Y(p) - p(-3) - 5 - 3[p Y(p) - (-3)] + 2 \cdot Y(p) = \frac{4}{p-2}$

$(p^2 - 3p + 2) Y(p) = \frac{4}{p-2} - 3p + 14$

$Y(p) = \frac{4}{(p-2)(p^2-3p+2)} - \frac{3p+14}{p^2-3p+2}$

$= \frac{4}{(p-2)(p-1)(p-2)} - \frac{3p+14}{(p-1)(p-2)}$

$= \frac{4}{(p-2)^2(p-2+1)} + \frac{8 - 3(p-2)}{(p-2)(p-2+1)}$

Recall :- $\frac{dY}{dt} + P \cdot Y = Q$

$\Rightarrow I.F = e^{\int P \, dt}$ and $y(t) = \int Q \cdot I.F \cdot dt + C$

taking inverse laplace transform.

$$y(t) = e^{2t} L^{-1}\left[\frac{4}{p^2(p+1)} + \frac{8-3p}{p(p+1)}\right]$$

$$= e^{2t} L^{-1}\left[\frac{4+8p-3p^2}{p^2(p+1)}\right] \quad \cdots (1)$$

we have

$$\frac{4+8p-3p^2}{p^2(p+1)} = \frac{A}{p} + \frac{B}{p^2} + \frac{C}{p+1}$$

$$\Rightarrow 4+8p-3p^2 = A\,p(p+1) + B(p+1) + C \cdot p^2$$

$$= Ap^2 + Ap + Bp + B + Cp^2$$

$$= (A+C)p^2 + (A+B)p + B$$

$\Rightarrow A+C = -3$, $A+B = 8$, $B = 4$

$\Rightarrow A = 8-4 = 4 \quad \Rightarrow A = 4$

and $C = -7$

$\therefore$ from eqn (1) becomes.

$$y(t) = e^{2t} L^{-1}\left[\frac{4}{p} + \frac{4}{p^2} - \frac{7}{p+1}\right]$$

$$= e^{2t}(4 + 4t - 7e^{-t}).$$

7) $ty'' + y' + ty = 0$, $y(0) = 1$ $y'(0) = 0$.

taking laplace transform on both sides we have

$$(-1)\frac{d}{dp}[p^2 \cdot Y(p) - p \cdot y(0) - y'(0)] + p \cdot Y(p) - y(0) + (-1)\frac{d}{dp}[Y(p)] = 0$$

$$-[2p\,Y(p) + p^2\,Y'(p) - 1] + p \cdot Y(p) - 1 - Y'(p) = 0$$

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$$-(p^2+1)\cdot Y'(p) + Y(p)[p-2p] = 0.$$

$$\Rightarrow (p^2+1)Y'(p) + pY(p) = 0$$

$$Y'(p) + \frac{p}{p^2+1}\cdot Y(p) = 0$$

$$\therefore I.F = e^{\int \frac{p}{p^2+1}\cdot dp}$$

$$= e^{\frac{1}{2}\log(p^2+1)} = e^{\log(p^2+1)^{1/2}}$$

$$= (p^2+1)^{\frac{1}{2}} = \sqrt{p^2+1}$$

and $Y(p)\cdot\sqrt{p^2+1} = \int 0\cdot dp + c$

$$Y(p) = \frac{c}{\sqrt{p^2+1}}.$$

taking inverse laplace transform.

$$y(t) = c\cdot J_0(t)$$

using the condition $y(0)=1$, we get

$$1 = c\cdot J_0(0) = c\cdot 1 \Rightarrow c=1$$

$\therefore$ The required solution is

$$y(t) = J_0(t).$$

5) $y'' + ty' - y = 0$, $y(0)=0$. $y'(0)=1$

$$\Rightarrow [p^2 Y(p) - 1] - \frac{d}{dp}[p\cdot Y(p) - 0] - Y(p) = 0$$

$$p^2Y(p) - p\cdot Y'(p) - Y(p) - Y(p) = 1$$

$$(p^2-2)Y(p) - p\cdot Y'(p) = 1$$

$$Y'(p) - \frac{(p^2-2)}{p} \cdot Y(p) = \frac{-1}{p}$$

$$I.F. = e^{\int\left(\frac{p^2-2}{p}\right)dp} = e^{-\frac{p^2}{2} + 2.\log p}$$

$$= p^2 \cdot e^{-\frac{p^2}{2}}$$

$$Y(p)\left[p^2 \cdot e^{-\frac{p^2}{2}}\right] = \int \frac{-1}{p} \cdot \left(p^2 \cdot e^{-\frac{p^2}{2}}\right) \cdot dp + c.$$

$$= \int -p \cdot e^{-\frac{p^2}{2}} \cdot dp + c.$$

$$= \int e^{x} \cdot dx + c \qquad \text{put } x = -\frac{p^2}{2}$$

$$= e^{x} + c$$

$$= e^{-\frac{p^2}{2}} + c.$$

$$\Rightarrow \quad Y(p) = \frac{1}{p^2} + c \cdot e^{\frac{p^2}{2}}$$

9) $y'' + 2y' + 5y = e^{-t} \cdot \sin t \quad , \; y(0) = 0 \; , \; y'(0) = 1 .$

$$\Rightarrow \left[p^2 Y(p) - p\,y(0) - y'(0)\right] + 2\left[p \cdot Y(p) - y(0)\right] + 5\left[Y(p)\right] = \frac{1}{(p+1)^2+1}$$

$$p^2 Y(p) - 1 + 2p\,Y(p) + 5\,Y(p) = \frac{1}{(p+1)^2+1}$$

$$(p^2+2p+5)\,Y(p) = \frac{1}{(p+1)^2+1} + 1.$$

$$Y(p) = \frac{1}{\left[(p+1)^2+1\right]\left[p^2+2p+5\right]} + \frac{1}{(p^2+2p+5)}$$

$$Y(p) = \frac{1}{\left[(p+1)^2+1\right]\left((p+1)^2+4\right)} + \frac{1}{(p+1)^2+4}$$

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taking inverse laplace transform.

$$y(t) = e^{-t} L^{-1}\left[\frac{1}{(p^2+1)(p^2+4)} + \frac{1}{p^2+4}\right]$$

$$= e^{-t} L^{-1}\left[\frac{p^2+2}{(p^2+1)(p^2+4)}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1/3}{p^2+1} + \frac{2/3}{p^2+4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t + \frac{2}{3}\cdot\frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}(\sin t + \sin 2t).$$

H.W.

1) $y''' - 3y'' + 3y' - y = t^2 \cdot e^{t}$ $\quad y(0) = 1, \; y'(0) = 0, \; y''(0) = -2$

$$\Rightarrow \quad y(t) = e^{t}\left(\frac{t^5}{60} - \frac{t^2}{2} - t + 1\right).$$

2) $2y''' + 3y'' - 3y' - 2y = e^{-t}$ $\quad y(0) = 0, \; y'(0) = 0, \; y''(0) = 1$

$$y(t) = \frac{1}{18}\left[8e^{t/2} - 7e^{t} - e^{-t} - 10e^{-2t}\right]$$

3) $y''' - y'' + 4y' - 4y = t$ $\quad y(0) = 0, \; y'(0) = 0, \; y''(0) = 1$

4) $y'' + 4y = f(t)$, $\quad f(t) = \begin{cases} \cos 4t, & 0 \le t \le \pi \\ 0, & t > \pi \end{cases}$ $\quad y(0) = 0, \; y'(0) = 1$

5) $y'' + 4y = \sin t - h(t-2\pi)\sin(t-2\pi)$. $\quad y(0) = y'(0) = 0$.

6) $y'' + ty' - 2ty = 1$, $\quad y(0) = 0, \; y'(0) = 1$.

7) $ty'' + (2t+3)y' + (t+3)y = 3e^{-t}$ $\quad y(0) = 0, \; y'(0) = 1$.

Impulse Response function and one sided Green's function -

Consider the general initial value problem

$y'' + ay' + by = f(t)$, $y(0) = K_0$, $y'(0) = K$ - - - (A)

where a and b are known constant

Denoting

$L[y(t)] = Y(P)$ and $L[f(t)] = F(P)$ and

taking laplace transform on both sides we get

$$[p^2 Y(P) - pK_0 - K_1] + a[pY(P) - K_0] + b \cdot Y(P) = F(P)$$

$$Y(P)[p^2 + ap + b] = F(P) + (P+a)K_0 + K_1$$

$$Y(P) = \frac{(P+a)K_0 + K_1}{p^2 + ap + b} + \frac{F(P)}{p^2 + ap + b}$$

taking the inverse laplace transform.

$$y(t) = L^{-1}\left[\frac{(P+a)K_0 + K_1}{p^2 + ap + b}\right] + L^{-1}\left[\frac{F(P)}{p^2 + ap + b}\right]$$

$$= y_H(t) + y_P(t) \quad \text{(say)}$$

Where, $y_H(t)$ is the solution of the initial value problem $y'' + ay' + by = 0$, $y(0) = K_0$, $y'(0) = K_1$

and,

$y_P(t)$ is the solution of $y'' + ay' + by = f(t)$

$y(0) = 0$, $y'(0) = 0$.

physically, the function $y_H(t)$ is the responce of the system under the absence of external disturbance $f(t)$. where as $y_P(t)$ represent's the responce of the system under external disturbance $f(t)$ at the rest position.

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Taking $y_H(t) = 0$ and $L^{-1}\left[\frac{1}{p^2+ap+b}\right] = g(t)$

we get, the solution

By convolution theorem,

$$y(t) = \int_0^t g(t-u) \cdot f(u) \cdot du$$

The function

$g(t) = L^{-1}\left(\frac{1}{p^2+ap+b}\right)$ is called as impulse response function and the function $g(t-u)$ is called one sided green's function.

Construct the one sided green's function for the system describe by $y'' - 2y' + 5y = f(t)$

⇒ comparing with $y'' + ay' + by = f(t)$

The impulse response function $g(t)$ is given by

$$g(t) = L^{-1}\left(\frac{1}{p^2+ap+b}\right) = L^{-1}\left(\frac{1}{p^2-2p+5}\right)$$

$$= L^{-1}\left[\frac{1}{(p-1)^2+4}\right]$$

$$= e^{t} \cdot L^{-1}\left[\frac{1}{p^2+4}\right]$$

$$= e^{t} \cdot \frac{1}{2} \cdot \sin 2t$$

$$= \frac{e^{t}}{2} \cdot \sin 2t$$

∴ The one sided green's function is

$$g(t-u) = \frac{e^{(t-u)}}{2} \sin 2(t-u)$$

\# Gives Laplace transform to construct impulse responce function and one sided green's function for the given diffⁿ operator M. where $D = \frac{d}{dt}$

1) $M = (D-a)^2$

$\Longrightarrow$ Given. $My = (D-a)^2 y = f(t)$

$$= (D^2 - 2aD + a^2)\, y = f(t).$$

$\therefore$ Impulse responce function $g(t)$ is.

$$g(t) = L^{-1}\left[\frac{1}{p^2+ap+b}\right]$$

$$= L^{-1}\left[\frac{1}{p^2 - 2ab + a^2}\right]$$

$$= L^{-1}\left[\frac{1}{(p-a)^2}\right]$$

$$= e^{at}.\, L^{-1}\left[\frac{1}{p^2}\right]$$

$$= t\, e^{at}$$

$\therefore$ Then one sided greens function is.

$$g(t-u) = (t-u).\, e^{a(t-u)}$$

2). $M = (D-a)(D-b) \quad a \neq b.$

$\Rightarrow$

$$g(t) = L^{-1}\left[\frac{1}{p^2+ap+b}\right] = L^{-1}\left[\frac{1}{p^2-(a+b)p+ab}\right]$$

$$= L^{-1}\left[\frac{1}{(p-\frac{a+b}{2})^2 + ab - \frac{(a+b)^2}{4}}\right]$$

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$$\therefore g(t) = e^{\left(\frac{a+b}{2}\right)t} L^{-1}\left[\frac{1}{p^2 + \left(ab - \frac{(a+b)^2}{4}\right)}\right]$$

$$= e^{\left(\frac{a+b}{2}\right)t} L^{-1}\left[\frac{1}{p^2 - \left(\frac{a-b}{2}\right)^2}\right]$$

$$= e^{\left(\frac{a+b}{2}\right)t} \cdot \frac{2}{a-b} \cdot \sinh\left(\frac{a-b}{2}\right) \cdot t$$

$\therefore$ one sided green's function is.

$$g(t-u) = \left[e^{\left(\frac{a+b}{2}\right)(t-u)} \frac{2}{a-b} \sinh\left(\frac{a-b}{2}\right)(t-u) \right]$$

3) $M = D^2 + 5$

4) $M = D^2 + 4t + 7$

5) $M = 4D^2 - 8D + 5$

6) $M = D^2 - D - 2$

Note - $L^{-1}\left[e^{-a\sqrt{p}}\right] = \frac{a}{2\sqrt{\pi t^2}} \cdot e^{-a^2/4t}$

★ Solution of Partial Differential Equations-

let $u(x,t)$ denote function of two variable x and t let

$$L[u(x,t)] = U(x,p)$$

Where it can be shown that

$$L\left[\frac{\partial u}{\partial x}\right] = \frac{d}{dx}U(x,p) = \frac{d}{dx}U = U_x$$

$$L\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[U(x,p)\right] = \frac{d^2U}{dx^2} = U_{xx}$$

$$L\left[\frac{\partial u}{\partial t}\right] = p\,U(x,p) - u(x,0)$$

$$L\left[\frac{\partial^2 u}{\partial t^2}\right] = p^2 U(x,p) - p\cdot u(x,0) - u_t(x,0)$$

Note - $L^{-1}\left[e^{-a\sqrt{p}}\right] = \frac{2a}{2\sqrt{\pi t^3}} \cdot e^{\frac{-a^2}{4t}}$

Solve Heat conduction equation

$$u_{xx} = a^{-2}u_t \quad 0 < x < \infty,\ t > 0 \quad \text{---- (I)}$$

Boundary condⁿ - $u(0,t) = f(t)$, $u(x,t) \to 0$ as $x \to \infty$ - (II)

Initial condⁿ - $u(x,0) = 0$ ---- (III)

⇒ Denoting.

$$L[u(x,t)] = U(x,p)\ldots\ L[f(t)] = F(p).$$

and taking laplace transform of equation (I) we get

$$U_{xx} = a^{-2}\left[p\cdot U(x,p) - u(x,0)\right]$$

$$U_{xx} - \frac{p}{a^2}U = 0.$$

The auxillary eqⁿ's is $D^2 - \frac{p}{a^2} = 0$

$$D = \pm\frac{\sqrt{p}}{a}.$$

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$$U(x,p) = A(p)\cdot e^{\frac{\sqrt{p}}{a}x} + B(p)\cdot e^{-\frac{\sqrt{p}}{a}x}$$

where.

A(p) and B(p) are functions of p.

Given that

$$u(x,t) \to 0 \quad \text{as} \quad x \to \infty$$

Hence $U(x,p) \to 0$ as $x \to \infty$

The condition's give's

$$0 = A(p)\cdot e^{\frac{\sqrt{p}}{a}x}$$

$$\Rightarrow A(p) = 0$$

Hence the solution becomes.

$$U(x,p) = B(p)\cdot e^{-\frac{\sqrt{p}}{a}\cdot x}$$

Also.

$u(0,t) = f(t)$ gives

$$U(0,p) = F(p)$$

$$F(p) = B(p)\cdot e^{0}.$$

$$\Rightarrow B(p) = F(p).$$

$$U(x,p) = F(p)\cdot e^{-\frac{\sqrt{p}}{a}\cdot x}$$

We know that.

$$L^{-1}\left[e^{-a\sqrt{p}}\right] = \frac{a}{2\sqrt{\pi t^3}}\cdot e^{-\frac{a^2}{4t}}$$

$$\therefore L^{-1}\left[e^{-\frac{x}{a}\sqrt{p}}\right] = \frac{x/a}{2\sqrt{\pi t^3}}\cdot e^{-\frac{x^2}{4a^2t}} = g(t)$$

$\therefore$ By convolution theorem. taking the inverse laplace transform.

$$u(x,t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \frac{x}{2a\cdot\sqrt{\pi}}\int_0^t \frac{f(\tau)}{(t-\tau)}\cdot e^{-\frac{x^2}{4a^2(t-\tau)}}\cdot d\tau.$$

2) Solve the boundary value Problem.

$U_{xx} = c^{-2} U_{tt}$, $0 < x < \infty$, $t > 0$ --- (I)

Boundary condⁿ :- $U(0,t) = f(t)$, $U(x,t) \to 0$ as $x \to \infty$ - (II)

Initial condition - $U(x,0) = 0$, $U_t(x,0) = 0$, $0 < x < \infty$ --- (III)

⇒ Denoting

$$L[U(x,t)] = U(x,p) - U_t(x,0)]$$

taking laplace transform of equation (I) we get

$$U_{xx} = c^{-2}[p^2 U(x,p) - pU(x,0) - U_t(x,0)]$$

$$= c^{-2}[p^2 U - 0 - 0]$$

$$\Rightarrow U_{xx} - \frac{p^2}{c^2} = 0$$

The solution of this equation is given by

$$U(x,p) = A(p) \cdot e^{\frac{p}{c}x} + B(p) \cdot e^{-\frac{p}{c}x}$$

Given that.

$$U(x,t) \longrightarrow 0 \text{ as } x \to \infty$$

This gives.

$$U(x,p) \longrightarrow 0 \text{ as } x \longrightarrow 0$$

Hence we have $0 = A(p) \cdot e^{\infty} + 0$

$$\Rightarrow A(p) = 0$$

Also .

$$U(0,t) = f(t) \Rightarrow U(x,p) = F(p)$$

Hence we get

$$B(p) = F(p)$$

$$U(x,p) = F(p)\, e^{-\frac{p}{c}x}$$

We have

$$L^{-1}[F(p)] = f(t)$$

$$L^{-1}[e^{-\frac{x}{c}p} F(p)] = f(t - \tfrac{x}{c}) \cdot h(t - \tfrac{x}{c})$$

$$\Rightarrow U(x,t) = f(t - \tfrac{x}{c}) \cdot h(t - \tfrac{x}{c})$$

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1) Solve the boundary value Problem.

$2u_{xx} = u_t \quad 0 < x < 5 \quad t > 0 \quad \text{--- (I)}$

boundary cond^n - $u(0,t) = 0, \quad u(5,t) = 0 \quad \text{---- (II)}$

Initial cond^n - $u(x,0) = 10 \sin 4\pi x \quad \text{---- (III)}$

$\Rightarrow$ Denoting

$$L[u(x,t)] = U(x,p)$$

$$2U_{xx} = pU(x,p) - u(x,0)$$

$$= pU(x,p) - 10 \sin 4\pi x$$

$$U_{xx} - \frac{p}{2}U = -5 \sin 4\pi x$$

$$C.F. = A(p) e^{\sqrt{\frac{p}{2}}x} + B(p) \cdot e^{-\sqrt{\frac{p}{2}}x}$$

$$P.I = \frac{1}{D^2 - \frac{p}{2}} (-5 \sin 4\pi x)$$

$$= \frac{-5 \sin 4\pi x)}{-16\pi^2 - p/2}$$

$$= \frac{10 \sin 4\pi x}{p + 32\pi^2}$$

$$U(x,p) = A(p) \cdot e^{\sqrt{\frac{p}{2}}x} + B(p) \cdot e^{-\sqrt{\frac{p}{2}}x} + \frac{10 \sin 4\pi x}{p + 32\pi^2}$$

$u(0,t) = 0$ gives.

$$U(0,p) = 0$$

$$\Rightarrow 0 = A(p) + B(p) \quad \text{---- (4)}$$

$$u(5,t) = 0 \Rightarrow U(5,p) = 0$$

$$0 = A(p) \cdot e^{\sqrt{\frac{p}{2}} \cdot 5} + B(p) \cdot e^{-\sqrt{\frac{p}{2}} \cdot 5} \quad \text{--- (5)}$$

solving (4) and (5) we get,

$$A(p) = B(p) = 0.$$

$$U(x,p) = \frac{10}{p + 32\pi^2} \sin 4\pi x.$$

taking inverse laplace transform we get

$$u(x,t) = 10\sin 4\pi x \cdot e^{32\pi^2 t}.$$

4). solve the boundary value problem.

$u_{xx} = a^{-2} u_t \quad 0<x<1, \quad t>0$ --- (I)

B.C. - $u(0,t) = T_0 \quad u_x(1,t) = 0$ --- (II)

I.C. - $u(x,0) = 0$ --- (III)

5). solve the boundary value problem.

$u_{xx} = a^{-2} u_t \quad 0<x<1 \quad t>0$ --- (I)

B.C - $u(0,t) = 0 \quad u(1,t) = 0$

I.C. - $u(x,0) = T_0$

6) solve the boundary value problem.

$0.25\, u_{xx} = u_t - 1 \quad 0<x<10 \quad t>0.$

B.C. - $u_x(0,t) = 0. \quad u(10,t) = 20.$

I.C. - $u(x,0) = 50$

7) $u_{xx} = u_t - 2x, \quad 0<x<1, \quad t>0.$

B.C. $u(0,t) = 0 \quad u(1,t) = 0$

I.C. $u(x,0) = x(1-x)$

8) $u_{xx} = c^{-2} u_{tt} \quad 0<x<\infty, \quad t>0$

B.C. - $u(0,t) = 0. \quad u_x(x,t) \to 0$ as $x \to \infty$

I.C. - $u(x,0) = 0. \quad u_t(x,0) = V_0$

9) $u_{xx} = c^{-2} u_{tt} \quad 0<x<\infty \quad t>0$

B.C. $u(0,t) = 0 \quad u(x,t) \to 0$ as $n \to \infty$

I.C. $u(x,0) = A, \quad u_t(x,0) = 0.$

10) $u_{xx} = c^{-2} u_{tt} - A\sin \pi x, \quad 0<x<1 \quad t>0$

B.C. $u(0,t) = 0 \quad u(1,t) = 0$

I.C. $u(x,0) = 0 \quad u_t(x,0) = 0.$