Unit 4 :- Relativity

Frames of refrence :-

To understand the concept of frame of reference, consider an example, when we say that something is moving this means that its position relative to something else is changing. A passenger moves relative to an airplane; the airplane moves relative to the earth, the earth moves relative to the sun, the sun moves relative to the galaxy and so on. In each case the frame of reference is part of the description of the motion.

Defn :- The frame of reference is a set of axes which ~~can~~ enable an observer to measure the aspect, position and motion of all points in a system relative.

The frame of reference are classified into two types, namely

1. Inertial frames of reference or inertial frames
2. Non-inertial frames of reference or non-inertial frames.

→ An inertial frame of reference is one in which Newton's laws of motion holds good or applicable.

→ The non-inertial frame of reference is a frame of reference in which Newton's laws of motion does not holds good.

Postulates of Special Theory of Relativity :-

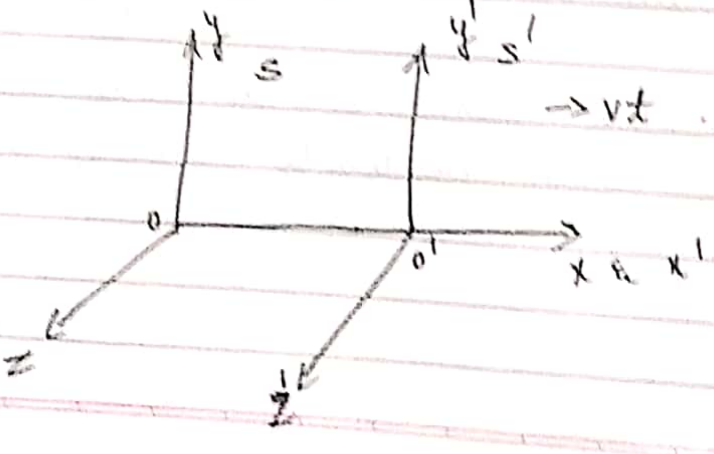
There are two postulates of special theory of relativity. They are.

1. The laws of physics are the same in all inertial frames of reference.
2. The speed of light in free space has the same value in all inertial frames of reference. The speed of light is 3×10^8 m/s.

Galilean Transformation equations :-

A physical phenomenon observed simultaneously in two separate reference will have ~~two~~ two separate ~~six~~ sets of coordinates corresponding to the two frames of reference.

Eqns relating the ~~w~~ two sets of co-ordinates of the event in which the two systems are called transformation equations.
As they enable the observations made in one system to be transformed into ~~those~~ ~~those~~ those made in other.
If the two systems are inertial or Galilean then the equations are called Galilean transformation equations.



Let s and s' be the two inertial frames of reference with the observers stationed at their origins O & O' respectively, moving with a relative velocity v along positive x-direction.

Suppose some event occurs at A then measuring time from the instant the two observers were just opposite each other i.e. setting the clocks in s and s' to zero when origins O & O' of the two frames coincides. The observer in frame s determines the position of the event by co-ordinates x, y, z and observer in s' by the coordinates x', y', z'.

Since s' is in uniform motion relative to s, the x-measurement in s exceed those in s' by vt, the distance covered by s' in time t in the positive x-direction have the value

$$x' = x - vt$$

and since s' is no relative motion between s & s' along axes of y and z, Then $y = y'$ & $z = z'$.
If we assume time to be independent of any frame of reference, we have the four transformation equations from s and s' are

$$x' = x - vt \rightarrow (1)$$

$$y' = y \rightarrow (2)$$

$$z' = z \rightarrow (3)$$

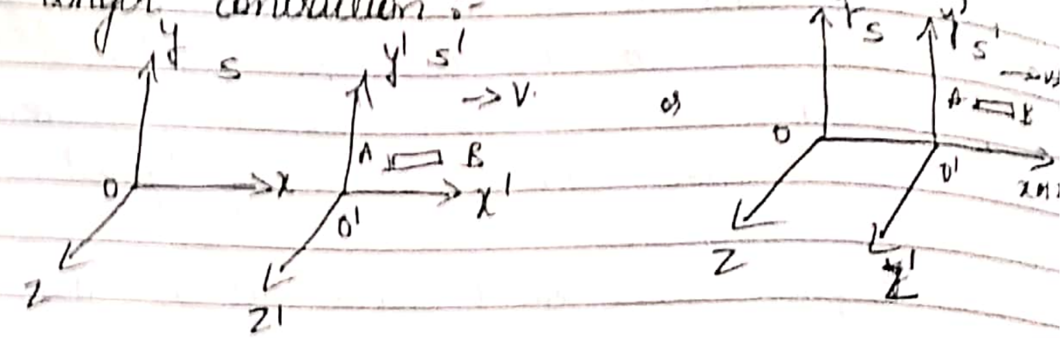
$$t' = t \rightarrow (4)$$

These are called Galilean transformation equations

The inverse transformation from s to s' are

$x = x' + vt$, $y = y'$ and $z = z'$, $t = t'$

length contraction :-



consider two frames of reference S and S'. The S frame is at rest and S' is moving with some velocity V.

Consider a rod AB to be carried in frame S' along X-axis. Let it be at rest.

If x_1' and x_2' are the ends of the rod in S' then its length is given by

$$l' = x_2' - x_1' \rightarrow (1)$$

Let an observer in S measure the length of the same rod when S' in motion. Let x_1 & x_2 be the ends of the rod for observer in S at time t then the length of the rod is given by

$$l = x_2 - x_1 \rightarrow (2)$$

From transformation equation we have

$$x_1' = A(x_1 - Vt) \rightarrow (3)$$

$$x_2' = A(x_2 - Vt) \rightarrow (4)$$

Substitute eqn (3) & (4) in (1) we get

$$l' = x_2' - x_1'$$

$$l' = A[(x_2 - Vt) - (x_1 - Vt)]$$

$$l' = A[x_2 - \cancel{Vt} - x_1 + \cancel{Vt}]$$

$$l' = A(x_2 - x_1)$$

$$l' = Al \qquad [\text{from eqn}(2)]$$

where $A = \dfrac{1}{\sqrt{1-\frac{V^2}{c^2}}}$

$$l' = \frac{l}{\sqrt{1-\frac{V^2}{c^2}}} \quad \text{or} \quad l = l'\sqrt{1-\frac{V^2}{c^2}}$$

This shows that l is less than l'
ie when the length of same rod for an observer in S is less than that for an observer in S' who is moving with the rod.
Hence for an observer who measure the length of the rod in motion, it is shorter than for an observer who measures it at rest. This leads to the length contraction.

Time dilation :-

The time that would be recorded by a clock moving with an object is called proper time.
A moving clock ticks more slowly than a clock at rest. or the clock at rest moves faster than the moving clock.

Measurement of time intervals are affected by relative motion between an observer and the thing observed.
If someone in a moving spacecraft finds that the time interval between two events in the spacecraft is t_0, we on the ground would find that the same interval has the longer duration t.
The quantity t_0 which is determined by events that occur at the same place in an observer's frame of reference is called the "proper time" of the interval between the events.
When the duration of the interval appears longer than the proper time, this effect is called 'time dilation'

~~The relation between the pre~~

The relation or equation for time dilation is given by

$$T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}} \quad \text{or} \quad t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

T is the dilated time, T_0 is proper Time.

The corresponding time t' as measured by the observer in S' is given by

$$t' = A\left[t - \frac{xv}{c^2}\right] \quad \text{(from lorentz transformation)}$$

$$t' = A\left[t - \frac{v^2 t}{c^2}\right] \quad (\because x = vt)$$

$$t' = At\left[1 - \frac{v^2}{c^2}\right]$$

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}\left[1-\frac{v^2}{c^2}\right] \quad \left(\because A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$t' = t\sqrt{1-\frac{v^2}{c^2}}.$$

$$\text{or} \quad t = \frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}.$$

This shows that $t' < t$.

The time measured by an observer of any event in relative motion appears to be more or dilated.

Hence the concept of time dilation.

Einstein's Mass Energy Relation :-

According to classical mechanics, the energy is defined in terms of work, work is force x distance and the force as the rate of change of momentum

Hence $F = \frac{dp}{dt}$ $(p = mv)$

$$F = \frac{d(mv)}{dt} \rightarrow (1)$$

According to theory of relativity the mass as well as velocity are variables.

Thus, $F = m\frac{dv}{dt} + v\frac{dm}{dt} \rightarrow (2)$

When the particle is displaced through a distance by the application of a force F, then the increase in KE is given by.

$$KE = F.dx$$

substitute the value of F from eqn (2)

$$KE = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$$

$$KE = mv\,dv + v^2 dm \rightarrow (3) \qquad \left(\frac{dx}{dt} = v\right)$$

The variation of mass with velocity is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

squaring on both sides

$$m^2 = \frac{m_0^2}{1-\frac{v^2}{c^2}}$$

$$m^2 = \frac{m_0^2 c^2}{c^2 - v^2}$$

$$m^2(c^2 - v^2) = m_0^2 c^2$$

$$m^2c^2 - m^2v^2 = m_0^2c^2$$

differentiating the equation

$$c^2 2m\,dm - v^2 2m\,dm - m^2 2v\,dv = 0$$

$$2m\left[c^2 dm - v^2 dm - mv\,dv\right] = 0$$

$$c^2 dm - v^2 dm - mv\,dv = 0$$

$$c^2 dm = v^2 dm + mv\,dv \rightarrow (4)$$

comparing eqns (3) & (4) we have

$$KE = c^2 dm \rightarrow (5)$$

consider the body is at rest initially and by applying force it acquires a velocity v. The mass of the body increases from m_0 to m. The KE acquired by the body is given by

$$KE = \int_{m_0}^{m} c^2 dm$$

$$KE = c^2(m - m_0) \rightarrow (6)$$

Therefore, the total energy is given by

$$E = KE + m_0c^2$$

$$E = c^2(m - m_0) + m_0c^2$$

$$E = mc^2 - m_0c^2 + m_0c^2$$

$$E = mc^2 \rightarrow (7)$$

This gives the mass energy relation.

velocity addition

let the velocity measurement be u & u' by the two observer in the frame of reference S & S' respectively. Thus $x_1 t_1$ & $x_2 t_2$ be the coordinates of the initial and final events representing the displacements in frame S and $x_1' t_1'$ & $x_2' t_2'$ be the corresponding co-ordinates in frame S'.

Then velocity relative to frame S is given by

$$u = \frac{\Delta x}{\Delta t} \quad \& \quad u' = \frac{\Delta x'}{\Delta t'}$$

we have
$$x_1 = x_1' + vt_1'$$
$$x_2 = x_2' + vt_2'$$
$$t_1 = t_1'$$
$$t_2 = t_2'$$

$$\therefore u = \frac{\Delta x}{\Delta t}$$

$$u = \frac{x_2 - x_1}{t_2 - t_1} \qquad (\because \Delta x = x_2 - x_1 \ \& \ \Delta t = t_2 - t_1)$$

$$u = \frac{(x_2' + vt_2') - (x_1' + vt_1')}{t_2' - t_1'}$$

$$= \frac{x_2' - x_1'}{t_2' - t_1'} + \frac{v\cancel{(t_2' - t_1')}}{\cancel{t_2' - t_1'}}$$

$$= \frac{x_2' - x_1'}{t_2' - t_1'} + v$$

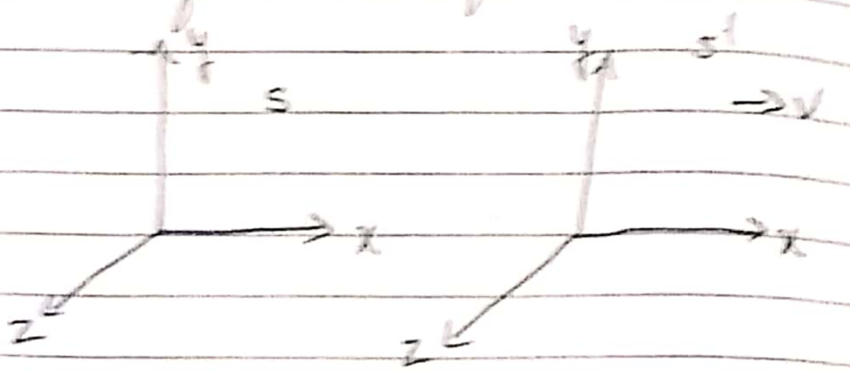
$$u = \frac{\Delta x'}{\Delta t'} + v$$

$$u = u' + v \quad \rightarrow (1)$$

or $$u' = u - v \quad \rightarrow (2)$$

This indicates that the velocities measured by the observers in the two frames of reference are not the same. we say that the velocity is not invariant to Galilean Transformation equations.
The eqns (1) & (2) referred as Galilean law of addition of velocities.

Lorentz Transformation Equations :-



consider S and S' are the two frames of references having the axes as OX, OY, OZ and O'X', O'Y' and O'Z' respectively.
Let frame S' is moving with constant velocity v relative to S.
Let the two observers be placed at the O & O' of the two frames and time be counted from the instant when the origins of these two frames coincide momentarily. when O & O' coincide, let a flash of light be emitted which gives rise to a spherical wavefront.

For observer O, after a time t spherical wave is defined by

$$x^2+y^2+z^2=c^2t^2$$

For observer O', it is

$$(x')^2 + (y')^2 + (z')^2 = c^2(t')^2$$

where t' is time measured by observer O'

If the motion is assumed to be only along x-axis then

$$y' = y$$
$$z' = z$$

Hence we have

$$x^2 - c^2t^2 = 0$$
$$(x')^2 - c^2(t')^2 = 0$$

combining or comparing the above equations

$$x^2 - c^2t^2 = (x')^2 - c^2(t')^2 \rightarrow (1)$$

For observer O, the origin O' of S' is given by.

$$x = vt$$
$$x - vt = 0$$

But for observer O' in S', the origin O' is given by

$$x' = 0$$
$$x' = x - vt$$
$$x' = A(x - vt) \rightarrow (2)$$

|||ly $\quad x = A(x' + vt') \rightarrow (3)$

where A is a constant used as same for the two frames.

From equation (3)

$$x = A(x' + vt')$$
$$x = A[A(x - vt) + vt']$$
$$\cancel{x = A^2x}$$
$$x = A[Ax - Avt + vt']$$
$$x = A^2x - A^2vt + Avt'$$

$$\cancel{x(1-} \quad x = A^2x - A^2vt + Avt'$$

$$x - A^2x + A^2vt = Avt'$$

$$x[1-A^2] + A^2vt = Avt'$$

$$t' = \frac{x(1-A^2) + A^2vt}{A^2v}$$

$$t' = \frac{x(1-A^2)}{Av} + \frac{A^2vt}{Av}$$

$$t' = \frac{x(1-A^2)}{Av} + At \rightarrow (4)$$

substituting the values of x' & t' from eqn (2) & (4) in eqn (1) we get

$$(1) \Rightarrow \quad x^2 - c^2t^2 = (x')^2 - c^2(t')^2$$

$$x^2 - c^2t^2 = \underbrace{A^2[x - vt]^2}_{(a-b)^2} - c^2\underbrace{\left[\frac{x(1-A^2)}{Av} + At\right]^2}_{(a+b)^2}$$

$$x^2 - c^2t^2 = A^2[x^2 + v^2t^2 - 2xvt] - c^2\left[\frac{x^2(1-A^2)^2}{A^2v^2} + A^2t^2 + \frac{2x(1-A^2)t}{v}\right] \rightarrow (5)$$

comparing the coefficients of x^2 on both sides of eqn (5)

$$1 = A^2 - \frac{c^2(1-A^2)^2}{A^2v^2}$$

$$(1-A^2) = \frac{-c^2(1-A^2)^2}{A^2v^2}$$

$$\frac{\cancel{(1-A^2)}}{(1-A^2)^{\cancel{2}}} = \frac{-c^2}{A^2v^2}$$

$$\frac{1}{1-A^2} = \frac{-c^2}{A^2v^2}$$

$$-(1-A^2) = \frac{A^2v^2}{c^2}$$

$$\frac{A^2v^2}{c^2} = -1 + A^2$$

$$1 = A^2 - \frac{A^2v^2}{c^2}$$

$$1 = A^2\left[1 - \frac{v^2}{c^2}\right]$$

$$A^2 = \frac{1}{\left[1 - \frac{v^2}{c^2}\right]}$$

$$A = \pm\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow (6)$$

comparing the coefficients of t^2 on both sides of eqn (5)

$$-c^2 = A^2v^2 - c^2A^2$$

$$-c^2 = \cancel{0} + A^2v^2 - c^2A^2$$

$$-\cancel{c^2} = -A^2c^2\left[1 - \frac{v^2}{c^2}\right]$$

$$A^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

$$A = \pm\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \rightarrow (7)$$

comparing the coefficients of xt on both sides of eqn (5)

$$0 = -2A^2v^2 - 2\frac{(1-A^2)}{v}c^2$$

$$2A^2v^2 = -2c^2(1-A^2)$$

$$\frac{\cancel{2}A^2v^2}{\cancel{2}c^2} = -(1-A^2)$$

$$\frac{A^2v^2}{c^2} = -1 + A^2$$

$$1 = A^2\left(1-\frac{v^2}{c^2}\right)$$

$$A^2 = \frac{1}{\left(1-\frac{v^2}{c^2}\right)}$$

$$A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \rightarrow (8)$$

The eqns (6), (7) & (8) justifies the choice of constant. The transformation equation are

$$x' = A(x - vt) \rightarrow (9)$$

$$y' = y \rightarrow (10)$$

$$z' = z \rightarrow (11)$$

$$t' = \frac{x(1-A^2)}{Av} + At$$

$$t' = \frac{-A^2v^2x}{c^2Av} + At \qquad \left(\because \frac{-A^2v^2}{c^2} = x(1-A^2)\right)$$

$$t' = -\frac{Avx}{c^2} + At.$$

$$t' = A\left[t - \frac{xv}{c^2}\right] \rightarrow (12)$$

The eqns (9), (10), (11) & (12) are known as Lorentz transformation equation.