# Unit - IV

Page No.
Date

2018. Define. of Power Green thm.

## * Line integral :-

Suppose $c: z(t) = x(t) + iy(t)$ $t \in [a,b]$ be a contour of $f(z)$ is continuous on c then

$$\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t) dt \quad \cdots ①$$

Let

$$f(z) = u + iv$$

& $z'(t) = x'(t) + iy'(t)$

$$\therefore z' = x' + iy'$$

$\therefore$ ① becomes,

$$\int_c f(z)dz = \int_a^b (u+iv)(x'+iy')dt$$

$$\therefore \int_c f(z)dz = \int_a^b [ux' + iuy' + ivx' + i^2 vy']dt$$

$$= \int_a^b \{[ux' - vy'] + i[uy' + vx']\} dt \quad (\because i^2 = -1)$$

$$= \int_a^b (ux' - vy') dt + i\int_a^b (uy' + vx') dt \quad \cdots ②$$

Now consider,

$$\int_a^b (ux' - vy')dt = \int_a^b ux' dt - \int_a^b vy' dt$$

Date

where, $x' = x'(t)$ & $y' = y'(t)$

$= \frac{dx}{dt}$ $\quad = \frac{dy}{dt}$

$\therefore x'dt = dx \qquad y'dt = dy$

Hence we have,

$$\int_a^b (ux' - vy')dt = \int_a^b u\,dx - \int_a^b v\,dy = \int_a^b u(x,y)dx - v(x,y)dy$$

Similarly, we have,

$$\int_a^b (uy' + vx')dt = \int_a^b v(x,y)dx + u(x,y)dy.$$

The integral of the form,

$$\int_a^b P(x,y)dx + Q(x,y)dy$$

is called as line integral.

Thus the complex integral is the sum of two line integral.

imp 2016 10M 2016 8M

## * Fundamental theorem of integration :

**Statement :** Suppose $f(z)$ is continuous function in domain D. and suppose there exist a differentiable function $F(z)$ such that $F'(z) = f(z)$ $\forall$ z in D. Then for any contour c in D parametrised by

$$c : z(t) : \quad t \in [a, b]$$

We have $\int_c f(z)dz = F(z(b)) - F(z(a))$.

**Proof :-** Given $f(z)$ is continuous function in D.
& $F(z)$ is differentiable function in domain D.
such that,

$$F'(z) = f(z) \quad \forall z \text{ in } D.$$

Given c is any contour ~~the~~ lying in D which is parametrised as

$$c : z(t) \quad t \in [a, b]$$

$$\therefore z'(t) = \frac{dz}{dt}$$

ie, $dz = z'(t) dt$.

We know that $\int_c f(z)dz = \int_a^b f(z(t)) \cdot z'(t) dt$

$$= \int_a^b F'(z(t)) \cdot z'(t) dt \quad \ldots (1)$$

Here,

$F = F(z)$ & $z = z(t)$

$$\frac{dF}{dt} = \frac{dF}{dz} \cdot \frac{dz}{dt}$$

$$= F'(z) \cdot z'(t)$$

$$= F'(z(t)) \cdot z'(t)$$

$\therefore$ eqⁿ ① becomes,

$$\int_C f(z)\,dz = \int_a^b \frac{dF}{dt}\,dt$$

$$= \Big[F(z(t))\Big]_a^b$$

$$= F(z(b)) - F(z(a))$$

$$\therefore \int_C f(z)\,dz = F(z(b)) - F(z(a))$$

This is the fundamental theorem of integration

Hence proved.

Problem:- By using F.T.O.I evaluate $\int_C f(z)dz$

where $f(z) = z^2$ & $C : z(t) = t + it \quad 0 \le t \le 1$.

Solution:- Given $f(z) = z^2$.

$\therefore$ $f(z)$ is continuous function in $\mathbb{C}$

& $C : z(t) = t + it \quad t \in [0,1]$

Here $C$ is not closed curve

$\therefore$ We can't apply cauchy's weak thm.

$\therefore$ We will use F. T. O. I.

choose $F(z) = \frac{z^3}{3}$

Clearly $f(z)$ is differentiable fun^n. in complex plane

Moreover

$$F'(z) = z^2$$

i.e, $F'(z) = f(z) \quad \forall z \text{ in } \mathbb{C}$.

$\therefore$ By fundamental theorem of integration, we have,

$$\int_c z^2 dz = F(z(b)) - F(z(a))$$

Now here, $a = 0 \Rightarrow z(a) = z(0) = 0$

$\quad$ & $b = 1 \Rightarrow z(b) = z(1) = 1+i$

& ~~f(z) =~~ $F(z) = \frac{z^3}{3}$ $\therefore F(z(a)) = F(0) = 0$.

& $F(z(b)) = F(1+i) = \frac{(1+i)^3}{3}$

$$\therefore \int_c z^2 dz = \frac{(1+i)^3}{3}$$

$$= \frac{1^3 + 3 \cdot 1^2 i + 3 \cdot 1 i^2 + i^3}{3}$$

$$= \frac{1 + 3i - 3 - i}{3}$$

$$= \frac{-2 + 2i}{3}$$

$$\int_c f(z) dz = -\frac{2}{3} + \frac{2}{3} i$$

Scanned by CamScanner

## * Fundamental theorem of Calculus :-

**Statement :-** If $f(x)$ is continuous function defined on $[a,b]$ then & iff there exist a differentiable function $F(x)$ on $[a, b]$ with $F'(x) = f(x)$ $\forall x \in [a,b]$ then

$$\int_a^b f(x)dx = F(b) - F(a).$$

2014 8M
8M 2015 10M Define line integral

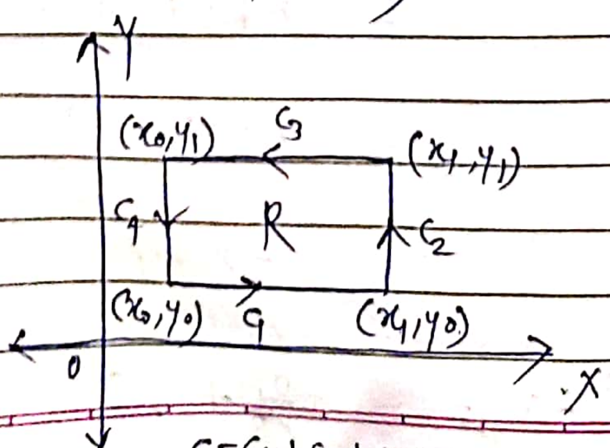
## Imp * Green's Theorem :-

**Statement :-** Let $P(x,y)$ & $Q(x,y)$ be continuous functions with continuous partials in a simply connected closed region R whose boundary is contour C then prove that

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \cdot dy.$$

**Proof :-** Given $P(x,y)$ & $Q(x,y)$ are continuous functions with continuous partials in a ~~sys~~ simply connected closed region R whose boundary is contour C.

We will prove this result for the special case that R is a rectangle whose sides are $\parallel$ to co-ordinate axes (ie, x & y -axis) as shown fig.




$C = C_1 + C_2 + C_3 + C_4$

Clearly we have $C = C_1 + C_2 + C_3 + C_4$.
We can observe that on the curves $C_1$ & $C_3$ $y$-co-ordinate is constant i.e, $dy = 0$ on $C_1$ & $C_3$.
Also we can observe that on the curves $C_2$ & $C_4$ $x$-coordinate is constant i.e, $dx = 0$ on $C_2$ & $C_4$.

Consider,

$$\int_C Pdx + Qdy = \int_{C_1+C_2+C_3+C_4} P(x,y)dx + Q(x,y)dy$$

$$= \int_{C_1} Pdx + Qdy + \int_{C_2} Pdx + Qdy + \int_{C_3} Pdx + Qdy + \int_{C_4} Pdx + Qdy$$

We have, $dx = 0$ on $C_2$ & $C_4$ & $dy = 0$ on $C_1$ & $C_3$
Hence we get,

$$\int_C Pdx + Qdy = \int_{C_1} Pdx + \int_{C_2} Qdy + \int_{C_3} Pdx + \int_{C_4} Qdy$$

Consider,

$$\int_C Pdx + Qdy = \int_{x_0}^{x_1} P(x,y_0)dx + \int_{y_0}^{y_1} Q(x_1,y)dy + \int_{x_1}^{x_0} P(x,y_1)dx + \int_{y_1}^{y_0} Q(x_0,y)dy$$

$$= \int_{x_0}^{x_1} P(x,y_0)dx + \int_{y_0}^{y_1} Q(x_1,y)dy - \int_{x_0}^{x_1} P(x,y_1)dx - \int_{y_0}^{y_1} Q(x_0,y)dy$$

$$\int_C P\,dx+Q\,dy = \int_{x_0}^{x_1}[P(x,y_0)-P(x,y_1)]\,dx + \int_{y_0}^{y_1}[Q(x_1,y)-Q(x_0,y)]\,dy \quad \ldots (1)$$

Given $P(x,y)$ & $Q(x,y)$ are continuous in $R$ with continuous partials.

$\therefore$ $P(x,y)$ & its partials are continuous in $[y_0,y_1]$

Also, $Q(x,y)$ and its partials are continuous in $[x_0,x_1]$

$\therefore$ By using fundamental theorem of calculus we have,

$$\int_{y_0}^{y_1}\frac{\partial P}{\partial y}\,dy = P(x,y_1)-P(x,y_0)$$

& $$\int_{x_0}^{x_1}\frac{\partial Q}{\partial x}\,dx = Q(x_1,y)-Q(x_0,y).$$

Hence eqn (1) becomes,

$$\int_C P\,dx+Q\,dy = \int_{x_0}^{x_1}\left[-\int_{y_0}^{y_1}\frac{\partial P}{\partial y}\,dy\right]dx + \int_{y_0}^{y_1}\left[\int_{x_0}^{x_1}\frac{\partial Q}{\partial x}\,dx\right]dy$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1}-\frac{\partial P}{\partial y}\,dx\,dy + \int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{\partial Q}{\partial x}\,dx\,dy$$

$$= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

$$\int_C P\,dx+Q\,dy = \iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$$

Problem: Find $\int_C (xy)dx + (x^2+y^2)dy$ where $0 \le x \le 1$ & $0 \le y \le 1$

Solution:- Given integral is

$$I = \int_C (xy)dx + (x^2+y^2)dy$$

Compare $I$ with respect $\int_C P\,dx + Q\,dy$.

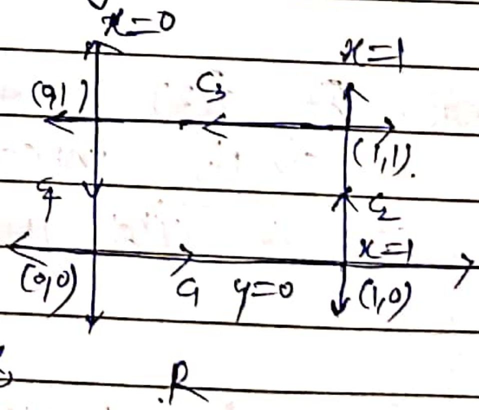
$\therefore$ We have $P = xy$ & $Q = (x^2+y^2)$

$$\frac{\partial P}{\partial y} = yx \quad \& \quad \frac{\partial Q}{\partial x} = 2x$$

$$\frac{\partial P}{\partial x} = y \quad \& \quad \frac{\partial Q}{\partial y} = 2y$$

Also given that, $0 \le x \le 1$ & $0 \le y \le 1$

Clearly the region bounded by these two interval is a rectangle $R$ as shown in fig.




Clearly the functions $P(x,y)$ & $Q(x,y)$ are continuous with continuous partials in $R$

$\therefore$ By Green's theorem, we have

$$\int_C (xy)dx + (x^2+y^2)dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$$

$$= \int_0^1\int_0^1 (2x - x)\,dx\,dy$$

$$\therefore \int_C (2xy)dx+(x^2+y^2)dy = \int_0^1\left[\int_0^1 x\,dx\right]dy$$

$$= \int_0^1\left[\frac{x^2}{2}\right]_0^1 dy$$

$$= \int_0^1 [1/2]\,dy$$

$$= \frac{1}{2}\int_0^1 dy$$

$$= \frac{1}{2}[y]_0^1$$

$$= \frac{1}{2}(1)$$

$$\int_C (xy\,dx+(x^2+y^2)dy) = \frac{1}{2}$$

imp * Cauchy's Weak theorem:-

Statement:- If $f(z)$ is analytic with continuous derivative in a simply connected domain D & C is closed contour lying in D then

$$\int_C f(z)dz = 0.$$

Proof:- Given $f(z) = u(x,y)+iv(x,y)$ is an analytic function in D. where D is simply connected domain.

Also given that C is closed contour ~~in~~ lying in D.

As $f(z)$ is analytic therefore partials of $u$ & $v$ satisfies the C-R equations for all points in D.
i.e, We have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \forall z \text{ in } D \quad \cdots (1)$$

We have, $z = x+iy$

$$\therefore dz = dx + i\,dy$$

$$\therefore f(z)dz = (u+iv)(dx+idy)$$

$$= u\,dx + i\,u\,dy + iv\,dx + i^2 v\,dy$$

$$= u\,dx + i\,u\,dy + iv\,dy - v\,dy$$

$$= u\,dx + i(u\,dy + v\,dy) - v\,dy$$

$$f(z)dz = (u\,dx - v\,dy) + i(u\,dy + v\,dy)$$

$$\therefore \int_c f(z)dz = \int_c (u\,dx - v\,dy) + i(u\,dy + v\,dy)$$

$$= \int_c u\,dx - v\,dy + i\int_c v\,dx + u\,dy \quad \cdots (2)$$

As $f(z) = u+iv$ is analytic ~~on~~

$\therefore$ $f(z)$ is continuous

$\therefore$ $u(x,y)$ & $v(x,y)$ are continuous in D.

Thus, as $f(z)$ is analytic $\therefore$ We have

$$f'(z) = -i\frac{\partial f}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\,i$$

$\int 0 \, dx = \text{const}$

But in Cauchy W.T. $\iint_R 0 \cdot dx \cdot dy$ is arbitary Hence it is 0.

Page No. Date

& $f'(z) = \frac{\partial f}{\partial x}$

$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

As $f'(z)$ is continuous in D.

$\therefore \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in D.

Thus, u, v & partials of u & v are continuous in D

We can apply Green's theorem to both integrals on R.H.S of ②

then we get,

$$\int_C u\,dx - v\,dy = \iint_R \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx \cdot dy$$

$\therefore$ Here $P = u$ & $Q = -v$

$$\therefore \int_C u\,dx - v\,dy = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx \cdot dy \quad \because \text{by ①}$$

$$\therefore \int_C u\,dx - v\,dy = \iint_R (0)\,dx \cdot dy = 0 \quad \text{--- ③}$$

Also we have,

$$\int_C v\,dx + u\,dy = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \cdot dy$$

$\therefore P = v$ & $Q = u$

$$\therefore \int_C v\,dx + u\,dy = \iint_R \left[\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right] dx \cdot dy \quad \because \text{by ①}$$

Date

$$\int_C v\,dx + u\,dy = 0 \quad \text{--- (4)}$$

by (2), (3) & (4) we have,

$$\int_C f(z)\,dz = 0.$$

**Theorem:** If $f(z)$ is analytic ~~in~~ with continuous derivative a simply connected domain $D$ & $C_1, C_2$ be any contours in domain $D$. having same initial & terminal points then prove that,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

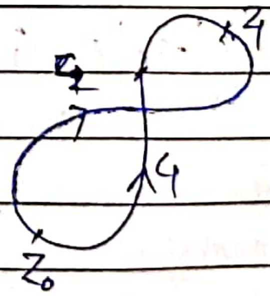
**Proof:-** Given $f(z)$ is analytic & $f'(z)$ is continuous in simply connected domain $D$.

Given that $C_1$ & $C_2$ are lying in $D$ such that $C_1$ & $C_2$ has same initial & terminal points say $z_0$ & $z_1$ respectively as shown in fig.

Define, $C = C_1 - C_2$

Where $-C_2$ is contour in $D$.

Have have $z_1$ as initial point & $z_0$ as terminal point.




Hence $C$ is a closed contour lying in $D$.

$\therefore$ By Cauchy's Weak theorem, we have,

$$\int_C f(z)\,dz = 0$$

$$\int_{C_1-C_2} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$

$$\int_{C} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\int_{C} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Hence proved.

Note:- [1] $f(z) = z^2$ is analytic & $f'(z) = 2z$ is continuous in complex plane.

Let $C_1 : z(t) = t + it \quad t \in [0,1]$
& $C_2 : z(t) = t + it^2 \quad t \in [0,1]$

Clearly $C_1$ & $C_2$ has same initial & terminal points. Hence by above theorem, we have,

$$\int_{C_1} z^2\,dz = \int_{C_2} z^2\,dz.$$

[2] $f(z) = |z|^2$ is not analytic in D (complex plane) & we have,

$$\int_{C_1} |z|^2\,dz \neq \int_{C_2} |z|^2\,dz.$$

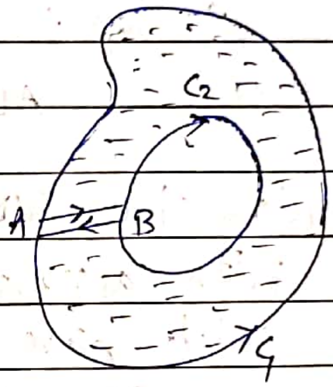
Page No.
Date

## * Chauchy's Weak theorem for Multiply connected domain:-

**Statement:-** If $f(z)$ is analytic with continuous derivative in multiply connected domain D & on its boundary c then

$\int_c f(z)dz = 0$ where the integration is performed along c in positive sence.

**Proof:-** Given $f(z)$ is an analytic function with continuous derivative in a domain D & on C. Where D is a multiply connected domain whose boundary is C.




$C : C_1 + AB + C_2 + BA.$

We will prove the result by converting the multiply connected domain into simply connected domain.

Now suppose the multiply connected domain D has only one whole.

Now construct a line segment AB joining two curves $C_1$ & $C_2$ as shown in fig.

Thus the contour C consist of four curves $C_1, C_2, C_3$ & $C_4$ line segment AB & line segment BA

i.e $C = C_1 +$ line segment $AB + C_2 +$ line segment BA.

$$C = C_1 + AB + C_2 + BA.$$

Thus, C is closed contour & the domain bounded by C is ~~the~~ a simply connected domain.

$\therefore$ We can apply the cauchy's Weak theorem.

Hence by cauchy's weak theorem, we have

$$\int_{C} f(z)dz = 0$$

i.e, $$\int_{C_1+AB+C_2+BA} f(z)dz = 0$$

$$\int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_{C_2} f(z)dz + \int_{BA} f(z)dz = 0 \quad \cdots (1)$$

Now the curves AB & BA are the same curves with apposite directions.

∴ We have,

$AB = -BA \cdots$

$$\therefore \int_{AB} f(z)dz = \int_{-BA} f(z)dz$$

$$\therefore \int_{AB} f(z)dz = -\int_{BA} f(z)dz$$

$$\int_{AB} f(z)dz + \int_{BA} f(z)dz = 0$$

∴ (1) becomes,

$$\int_{C_1} f(z)dz + \int_{C_2} f(z)dz = 0$$

i.e, $$\int_{C_1+C_2} f(z)dz = 0$$

i.e, $\int_{c'} f(z)\,dz = 0$.

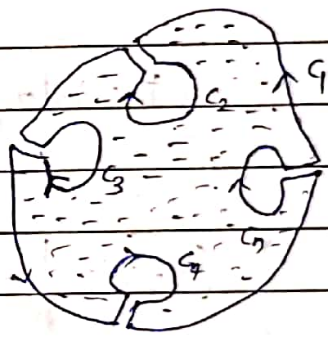
where $c' = c_1 + c_2$.

Clearly the domain bounded by $c'$ is multiply connected ~~com~~ domain with one whole.

Hence, the Cauchy's theorem is true for multiply connected domain having only one whole.

We can generalize the result for the multiply connected domain having $(n-1)$ holes. as

$$\int_{c_1 + c_2 + \cdots + c_n} f(z)\,dz = 0$$





10M. (All thm's imp) 2014

Vimp * Cauchy's Theorems :-

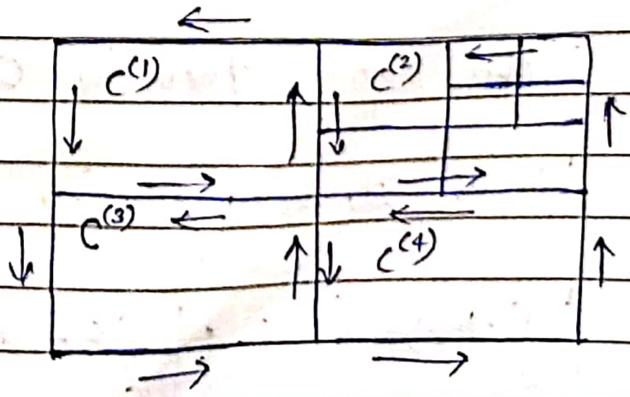
2014 10M

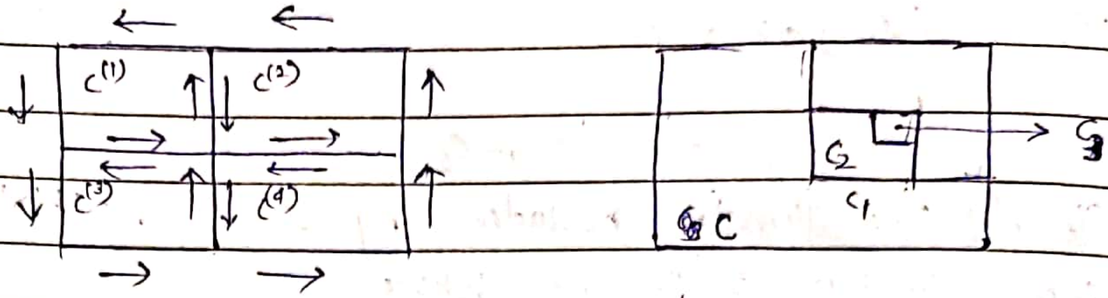
1] Cauchy's theorem for rectangle :

Statement :- Let $f(z)$ be analytic in domain $D$, containing a rectangle $C$ and its interior then

$$\int_C f(z)\,dz = 0$$

Proof :- Given $f(z)$ is analytic in domain $D$. containing a rectangle $C$ & its interior.






Divide a rectangle C into four congruent rectangles $C^{(1)}, C^{(2)}, C^{(3)}$ & $C^{(4)}$ as shown in fig.
Here the integral over the common sides have the opposite orientation as shown in fig.

∴ The integral over common sides cancells each other. thus we have,

$$\int_C f(z)dz = \int_{C^{(1)}} f(z)dz + \int_{C^{(2)}} f(z)dz + \int_{C^{(3)}} f(z)dz + \int_{C^{(4)}} f(z)dz$$

$$\left|\int_C f(z)dz\right| = \left|\int_{C^{(1)}} f(z)dz + \int_{C^{(2)}} f(z)dz + \int_{C^{(3)}} f(z)dz + \int_{C^{(4)}} f(z)dz\right|$$

$$\leq \left|\int_{C^{(1)}} f(z)dz\right| + \left|\int_{C^{(2)}} f(z)dz\right| + \left|\int_{C^{(3)}} f(z)dz\right| + \left|\int_{C^{(4)}} f(z)dz\right|$$

But we know that the rectangles $C^{(1)}, C^{(2)}, C^{(3)}$ & $C^{(4)}$ are the congruent rectangles.

∴ We denote them by $C_1$

$$C^{(1)} = C^{(2)} = C^{(3)} = C^{(4)} = C_1 \quad \text{(say)}$$

Hence, above inequality becomes,

$$\left|\int_C f(z)dz\right| \leq 4\left|\int_{C_1} f(z)dz\right|$$

nested = Rectangles inside rectangles.

Page No.
Date

$$\frac{1}{4}\left|\int_C f(z)dz\right| \le \left|\int_{C_1} f(z)dz\right|$$

$$\therefore \left|\int_{C_1} f(z)dz\right| \geq \frac{1}{4}\left|\int_C f(z)dz\right| \quad \cdots (1)$$

Now divide the rectangle $C_1$ into four congruent rectangles (say $C_1^{(1)}, C_1^{(2)}, C_1^{(3)}, C_1^{(4)}$.

Put $C_2 = C_1^{(1)} = C_1^{(2)} = C_1^{(3)} = C_1^{(4)}$ then we get

$$\therefore \left|\int_{C_1} f(z)dz\right| \le 4\left|\int_{C_2} f(z)dz\right|$$

$$\therefore \left|\int_{C_2} f(z)dz\right| \geq \frac{1}{4}\left|\int_{C_1} f(z)dz\right| \geq \frac{1}{4^2}\left|\int_C f(z)dz\right|$$

($\because$ by (1))

Continuing the process we obtain ~~a~~ a nested sequence of rectangles $C_n$ each satisfying satisfying

$$\left|\int_{C_n} f(z)dz\right| \geq \frac{1}{4^n}\left|\int_C f(z)dz\right| \quad \cdots (2)$$

Then we have,

$$C \supset C_1 \supset C_2 \supset C_3 \cdots \supset C_n \supset C_{n+1} \cdots$$

Where,

$\bigcap_{n\in N} C_n$ is non-empty i.e, $\bigcap_{n\in N} C_n \neq \phi$

ie, there exist a point $z_0$ in $D$ such that,

$$z_0 \in \bigcap_{n\in N} C_n$$

i.e, $z_0 \in C_n \ \forall n \in N$

i.e. for each $n$ the point $z_0$ lies in $C_0$
As $f(z)$ is analytic in D & C lies in D
$\therefore$ $f(z)$ is analytic at every point in and on C.
Thus,
$f(z)$ is analytic at $z_0$,
$\therefore$ $f'(z_0)$ is exist.

i.e, $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exist.

Moreover, we have,

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Thus, by definition of limit, for every $\epsilon>0$ $\exists$ $\delta>0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon \quad \text{whenever } |z-z_0|<\delta \quad \ldots (3)$$

put $$\eta(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \quad \text{--- } (4)$$

$\therefore$ by (3) we have,

$$|\eta(z)| < \epsilon \qquad \text{whenever } |z-z_0|<\delta \quad \cdots (5)$$

by (4) we have,

$$\eta(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$$

$$\Rightarrow \eta(z) + f'(z_0) = \frac{f(z)-f(z_0)}{z-z_0}$$

$\Rightarrow [\eta(z)+f'(z_0)](z-z_0) = f(z) - f(z_0)$

$\Rightarrow f(z) = [\eta(z)+f'(z_0)].(z-z_0) + f(z_0)$

Now,

$$\int_{C_n} f(z)dz = \int_{C_n}\{[\eta(z)+f'(z_0)](z-z_0)+f(z_0)\}dz$$

$$= \int_{C_n}[f(z_0)+f'(z_0)(z-z_0)]dz + \int_{C_n}\eta(z).(z-z_0)dz$$

$$\int_{C_n} f(z)dz = \int_{C_n}\eta(z).(z-z_0)dz \quad \cdots (6)$$

Here $h(z) = f(z_0)+(z-z_0)f'(z_0)$ is analytic function in D. ($\because f(z_0)$ & $f'(z_0)$ are constants)

Moreover,

$h'(z) = f'(z_0)$ is continuous function in D

& Rectangle $C_n$ is closed contour in D.

$\therefore$ By Cauchy's weak theorem,

$$\int_{C_n} h(z)dz = 0$$

i.e, $\int_{C_n}[f(z_0)+(z-z_0)f'(z_0)]dz = 0.$

from (6), we have,

$$\left|\int_{C_n} f(z)dz\right| = \left|\int_{C_n}\eta(z).(z-z_0)dz\right|$$

$$\left|\int_{C_n} f(z)\,dz\right| \leq \left|\int_{C_n} h(z)(z-z_0)\,dz\right|$$

$$\leq \int_{C_n} |h(z)|\cdot|z-z_0|\,|dz| \quad \cdots (7)$$

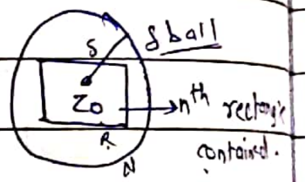
Choose $n$ so large, such that, $C_n \subset N(z_0,\delta)$

Also choose,

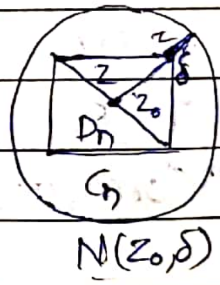
$$z \in C_n \subset N(z_0,\delta) \quad \therefore |z-z_0|<\delta.$$

$\therefore$ By (5) we have $|h(z)|<\epsilon$ whenever $|z-z_0|<\delta$

$\therefore$ eqn (7) becomes,

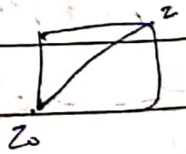



$$\left|\int_{C_n} f(z)\,dz\right| \leq \epsilon \int_{C_n} |z-z_0|\,|dz|$$




Now, $\left|\int_{C_n} f(z)\,dz\right| \leq \epsilon\cdot D_n \int_{C_n} |dz|$

Whenever $D_n$ = Diagonal of $C_n$




$|z-z_0| < D_n$

$\therefore \left|\int_{C_n} f(z)\,dz\right| \leq \epsilon\cdot D_n$ (length of $C_n$)

$L_n$ = Perimeter of rectangle

$\therefore \left|\int_{C_n} f(z)\,dz\right| \leq \epsilon D_n L_n$

where $L_n$ = length of $C_n$.

Let $L$ = length of $C$ & $D$ = digonal of $C$

$\therefore$ We have,

$$D_n = \frac{D}{2^n}$$

& $L_n = \frac{L}{2^n}$

Hence we get,

$$\left|\int_{C_n} f(z)\,dz\right| \leq \epsilon \frac{D}{2^n} \cdot \frac{L}{2^n}$$

$$\left|\int_{C_n} f(z)\,dz\right| \leq \frac{\epsilon D L}{4^n}$$

We have

$$\left|\int_{C_n} f(z)\,dz\right| \geq \frac{1}{4^n}\left|\int_C f(z)\,dz\right| \quad (\because \text{ by } (2))$$

$$\therefore \frac{1}{4^n}\left|\int_C f(z)\,dz\right| \leq \left|\int_{C_n} f(z)\,dz\right|$$

$$\therefore \frac{1}{4^n}\left|\int_C f(z)\,dz\right| < \frac{\epsilon D L}{4^n}$$

$$\left|\int_C f(z)\,dz\right| < \epsilon D L$$

As $\epsilon$ is arbitary, we have,

$$\cancel{\left|\int_C f(z)\,dz\right|} \qquad \int_C f(z)\,dz = 0.$$

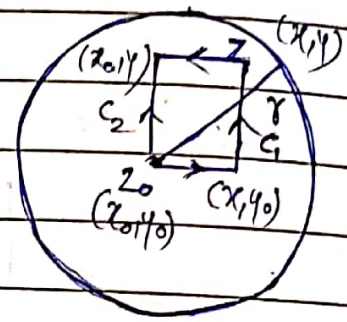
Hence proved.

* Cachy's theorem for disk:-

Statement: Let $f(z)$ be analytic in a domain containing the closed the disk $|z-z_0| \leq r$ then P.T.

$$\int_{|z-z_0| \leq r} f(z)dz = 0.$$

Proof:- Given $f(z)$ is analytic in a domain D containing a closed disk $|z-z_0| \leq r$.




Thus, $f(z)$ is continuous at every point of D.

Now we have to find a function $F(z)$ such that

$$f(z) = F'(z) \quad \forall z \in D \text{ in } |z-z_0| \leq r.$$

Choose an arbitary point $z = x+iy$ in the disk $|z-z_0| < r$

We know that the center of disk is $z_0 = x_0 + iy_0$

Now join $z_0$ & $z$ by two polygonal lines $c_1$ & $c_2$ where $c_1$ is a contour consisting of a line segments joining $(x_0, y_0)$ to $(x, y_0)$ followed by another line segment joining $(x, y_0)$ to $(x, y)$ as shown in fig.

$c_2$ is a contour consisting of a line segment joining $(x_0, y_0)$ to $(x_0, y)$ followed by another line segment joining $(x_0, y)$ to $(x, y)$ as shown in fig.

Thus, $c_1$ & $c_2$ are the curves having same initial & terminal points

$\therefore$ put $c' = c_1 - c_2$ which represents a closed ~~curve~~ rectangle. (for rectangle)
$\therefore$ By theorem, we have

$$\int_{c'} f(z)dz = 0.$$

$$\therefore \int_{c_1 - c_2} f(z)dz = 0 \Rightarrow \int_{c_1} f(z)dz - \int_{c_2} f(z)dz = 0.$$

$$\therefore \int_{c_1} f(z)dz = \int_{c_2} f(z)dz. \quad \ldots (1)$$

Now define,

$$F(z) = \int_{c_1} f(z)dz$$

$$\therefore F(z) = \int_{c_1} f(z)dz = \int_{c_2} f(z)dz \quad \therefore \text{by } (1)$$

Consider,

$$F(z) = \int_{c_1} f(z)dz$$

On 1st line segment of $c_1$, $y$ is constant & $x$ varies from $x_0$ to $x$

$\therefore$ ~~$z = x + iy$ $t \in [$~~

$$\therefore z = t + iy_0 \quad t \in [x_0, x]$$

$$\frac{dz}{dt} = 1 + i0$$

$$\therefore dz = dt$$

change [illegible] t through f [illegible] axis [illegible] coordinate [illegible]
constant and to [illegible] const. [illegible]

Page No. | Date

On the 2nd line segment, $x$ is constant, & $y$ varies from $y_0$ to $y$.

We have, $z = x + it \qquad t \in [y_0, y]$

$$\therefore \frac{dz}{dt} = 0 + i(1)$$

$$\therefore dz = i\,dt$$

Thus, we have, $F(z) = \int_{C_1} f(z)\,dz$

$$= \int_{x_0}^{x} f(t + iy_0)\,dt + \int_{y_0}^{y} f(x + it) \cdot i\,dt$$

diff. with respect to $y$, partially,

$$\frac{\partial F}{\partial y} = 0 + i \frac{\partial}{\partial y} \int_{y_0}^{y} f(x+it)\,dt$$

$$= i f(x + iy)$$

$$= i f(z)$$

integral derivative vanishes & $[f(x+iy_0) - f(x+iy_0)]$ but $y_0$ = const hence derivative 0

$$f(z) = \frac{1}{i} \frac{\partial F}{\partial y} = -i \frac{\partial F}{\partial y} \quad \text{--- (2)}$$

Now

$$\cancel{f(z)} = F(z) = \int_{C_2} f(z)\,dz$$

On first line segment, $x$ is constant. & $y$ varies from $y_0$ to $y$.

Scanned by CamScanner

$\therefore z = x_0 + it \qquad t \in [y_0, y]$

$\therefore dz = i\,dt$

On second line segment, $y$ is constant & $x$ varies from $x_0$ to $x$.

$\therefore z = t + iy \qquad t \in [x_0, x]$

$\therefore dz = dt.$

$\therefore F(z) = \int_{C_2} f(z)\,dz.$

$$\Rightarrow \quad F(z) = \int_{y_0}^{y} f(x_0 + it)\,i\,dt + \int_{x_0}^{x} f(t + iy)\,dt$$

$$F(z) = \int_{x_0}^{x} f(t+iy)\,dt + i\int_{y_0}^{y} f(x_0 + it)\,dt$$

diff. with resp. to $x$, partially, then we get,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_{x_0}^{x} f(t+iy)\,dt$$

$$\frac{\partial F}{\partial x} = f(x+iy).$$

$$\therefore \frac{\partial F}{\partial x} = f(z) \quad \cdots \text{③}$$

$\therefore$ by ② & ③ we have,

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

These are C-R eqn's for F(z) at the point z in the disk

$\therefore$ F(z) is differentiable at the pt z.

Moreover,

$$F'(z) = \frac{\partial F}{\partial z}$$

$$\therefore F'(z) = f(z) \quad \text{at } z$$

But z is arbitrary point of disk.

$\therefore$ We have,

$$F'(z) = f(z) \quad \forall\ z \text{ in the disk } |z-z_0| \le r$$

Thus, there exist a differentiable function F(z) such that, $F'(z) = f(z)$ $\forall$ z in the disk $|z-z_0| \le r$.

$\therefore$ By fundamental theorem of integration, We have,

$$\int_C f(z)dz = F(z(b)) - F(z(a)) \quad \text{where } C: |z-z_0| = r$$

But c is a closed curve, therefore we have,

$$z(a) = z(b)$$ {initial & terminal pt of closed curve is same

$$\therefore \int_C f(z)dz = F(z(a)) - F(z(a))$$

$$\therefore \int_{|z-z_0| \le r} f(z)dz = 0.$$

H.P.

2015 10M

2016 10M * Cauchy's Main theorem:-

Statement:- If $f(z)$ is analytic in a simply connected domain D. and C is closed contour lying in D then

$$\int_C f(z)dz = 0$$

Proof: Given $f(z)$ is analytic function in simply connected domain D & C is any ~~simp~~ closed contour lying in D.

Clearly $f(z)$ is continuous function in D.

claim: TPT $\int_C f(z)dz = 0$.

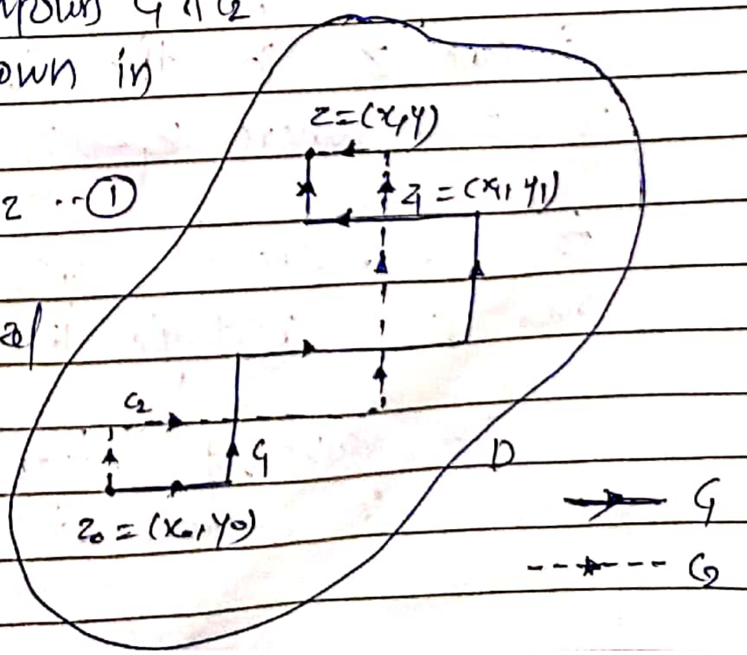
To prove this we will prove that there exist a function $F(z)$ such that $F'(z) = f(z)$ $\forall z$ in D.

Consider, ~~any~~ two points $z_0$ an arbitary point z in D. & suppose $z_0$ as some fixed point in D.

Construct the two contours $C_1$ & $C_2$ joining $z_0$ & z as shown in fig.

& define $F(z) = \int_{C_1} f(z)dz$ ..(1)

$C_1$ & $C_2$ are the polygonal lines ~~joi~~ having same initial & terminal points as shown in fig.




Scanned by CamScanner

Now put $c' = c_1 - c_2$

$\therefore$ $c'$ is closed curve in D.

Clearly,

$$\int_{c'} f(z)\,dz = 0$$

$\because$ integral over each rectangle is zero.

Thus,

$$\int_{c_1 - c_2} f(z)\,dz = 0$$

$$\therefore \int_{c_1} f(z)\,dz - \int_{c_2} f(z)\,dz = 0$$

$$\therefore \int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

$\therefore$ by ① we have,

$$F(z) = \int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz.$$

Suppose $z_1$ is the last point of intersection betn $z_0$ & $z$. where $z_1 = (x_1, y_1)$.

As $c_1$ & $c_2$ are the contours joining $z_0$ & $z_1$ & integral over each rectangle is again zero.

$\therefore$ The value of integral is same of $F(z)$ from $z_0 - z_1$ is same along $c_1$ & $c_2$

Scanned by CamScanner

Suppose K be the common value of this integral.

~~Now, for 1a~~

Now, in the last rectangle $c_1$ contains a horizontal line segment followed by a vertical line segment and $c_2$ contains a vertical line segment followed by a horizontal line segment as shown in fig.

Now we have,

$$F(z) = \int_{c_1} f(z)dz$$

$$= K + \int_{x_1}^{x} f(t+iy_1)dt + \int_{y_1}^{y} f(x+it)\, i\, dt \quad \cdots (2)$$

$$F(z) = \int_{c_2} f(z)dz.$$

$$= K + \int_{y_1}^{y} f(x_1+it)\, i\, dt + \int_{x_1}^{x} f(t+iy)dt \quad \cdots (3)$$

diff. (2) with respect to y partially,

$$\frac{\partial F}{\partial y} = 0 + 0 + i\frac{\partial}{\partial y}\int_{y_1}^{y} f(x+it)dt$$

$$= i f(x+iy)$$

$$\frac{\partial F}{\partial y} = i f(z)$$

$$\therefore f(z) = -i\frac{\partial F}{\partial y} \quad \cdots (4)$$

Now, diff ③ with respect to $x$ partially, then we get,

$$\frac{\partial F}{\partial x} = 0 + 0 + \frac{\partial}{\partial x}\int_{x_1}^{x} f(t+iy)\,dt$$

$$= f(x+iy)$$

$$= f(z)$$

$$\therefore f(z) = \frac{\partial F}{\partial x} \quad \text{--- ⑤}$$

From ④ & ⑤ we have,

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y} \qquad (\text{C-R eqn for } F(z))$$

$\therefore$ By theorem, $F(z)$ is differentiable at $z \in D$.

$$\therefore F'(z) = \frac{\partial F}{\partial x}$$

$$\therefore F'(z) = f(z) \qquad \text{at } z \in D.$$

But $z \in D$ is arbitrary

$\therefore$ We have,

$$F'(z) = f(z) \qquad \forall z \in D.$$

Thus, there exist a differentiable funn, $F(z)$ s.t,

$$F'(z) = f(z) \quad \forall z \in D.$$

∴ By ~~fundt~~ fundamental theorem of integration,
We have,

$$\int_{C_1} f(z)dz = f(z(b)) - f(z(a))$$

Where $C_1$ is an arbitrary curve.
Given $c$ is any closed curve in $D$.

∴ For $c$ $\quad z(b) = z(a)$
∴ We have,

$$\int_c f(z)dz = 0$$

Hence proved

Example Note:- $f(z) = \sin z$ is analytic function in $D = \mathbb{C}$ (complex plane)
& let $c : |z-2| = 5$.
then,

$$\int_c \sin z\, dz = 0.$$

Note:- $f(z) = 1/z$ is analytic everywhere in punctural plane (ie, except origin) $(\mathbb{C} - \{0\})$ & let $c$ be any closed curve in $\mathbb{C} - \{0\}$ then

$$\int_c \frac{dz}{z} = 0.$$

We can not apply Cauchy's Theorem, if $f(z) = 1/z$ is defined in complex plane.
Because $f(z)$ is not analytic everywhere in complex plane

# Note :→ Suppose, $f(z)$ is analytic in domain $D$ containing a closed disk $|z-z_0| < r$. and $z_1$ is a point lying inside disk. then.

$$\int_C \frac{dz}{z-z_1} = 2\pi i \quad (i). \qquad C : |z-z_0| \le \varepsilon.$$

# Cauchy's Integral formula :→

Let, $f(z)$ be analytic in simply connected domain $D$ containing a simple closed contour $c$ then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)} dz$$ Where, $z_0$ is a point lying inside $c$.

→ proof :→ Given, $f(z)$ is analytic in simply connected domain $D$ containing the simple closed contour $c$.

$z_0$ is the point lying inside $c$

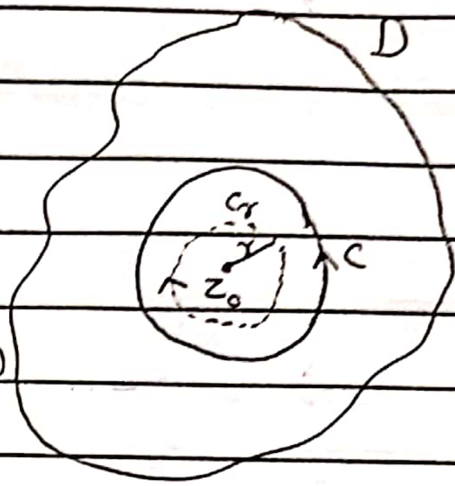
∴ the function.

$$g(z) = \frac{f(z)}{(z-z_0)}$$ is not analytic

at $z-z_0$.

Now, construct a circle $C_\gamma$ inside $c$ as,

$C_\gamma : |z - z_0| = \gamma$




Now, choose, $\epsilon > 0$ sufficiently small so that,

$|f(z) - f(z_0)| < \epsilon \;\; \forall \; z \text{ on } C_\gamma$ — ①

Now put,

$$c' = c - C_\gamma$$

clearely, the domain bounded by $c'$ is multiply connected domain which does not contain the point $z_0$

$\therefore$ $g(z)$ is analytic in the domain bounded by $c'$.

$$\therefore \int_{c'} g(z)\,dz = 0 \qquad (\because \text{ cauchy thm}).$$

$$\int_{c - C_\gamma} \frac{f(z)}{(z - z_0)}\,dz = 0$$

$$\int_c \frac{f(z)}{(z - z_0)}\,dz - \int_{C_\gamma} \frac{f(z)}{(z - z_0)}\,dz = 0.$$

$$\therefore \int_c \frac{f(z)}{(z - z_0)}\,dz = \int_{C_\gamma} \frac{f(z)}{(z - z_0)}\,dz \quad \text{— ②}$$

• Let,

$$\int_{C_r} \frac{f(z)}{(z-z_0)}\,dz = \int_{C_r} \frac{f(z)-f(z_0)+f(z_0)}{(z-z_0)}\,dz$$

$$= \int_{C_r} \frac{f(z)-f(z_0)}{z-z_0}\,dz + \int_{C_r} \frac{f(z_0)}{(z-z_0)}\,dz \quad \text{——— (3)}$$

Now, We have ~~the~~ to prove $1^{st}$ term on R.H.S. of $eq^n$ (3) tends to zero.

consider,

$$\left| \int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)}\,dz \right| \le \int_{C_r} \left| \frac{f(z)-f(z_0)}{(z-z_0)}\,dz \right|$$

$$\le \int_{C_r} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz|$$

$$< \epsilon \int_{C_r} \frac{|dz|}{|z-z_0|}$$

$$< \frac{\epsilon}{r} \int_{C_r} |dz| \qquad (\because \text{on } C_r, |z-z_0| = r).$$

$$< \frac{\epsilon}{r} \cdot \text{Length of } C_r$$

$$< \frac{\epsilon}{r} \cdot 2\pi r.$$

$$\left| \int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)}\,dz \right| < 2\pi\epsilon.$$

Scanned by CamScanner

hence, $\int_{C_r} \frac{f(z)-f(z_0)}{(z-z_0)} \to 0$ ($\because \epsilon$ is arbitrary small real no.).

$\therefore$ eqn (3) becomes,

$$\int_{C_r} \frac{f(z)}{(z-z_0)} dz = \int_{C_r} \frac{f(z_0)}{(z-z_0)} dz.$$

$$= f(z_0) \int_{C_r} \frac{1}{(z-z_0)} dz. \quad (\because f(z_0) \text{ is constant})$$

$$= f(z_0) \cdot 2\pi i \quad \left(\because \int_{C_r} \frac{dz}{(z-z_0)} dz = 2\pi i\right)$$

put this value in eqn (2) then we have,

$$\int_{C} \frac{f(z)}{(z-z_0)} dz = f(z_0)\, 2\pi i$$

$$\boxed{\therefore f(z_0) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z-z_0)} dz}.$$

Where, $c$ is a curve enclosing $z_0$.

Scanned by CamScanner

(2) Evaluate, $\int_C \frac{z^2+2}{z^2-6z+8}\, dz$ , $C : |z| = 3$

Given, $C : |z| = 3$.

→ and $\int_C \frac{z^2+2}{z^2-6z+8}\, dz$.

Now, $z^2-6z+8 = 0$

$\Rightarrow (z-4)(z-2) = 0$

$\Rightarrow z = 4,\ z = 2$.

here, $z=4$ Lies outside $C$ and $z=2$ Lies inside $C$.

$\therefore z_0 = 2$.

We have,

$$\frac{z^2+2}{z^2-6z+8} = \frac{z^2+2}{(z-4)(z-2)}$$

$$= \frac{z^2+2/(z-4)}{(z-2)}.$$

$$= \frac{f(z)}{(z-2)}.$$

$$\therefore f(z) = \frac{z^2+2}{z-4}.$$

$\therefore$ By cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{|z|=3} \frac{z^2+2/z-4}{z-2}$$

$$\int_{|z|=3} \frac{z^2+2/z-4}{(z-2)} = 2\pi i \times \frac{4+2}{2-4}$$

$$= 2\pi i \times \frac{6}{-2} = \underline{-6\pi i}$$

Scanned by CamScanner

① # problems :- Given, c is defined as $|z|=3$ then find $\int_c \frac{e^z}{z+2}\,dz$.

→ Given, $\int_c \frac{e^z}{z+2}\,dz$, and $c: |z|=3$

We have, $z - z_0 = z+2$.

$\Rightarrow z_0 = -2$.

$\therefore z_0$ lies inside c.

Also, We have,

$f(z) = e^z$

$\therefore$ By cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-z_0)}\,dz.$$

$$\int_c \frac{f(z)}{(z-z_0)}\,dz = 2\pi i\, f(z_0).$$

$$\int_{|z|=3} \frac{e^z}{z+2}\,dz = f(-2).2\pi i$$

$$= e^{-2} \times 2\pi i$$

Scanned by CamScanner

3) Evaluate,

$$\int_C \frac{e^z}{z-2}\,dz \qquad c : |z| = 3.$$

→ Here, $c : |z| = 3$ and $\int_C \frac{e^z}{z-2}\,dz.$

We have, $z - z_0 = z - 2.$

$\Rightarrow z_0 = 2$

$\therefore$ $z_0$ Lies inside the circle $(c : |z| = 3)$

Also, We have, $f(z) = e^z.$

$\therefore$ By cauchy integral formula, We have,

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)}\,dz.$$

$$\int_C \frac{f(z)}{z - z_0}\,dz = f(z_0)\, 2\pi i$$

$$\int_C \frac{e^z}{z-2}\,dz = f(2) \cdot 2\pi i$$

$$\boxed{\int_C \frac{e^z}{z-2}\,dz = e^2 \times 2\pi i.}$$

#

4) Evaluate, $\int_C \frac{1}{z^2-1} dz$. $C: |z| = 2$.

→ Given, $C: |z|=2$ and $\int_C \frac{1}{z^2-1} dz$.

consider,

$$\frac{1}{(z^2-1)} = \frac{1}{(z-1)(z+1)}.$$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{(z-1)} + \frac{B}{(z+1)}.$$

$$= \frac{A(z+1) + B(z-1)}{(z-1)(z+1)}.$$

$$1 = A(z+1) + B(z-1)$$

$A = \frac{1}{2}$ and $B = \frac{-1}{2}$.

$$\therefore \frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

$$\int_C \frac{1}{z^2-1} dz = \int \frac{1}{2(z-1)} - \frac{1}{2(z+1)} dz$$

$$= \frac{1}{2} \int \frac{1}{(z-1)} dz - \frac{1}{2} \int \frac{1}{(z+1)} dz$$

Here, $z - z_0 = z-1$ and $z - z_0 = z+1$

$\Rightarrow z_0 = 1$ and $z_0 = -1$

and $f(z) = 1$

By cauchy integration formula, We have,

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)} dz$$

$$\int_C \frac{f(z)}{(z-z_0)} dz = 2\pi i \, f(z_0)$$

$$\therefore \int_C \frac{dz}{z^2-1} = \frac{2\pi i}{2} f(1) - \frac{2\pi i}{2} f(-1).$$

$$= \frac{2\pi i}{2} - \frac{2\pi i}{2}$$

$$= 0.$$

$$\boxed{\therefore \int_C \frac{1}{z^2-1} dz = 0.}$$

# Cauchy's Generalised integral formula :-

Let, f(z) be analytic in a simply connected domain D containing a simple closed contour c and $z_0$ is a point lying inside c then f(z) has derivatives of all orders at a point $z_0$ in c Moreover,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

→ Given, f(z) is analytic in simply connected domain D containing a simple closed contour C and $z_0$ is a point lying inside C.

∴ By cauchy integral formula for the point $z_0+h$, We have,

$$f(z_0+h) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-(z_0+h)} dz.$$

$$= \frac{1}{2\pi i} \int_c \frac{f(z)}{z-z_0-h} dz \quad \text{—— (2).}$$

Now consider,

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{h}\left[\frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0-h} dz - \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0} dz\right]$$

$$= \frac{1}{2\pi i h} \int_C f(z) \left[ \frac{1}{z-z_0-h} - \frac{1}{z-z_0} \right] dz$$

$$\frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i h} \int_C f(z) \left[ \frac{z-z_0-z+z_0+h}{(z-z_0)(z-z_0-h)} \right] dz$$

$$= \frac{1}{2\pi i \cancel{h}} \int_C f(z) \frac{\cancel{h}}{(z-z_0)(z-z_0-h)} dz$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} dz$$

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \frac{1}{2\pi i} \lim_{h\to 0} \left[ \int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} dz \right] \quad (3)$$

Now, consider.

$$\left| \int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} dz - \int_C \frac{f(z)}{(z-z_0)^2} dz \right|$$

$$\leq \int_C \left| \frac{f(z)}{(z-z_0)(z-z_0-h)} - \frac{f(z)}{(z-z_0)^2} \right| |dz|$$

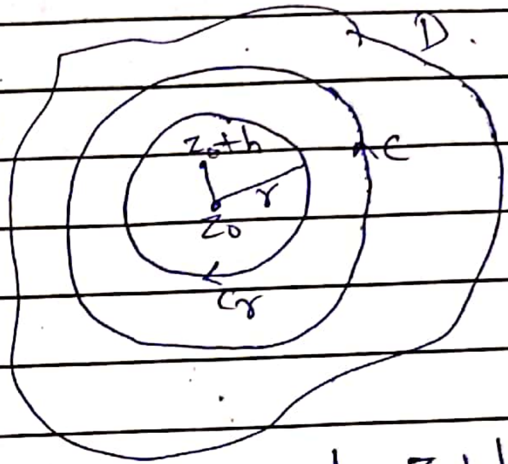
$$\leq \int_C |f(z)| \left| \frac{1}{(z-z_0)(z-z_0-h)} - \frac{1}{(z-z_0)^2} \right| |dz|$$

$$\leq \int_C |f(z)| \left| \frac{(z-z_0)-(z-z_0-h)}{(z-z_0)^2(z-z_0-h)} \right| |dz| \quad (4)$$

Now, construct a circle $C_r : |z-z_0|=r$ lying inside $C$ (choose $r$ such that $|h| \le r/2$) i.e. both the points $z_0$ and $z_0+h$ lie inside $C_r$ as shown in fig.

Now define,

$$c' = C - C_r$$




clearely the domain bounded by $c'$ is multiply connected domain.
(Annulus betn $c$ and $C_r$)
which does not contain both the points $z_0$ and $z_0+h$.

$\therefore$ By cauchy thm for multiply connected domain, we have,

$$\int_{c'} f(z) \left( \frac{(z-z_0)-(z-z_0-h)}{(z-z_0)^2 (z-z_0-h)} \right) dz = 0$$

$$\int_{C-C_r} f(z) \left[ \frac{h}{(z-z_0)^2 (z-z_0-h)} \right] dz = 0.$$

$$\therefore \int_{C} f(z) \left[ \frac{h}{(z-z_0)^2 (z-z_0-h)} \right] dz = \int_{C_r} \frac{f(z)\, h}{(z-z_0)^2 (z-z_0-h)} dz$$

$$\int \frac{|h|\,|f(z)|}{|z-z_0|^2 |z-z_0-h|} |dz| = \int_{C_r} \frac{|h| \cdot |f(z)|}{|z-z_0|^2 |z-z_0-h|} |dz|.$$

$$\le \frac{M|h|}{r^2} \int_{C_r} \frac{|dz|}{|z-z_0-h|}.$$

on $C_r$, $|z-z_0| = r$ and by maximum modulus theorem $|f(z)| \le M$.

Now,

$$\frac{1}{|z-z_0-h|} \le \frac{1}{|z-z_0|-|h|}$$

$$\le \frac{1}{r - r/2} \qquad |h| \le r/2 \text{ and } |z-z_0| = r.$$

$$\le \frac{1}{r/2}.$$

$$\therefore \frac{1}{|z-z_0-h|} \le \frac{1}{r/2}.$$

$$\therefore \int_c \frac{|h|.|f(z)|}{|z-z_0|^2 . |z-z_0-h|} |dz| \le \frac{M|h|}{r^2 \times r/2} \int_{C_r} |dz|$$

$$\le \frac{2M.|h|}{r^3} \int_{C_r} |dz|$$

$$\le \frac{2M.|h|}{r^3} 2\pi r.$$

$$\int_c \frac{|h|.|f(z)|}{|z-z_0|^2 . |z-z_0-h|} |dz| \le \left(\frac{2M}{r^2}\right) |h|. 2\pi.$$

put this value in eq$^n$ 4, we get

$$\left| \int_c \frac{f(z)}{(z-z_0)(z-z_0-h)} dz - \int_c \frac{f(z)}{(z-z_0)^2} dz \right|$$

$$\le \frac{2M}{r^2} |h|. 2\pi.$$

As $h \to 0$, $|h| \to 0$.

$\therefore$ By above inequality we have,

$$\left| \int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} dz - \int_C \frac{f(z)}{(z-z_0)^2} dz \right| \to 0$$

as $h \to 0$.

i.e.

$$\lim_{h \to 0} \int_C \frac{f(z)}{(z-z_0)(z-z_0-h)} dz = \int_C \frac{f(z)}{(z-z_0)^2} dz.$$

put this value in eq$^n$ 3 we get.

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$$

$$\therefore f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz.$$

Thus, the 1$^{st}$ order derivetive of $f(z)$ exist at the point $z_0$.

Illy, we can prove that all the derivatives of $f(z)$ exist at the point $z = z_0$.

Moreover,

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz}$$

Ex :-) Evaluate, $\int_c \frac{z-3\cos z}{(z-\pi/2)^2}\, dz$ , $c : |z| = 2$

→ Given, $f(z) = z - 3\cos z$.

$z_0 = \pi/2$.

$n = 1$

clearely, $z_0 = \pi/2$ lying inside c.

∴ By cauchy generalized integral formula, we have,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

$$\int_c \frac{z-3\cos z}{(z-\pi/2)^2} = \frac{2\pi i\, f'(\pi/2)}{1}$$

$$= 2\pi i\, f'(\pi/2)$$

$f(z) = z - 3\cos z$.

$f'(z) = 1 + 3\sin z$.

$= 1 + 3\sin(\pi/2)$

$f'(\pi/2) = 4$

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz = 2\pi i \cdot 4$$

$$= 8\pi i.$$

2) Evaluate,

$$\int_c \frac{3z^4+3z-6}{(z-2)^3}\,dz \quad , \quad c: |z| = 3.$$

→ Given, $f(z) = 3z^4+3z-6$

$z_0 = 2.$

$n = 2.$

clearly, $z_0 = 2$ Lies inside c.

∴ By cauchy generalized integral formula, we have,

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i\, f^{(n)}(z_0)}{n!}$$

$$\int_c \frac{3z^4+3z-6}{(z-2)^3}\,dz = \frac{2\pi i\, f''(2)}{2!}$$

$$= \frac{2\pi i\, f''(2)}{2}.$$

here, $f(z) = 3z^4+3z-6.$

$f'(z) = 12z^3+3.$

$f''(z) = 36z^2$

$f''(2) = 36\times4$

$= 144$

$$\boxed{\int_c \frac{3z^4+3z-6}{(z-2)^3}\,dz = 144\pi i}$$

3) Evaluate: $\int_{|z|=3} \frac{e^z \cdot \sin z}{(z-2)^2} dz$

→ Here, $\int_{|z|=3} \frac{e^z \cdot \sin z}{(z-2)^2} dz$

$f(z) = e^z \sin z$

$z_0 = 2$

and $n = 1$.

Here, $z_0 = 2$. is the point lies inside the circle.

By cauchy's generalized integral formula, we have,

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

We have,

$f(z) = e^z \sin z$

$f'(z) = e^z \cdot \cos z + \sin z \, e^z$

$= e^z (\sin z + \cos z)$.

$$\boxed{\int_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{1!} e^2 (\sin 2 + \cos 2)}$$

4)# Evaluate, $\int_c \frac{z}{(16-z^4)(z+i)} dz$

i) $c: |z| = 2$ 2) $|z-4| = 2$ iii) $|z+4| = 2$

4) $|z| = \frac{1}{2}$ 5) $|z| = 5$

5) Evaluate: $\int_C \frac{e^{z^2}}{(z-2)^2}\,dz$ $c: |z|=3$.

→ Given, $c: |z|=3$, and $\int_C \frac{e^{z^2}}{(z-2)^2}\,dz$.

compare this $eq^n$. with,

$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$

$\Rightarrow f(z) = e^{z^2}$, $z_0 = 2$, and $n=1$

Here, $f(z) = e^{z^2}$.

$\therefore$ $f'(z) = 2ze^{z^2}$.

We know that,

cauchy's generalized integral formula is,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

$$\therefore \int_C \frac{e^{z^2}}{(z-2)^2}\,dz = \frac{2\pi i}{1!} f'(z_0)$$

$$= 2\pi i\, f'(2)$$

$$= 2\pi i\, 4 \cdot e^4$$

$$= 8\pi i e^4.$$

$$\boxed{\therefore \int_C \frac{e^{z^2}}{(z-2)^2}\,dz = 8\pi i e^4}$$