# UNIT - III

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## Chapter: Analytic function

**Definition:** Let $g$ be a function of two real variables $x$ & $y$ then we say that

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = L$$

If for every $\epsilon>0$ $\exists$ $\delta>0$ such that

$$|g(x,y)-L| < \epsilon \text{ whenever } 0<d[(x,y)-(x_0,y_0)] < \delta$$

i.e, $|g(x,y)-L| < \epsilon$ whenever $0<\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

**Definition (Continuity):**

The function $g(x,y)$ is said tobe continuous at the point $(x_0,y_0)$ if

i] $g(x,y)$ is defined at $(x_0,y_0)$

ii] $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} g(x,y)$ exist.

iii] $\lim\limits_{\substack{x\to x_0 \\ y\to y_0}} g(x,y) = g(x_0,y_0)$

i.e, for every $\epsilon>0$ there exist $\delta>0$ such that,

$$|g(x,y)-g(x_0,y_0)| < \epsilon \quad \text{whenever } 0<\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$$

**Definition (Continuity of Complex function $f(z)$):-**

The function $f(z)$ is said tobe continuous at the point $z_0=(x_0,y_0)$ if,

i] $f(z)$ is defined at $z_0$

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ii] $\lim_{z \to z_0} f(z)$ is exist

iii] $\lim_{z \to z_0} f(z) = f(z_0)$

i.e, for every $\epsilon > 0$ there exist $\delta > 0$ such that,

$|f(z) - f(z_0)| < \epsilon$ whenever $0 \leq |z - z_0| < \delta$.

Theorem: P-7. the function $f(z) = u(x, y) + iv(x, y)$ is continuous at the point $z_0$ iff $u(x, y)$ & $v(x, y)$ are continuous at $(x_0, y_0)$

Proof:- Given function is, $f(z) = u(x, y) + iv(x, y)$

Claim: i] $f(z)$ is continuous at $z_0$ $\Rightarrow$ u and v are continuous at $(x_0, y_0)$

ii] u & v are cont. at $(x_0, y_0)$ $\Rightarrow$ $f(z)$ is cont. at $z_0$

ii] ~~Con~~ Suppose $u(x, y)$ & $v(x, y)$ are continuous at $(x_0, y_0)$ where $z_0 = (x_0, y_0)$

As $u(x, y)$ is cont. at $(x_0, y_0)$

$\therefore$ By definition

for some $\epsilon > 0$ $\exists$ $\delta_1 > 0$ such that,

$|u(x, y) - u(x_0, y_0)| < \epsilon/2$ whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_1$ ... (1)

Again,

$|v(x, y) - v(x_0, y_0)| < \epsilon/2$ whenever $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta_2$

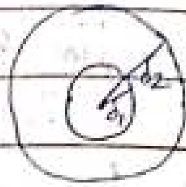
$v(x, y)$ is continuous at $(x_0, y_0)$

$\therefore$ By definition

for some $\epsilon > 0$ $\exists$ $\delta_2 > 0$ s.t.

$|v(x,y)-v(x_0,y_0)| < \epsilon/2$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta_2$ --- ②

choose $\delta = \max(\delta_1, \delta_2)$



$\therefore$ By ① & ② we have,

$|u(x,y)-u(x_0,y_0)| < \epsilon/2$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ -- ③

& $|v(x,y)-v(x_0,y_0)| < \epsilon/2$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ -- ④

Consider,

$$|f(z)-f(z_0)| = |u(x,y)+iv(x,y)-u(x_0,y_0)-iv(x_0,y_0)|$$
$$= |[u(x,y)-u(x_0,y_0)]+i[v(x,y)-v(x_0,y_0)]|$$
$$\leq |u(x,y)-u(x_0,y_0)|+|i|\,|v(x,y)-v(x_0,y_0)|$$
$$< \frac{\epsilon}{2}+\frac{\epsilon}{2} \quad \because \text{By ③ \& ④}$$

$\therefore$ $|f(z)-f(z_0)| < \epsilon$ whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ -- ⑤

$\therefore$ $|f(z)-f(z_0)| < \epsilon$ whenever $|z-z_0| < \delta$

$\therefore$ By defn, $f(z)$ is continuous at $z=z_0$.

ii]

Conversely, suppose that ~~f(z) is continuous at z₀=(x₀,y₀)~~
$f(z)$ is continuous at $z_0=(x_0,y_0)$

$\therefore$ By defn, for every $\epsilon>0$ there exist a $\delta>0$ s.t.

$|f(z)-f(z_0)| < \epsilon$ whenever $|z-z_0| < \delta$

$\Rightarrow |u(x,y) + iv(x_0,y_0) - u(x_0,y_0) + iv(x_0,y_0)| < \epsilon$

$\Rightarrow$ ~~$|[u(x,y) - u(x_0,y_0)] + [i(v(x,y) \pm v$~~

$\Rightarrow |[u(x,y) - u(x_0,y_0)] + [i(v(x,y) - v(x_0,y_0))]| < \epsilon \cdots (6)$

$\Rightarrow$ if $z = x+iy$

then we have, $|x| \leq |z|$

i.e., $|x| \leq |x+iy|$

$\therefore$ By using this inequality, we have,

$|u(x,y) - u(x_0,y_0)| \leq |[u(x,y) - u(x_0,y_0)] + [i[v(x,y) - v(x_0,y_0)]]|$

$\therefore |u(x,y) - u(x_0,y_0)| < \epsilon$ whenever $|z - z_0| < \delta$

i.e., $|u(x,y) - u(x_0,y_0)| < \epsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

$\therefore$ By definition, $u(x,y)$ is continuous at $(x_0,y_0)$

Similarly, by the inequality $|y| \leq |z|$ we get

$|v(x,y) - v(x_0,y_0)| < \epsilon$ whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

i.e., $v(x,y)$ is continuous at $(x_0,y_0)$

$\Rightarrow f(z)$ is continuous at $z_0 \Leftrightarrow u$ & $v$ are cont. at $(x_0,y_0)$

H.P.

Problem: Discuss the continuity of the function $f(x,y) = \frac{xy}{x^2+y^2}$,

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0)$$

at $(0,0)$

Solution:- Given function is

$$f(x,y) = \frac{xy}{x^2+y^2}, \quad (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0)$$

Clearly $f(x,y)$ is defined at $(0,0)$

Now we have to find $\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y)$

i] Suppose $(x,y) \to (0,0)$ i.e, $\Rightarrow x \to 0, y \to 0$ along $x$-axis.

On the $x$-axis, $y=0$

$\therefore f(x,y) = f(x,0)$.

$$\therefore \lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \lim_{x\to 0} \left. \frac{xy}{x^2+y^2} \right|_{y=0}$$

$$= \lim_{x\to 0} \frac{x\cdot 0}{x^2+0}$$

$$= 0$$

2] Suppose $(x,y) \to (0,0)$ i.e, $x \to 0, y \to 0$ along y-axis.

On the y-axis, $x=0$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{y\to 0} \left.\frac{xy}{x^2+y^2}\right|_{x=0}$$

$$= \lim_{y\to 0} \frac{0}{x^2+y^2}$$

$$= 0$$

3] Suppose $x\to 0, y\to 0$ along the lines $y=mx$, $m\in\mathbb{R}$

then we have

$$f(x,y) = f(x,mx)$$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ mx\to 0}} f(x,mx)$$

$$= \lim_{x\to 0} \left.\frac{xy}{x^2+y^2}\right|_{y=mx}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2(1+m^2)}$$

Hence $f(x,y)$ is not cont. at $(0,0)$

$$= \frac{m}{1+m^2}$$

$\therefore$ limit depends ~~up~~ on values of $m$. Hence limit has different values for various values of $m$

H.W.

Problem: $f(x,y) = \dfrac{x^2y^2}{(x+y^2)^3}$ $(x,y) \neq (0,0)$

$= 0$ $(x,y) = (0,0)$

Solution:- Given function is

$$f(x,y) = \frac{x^2y^2}{(x+y^2)^3} \quad (x,y) \neq (0,0)$$

$$= 0 \quad (x,y) = (0,0)$$

Clearly $f(x,y)$ is defined at $(0,0)$
Now we have to find

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y)$$

1) Suppose $(x,y) \to (0,0)$ i.e, $x \to 0$, $y \to 0$ along x-axis
On the x-axis $y=0$

$\therefore f(x,y) = f(x,0)$

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{x \to 0} \left. \frac{x^2y^2}{(x+y^2)^3} \right|_{y=0}$$

$$= \frac{x^2 \cdot 0^2}{(x^2+0^2)^3}$$

$$= 0$$

2) Suppose $(x,y) \to (0,0)$ i.e, $x \to 0$, $y \to 0$ along y-axis
On the y-axis $x=0$

$\therefore f(x,y) = f(0,y)$.

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{y\to 0} \left.\frac{x^2y^2}{(x^2+y^2)^3}\right]_{x=0}$$

$$= \frac{0\cdot y}{(0+y^2)^3}$$

$$= 0$$

3) Suppose $x\to 0$, $y\to 0$ along the lines $y=mx$, $m\in R$ then we have,

$$f(x,y) = f(x, mx)$$

$$\therefore \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ mx\to 0}} f(x, mx)$$

$$= \lim_{x\to 0} \left.\frac{x^2y^2}{(x+y^2)^3}\right]_{y=mx}$$

$$= \lim_{x\to 0} \frac{x^2(m^2x^2)}{(x+m^2x^2)^3}$$

$$= \lim_{x\to 0} \frac{m^2x^4}{x^3(1+m^2x)^3}$$

$$= \lim_{x\to 0} \frac{m^2x}{(1+m^2x)^3}$$

$$= 0$$

4] suppose $x\to 0$, $y\to 0$ along the st. line. But along parabola $x=my^2$ $(m\neq 0)$ or $y^2=4ax$.

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} f(my^2, y)$$

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$$= \lim_{y \to 0} \frac{m^2 y^4 y^2}{(m^2 y^2 + y^2)^3} = \frac{m^2 y^6}{y^6 (m^2+1)^3} = \frac{m^2}{(m^2+1)^3}$$

$$= \frac{m^2}{(m^2+1)^3}$$

Hence, limit depends on values of m. Hence limit has different values for various values of m

Hence f(x,y) does not exist at $\lim_{\substack{x \to 0 \\ y \to 0}}$

Problem: Show that the fn $f(x,y) = \frac{x^3 - 2y^3}{x^2+y^2}$ where $(x,y) \neq (0,0)$

$$f(x,y) = \frac{x^3-2y^3}{x^2+y^2} \qquad (x,y) \neq (0,0)$$
$$= 0 \qquad (x,y) = (0,0)$$

is constant at origin.

Solution :- Given, $f(x,y) = \frac{x^3 - 2y^3}{x^2+y^2} \qquad (x,y) \neq (0,0)$

$$= 0 \qquad (x,y) = (0,0)$$

is clearly here $f(0,0) = 0$.

We will prove that,

$$|f(x,y) - f(0,0)| < \epsilon \quad \text{whenever} \quad \sqrt{x^2+y^2} < \delta$$

ie, $\left|\frac{x^3-2y^3}{x^2+y^2}\right| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$

Consider, $|x^3-2y^3| \le |x^3| + |2y^3|$

$\le |x|^3 + 2|y|^3$

$\le |x| \cdot x^2 + 2|y|y^2 \quad \dots (1)$

Clearly, $x^2 \le x^2+y^2 \quad (\because y^2 \geq 0)$

$\Rightarrow |x| \le \sqrt{x^2+y^2}$

Similarly, $y^2 \le x^2+y^2 \quad x^2 \geq 0$

$\Rightarrow |y| \le \sqrt{x^2+y^2}$

$\therefore$ eq (1) becomes,

$|x^3-2y^3| \le (\sqrt{x^2+y^2})x^2 + 2(\sqrt{x^2+y^2})y^2$

$\le (\sqrt{x^2+y^2})2x^2 + (\sqrt{x^2+y^2})2y^2$

$|x^3-2y^3| \le 2[\sqrt{x^2+y^2}](x^2+y^2) \quad \dots (2)$

Now consider,

$|f(x,y)| = \left|\frac{x^3-2y^3}{x^2+y^2}\right|$

$= \left|\frac{x^3-2y^3}{x^2+y^2}\right|$

$$\leq \frac{2\sqrt{x^2+y^2}\,(x^2+y^2)}{x^2+y^2} \quad \because \text{ by } ②$$

$$\leq 2\sqrt{x^2+y^2}$$

$$\leq 2\delta \qquad \text{whenever } \sqrt{x^2+y^2} < \delta$$

$|f(x,y) - 0| < \epsilon$ whenever $\epsilon = 2\delta$

$\therefore$ By definition $f(x,y)$ is continuous at $(0,0)$ moreover

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Note:- The function $f(z) = \frac{x^3-2y^3}{x^2+y^2} + i0$

is continuous at $z=0$ because both the function

$$u(x,y) = \frac{x^3-y^3}{x^2+y^2} \text{ \& } v(x,y) = 0$$

at constant at $(0,0)$

## * Differentiable function:-

A function $f$ is said tobe differentiable at pt $z$ if $\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$ exist.

Moreover,

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

**Note:-** ① Here $h \to 0$ along every possible path passing through that point.

② Here $h = h_1 + ih_2$

$\therefore h \to 0 \Rightarrow h_1 + ih_2 \to 0 + i0$

$\Rightarrow h_1 \to 0,\ h_2 \to 0$

Where, $h$ is small change in $z$.

$h_1$ is small change in $x$

$h_2$ is small change in $y$

③ The functions $f(z) = z^n$, $n = 0, 1, 2 \ldots$ $\sin z$, $\cos z$, $e^z$, polynomials and constant function are every where differentiable.

④ Every differentiable function is continuous converse may not be true.

**Example:** e.g. The function $f(z) = |z|$ is continuous at origin but it is not differentiable at origin.

**Solution:-** Given $f(z) = |z|$

$$= \sqrt{x^2+y^2} + i0$$

Here $u(x,y) = \sqrt{x^2+y^2}$ & $v(x,y) = 0$

$u(x,y)$ is defined at $(x,y) = (0,0)$.

Here $u(0,0) = 0$.

$$|u(x,y) - u(0,0)| = |\sqrt{x^2+y^2} - 0| \quad ; \text{ whenever } \sqrt{x^2+y^2} < \delta.$$

$= \sqrt{x^2+y^2}$

$< \delta = \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$

$|u(x,y) - u(0,0)| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$.

$\therefore$ $u(x,y)$ is continuous at $(0,0)$
& $v(x,y)$ is constant function.
$\therefore$ $v(x,y)$ is continuous at $(0,0)$.
$\therefore$ By theorem,

$f(z) = u+iv$ is continuous at $z=0$

ie, $\lim\limits_{h \to 0} \frac{f(z+h)-f(z)}{h}$ does not exist.

let $h = h_1 + h_2$

Case (I): Consider $h \to 0$ along +ve $x$-axis.
On the $x$-axis $y=0$
ie, if $h = h_1 + ih_2$ then on the $x$-axis $h_2 = 0 \Rightarrow h = h_1$
Hence, here $h \to 0 \Rightarrow h_1 \to 0$

Now consider,

$$\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{f(h)-f(0)}{h} \quad \cdots (1)$$

We have,

$$f(z) = f(x,y) = \sqrt{x^2+y^2}$$
$$f(0) = f(0,0) = \sqrt{0+0} = 0$$

$f(h) = f(h_1, h_2) = \sqrt{h_1^2 + h_2^2}$

In this case $h_2 = 0$ $\therefore f(h) = \sqrt{h_1^2} = |h_1|$

$\therefore$ We have,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h_1\to 0} \frac{|h_1|-0}{h_1} \quad \because h = h_1$$

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h_1\to 0} \frac{|h_1|}{h_1}$$

Here $h \to 0$ i.e, $h_1 \to 0$
Along +ve x-axis
$\therefore h_1 > 0$ $\therefore |h_1| = h_1$
& hence we have,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h_1\to 0} \frac{h_1}{h_1} = 1$$

Case (II): Suppose $h \to 0$ along -ve x-axis

$$\therefore \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h_1\to 0} \frac{|h_1|}{h_1}$$

As $h \to 0$ i.e, $h_1 \to 0$ along -ve axis
$\therefore h_1 < 0$ $|h_1| = -h_1$
Hence we have,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{-h_1}{h_1}$$

$$= -1$$

$\therefore \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ are different along different directions.

$\therefore \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist.

$\therefore$ $f(z)$ is not differentiable $z=0$

$\therefore f(z)=|z|$ is continuous at $z=0$
But it is not differentiable at $z=0$.

Note:- ① The function $f(z)=\bar{z}$, $Re\,z$, $Im\,z$ are no where differentiable function.

② $f(z)=|z|$ is no where differentiable function.

③ $f(z)=|z|^2$ is differentiable only at origin.

④ $f(z)=1/z$ $(z\neq 0)$ is differentiable ~~only at~~ every-where except origin.

⑤ If $f(z)=z^n$ then $f'(z)=nz^{n-1}$.
suppose $g(z)$ any other differentiable function then we have :

(i) $\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$

(ii) $\frac{d}{dz}[f(z)\cdot g(z)] = f(z)\cdot g'(z)+f'(z)\cdot g(z)$.

(iii) $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\cdot f'(z)-f(z)g'(z)}{(g(z))^2}$

(6) If $g$ is differentiable at $z$ and if $f$ is differentiable at $g(z)=w$ then,

$$f(z) = (f \circ g)(z) = f[g(z)]$$

is differentiable at $z$.
Moreover,

$$f'(z) = f'(g(z)) \cdot g'(z).$$

* Neccessary Condition for the differentiability of the function $f(z)$.

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**Theorem:** P.T. the necessary condition for the differentiability of the function $f(z)=u+iv$ is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Where $u(x,y)$ & $v(x,y)$ are continuous at the pt $z$

(OR)

If $f(z)=u+iv$ is differentiable at the pt $z$ then partials of $u$ & $v$ satisfies the C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ at } z.$$

**Proof:-** Given $f(z) = u(x,y) + iv(x,y)$ is differentiable at the point $z$.

Hence $f(z) = u(x,y) + iv(x,y)$ is continuous at pt $z$.

$\therefore$ $u$ & $v$ are continuous at $(x,y)$.

As $f(z)$ is differentiable at $z$

$\therefore$ By definition, we have

$$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} = f'(z)$$

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Here $z = x + iy = (x, y)$
let $h = h_1 + ih_2$

$\therefore z+h = x+iy+h_1+ih_2$
$= (x+h_1) + i(y+h_2)$

$\therefore z+h = (x+h_1, y+h_2)$

Now,
$f(z) = u(x,y) + iv(x,y)$
$\therefore f(z+h) = u(x+h_1, y+h_2) + iv(x+h_1, y+h_2)$

$\therefore f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$

$$\therefore f'(z) = \lim_{h\to 0}\left[\frac{u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y)}{h}\right] \quad \cdots (1)$$

Case (I):
Suppose $h \to 0$ along $x$-axis.
On the $x$-axis, $y$-coordinate is zero i.e, $h_2 = 0$
$\therefore h = h_1 \quad \therefore$ as $h \to 0$, $h_1 \to 0$

$\therefore$ eqn (1) becomes,

$$f'(z) = \lim_{h_1\to 0}\left[\frac{u(x+h_1, y) + iv(x+h_1, y) - u(x,y) - iv(x,y)}{h_1}\right]$$

$$f'(z) = \lim_{h_1\to 0}\left[\frac{u(x+h_1, y) - u(x,y)}{h_1} + i\,\frac{v(x+h_1, y) - v(x,y)}{h_1}\right]$$

$$f'(z) = \lim_{h_1 \to 0}\left[\frac{u(x+h_1, y) - u(x,y)}{h_1}\right] + i \lim_{h_1 \to 0}\left[\frac{v(x+h_1,y) - v(x,y)}{h_1}\right].$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{at point } z \quad \cdots (2)$$

Case (II): Suppose $h \to 0$ along y-axis.
On the y-axis, x-coordinate is zero
ie, $h_1 = 0$

$\therefore h = ih_2 \quad \therefore h \to 0 \Rightarrow ih_2 \to 0 \Rightarrow h_2 \to 0$ ($\because i$ constant)

$\therefore$ (1) becomes,

$$f'(z) = \lim_{h_2 \to 0}\left[\frac{u(x, y+h_2) + iv(x, y+h_2) - u(x,y) - iv(x,y)}{ih_2}\right]$$

$$= \frac{1}{i}\lim_{h_2 \to 0}\left[\frac{u(x, y+h_2) - u(x,y)}{h_2} + i\frac{v(x,y+h_2) - v(x,y)}{h_2}\right]$$

$$= \frac{1}{i}\lim_{h_2 \to 0}\left[\frac{u(x,y+h_2) - u(x,y)}{h_2}\right] + \frac{1}{i}\lim_{h_2 \to 0} i\left[\frac{v(x,y+h_2) - v(x,y)}{h_2}\right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \because \frac{1}{i} = -i$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

$$\therefore f'(z) = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) \quad \text{at the pt } z \cdots (3)$$

$\therefore$ By ② & ③ We have,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(-\frac{\partial u}{\partial y}\right) \quad \text{at the pt } z.$$

$\therefore$

We have, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ at pt $z$.

Hence $f(z)$ is differentiable at $z$.

$\Rightarrow$ C-R equations hold at the point $z$.

H.P.

Note:- ① The equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

ie, $u_x = v_y$ & $v_x = -u_y$ are called as Cauchy-Riemann equations.

② $f(z)$ is differentiable at $z$ $\Rightarrow$ C-R equations hold at $z$.

equivalently,

C-R equations does not hold at $z$ $\Rightarrow$ $f(z)$ is not differentiable at $z$.

Example: Show that $f(z) = \bar{z}$ is no where differentiable function.

Solution:- Given $f(z) = \bar{z}$

$\therefore$ $u + iv = x - iy$

$\therefore$ $u = x$ & $v = -y$

$\frac{\partial u}{\partial x} = 1$ $\quad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad$ $\frac{\partial v}{\partial y} = -1$

As $|z| \to 1$

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

∴ C-R equations does not hold for all z.

∴ f(z) is not differentiable for all z.

∴ f(z) is no where differentiable function.

Note: ③ Converse of the theorem is not true.

i.e, C-R equations holds at $z_0 \not\Rightarrow f(z)$ is differentiable at $z_0$

Ex: The function $f(z) = \frac{xy^2}{x^2+y^2} + i.0$ whenever $z \neq 0$

$= 0$ whenever $z = 0$

is continuous at origin. Moreover the C-R equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ holds at $z=0$

but f(z) is not differentiable at $z=0$.

④ If f(z) is differential at z then value of $f'(z)$ along every direction exist and it is equal

Thus we have $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ (along x-axis)

$= \frac{\partial}{\partial x}(u+iv)$

$f'(z) = \frac{\partial f}{\partial x}$

Similarly,

$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ (along y-axis)

$$f'(z) = \frac{1}{i}\left[ i\frac{\partial v}{\partial y} - i^2\frac{\partial u}{\partial y}\right]$$

$$f'(z) = \frac{1}{i}\left[ i\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right] \qquad \because -i^2 = +1$$

$$\therefore f'(z) = (-i)\left[ \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] \qquad \because \frac{1}{i} = -i$$

$$f'(z) = (-i)\left[\frac{\partial}{\partial y}(u+iv)\right]$$

$$f'(z) = (-i)\frac{\partial f}{\partial y}$$

Hence we have,

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

(3) We know by above theorem $f(z)$ is differentiable at $z$.

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\left(\frac{-\partial u}{\partial y}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}(u+iv) = (-i)\left[\frac{1}{(-i)}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right]$$

$$\Rightarrow \frac{\partial f}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] \qquad \left(\because \frac{1}{-i} = i\right)$$

$\therefore$ $f(z)$ is differentiable at $z \Rightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ at $z$.

2016 8M / 2015 10M

**Theorem:** Necessary condition for the differentiable of $f(z)$

**Statement:** P.T. the neccessary condition for the function $f(z)$ to be differentiable at the pt $z=a$ is that its satisfies the equation $f_{\bar{z}} = 0$ at $z=a$.

**Proof:** Given $f(z)$ is differentiable at $z=a$

$\therefore$ By theorem, C-R equations holds at $z=a$ i.e, we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad at \ z=a \ldots (1)$$

We have,

$$f(z) = u(x,y) + iv(x,y)$$

Where $z = x+iy$

$$\therefore \bar{z} = x-iy$$

$\therefore$ We have,

$$z+\bar{z} = x+iy+x-iy$$

$$\therefore z+\bar{z} = 2x$$

$$\therefore \frac{z+\bar{z}}{2} = x$$

Hence $x = x(z,\bar{z})$

$$\frac{\partial x}{\partial z} = \frac{1}{2}(1+0) = \frac{1}{2}$$

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}(0+1) = \frac{1}{2}$$

Also we have,

$$z - \bar{z} = x + iy - x + iy$$

$$z - \bar{z} = 2iy$$

$$\frac{z - \bar{z}}{2i} = y$$

$$y = y(z, \bar{z})$$

$$\frac{\partial y}{\partial z} = \frac{1}{2i}(1 - 0) = \frac{1}{2i}$$

$$\frac{\partial y}{\partial \bar{z}} = \frac{1}{2i}(0 - 1) = \frac{-1}{2i}$$

We know that, $f = f(x, y)$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{1}{2} + \frac{\partial f}{\partial y} \left(\frac{-1}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y}\right] \qquad \left(\because \frac{-1}{i} = i\right) \cdots (2)$$

Now, $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$= \frac{\partial v}{\partial y} + i\left(\frac{-\partial u}{\partial y}\right) \quad \text{at } z = a$$

$$= (-i)\left[\frac{1}{(-i)} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\right]$$

$$= (-i)\left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right] \qquad i = \frac{1}{-i}$$

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$$\frac{\partial f}{\partial x} = (-i)\frac{\partial f}{\partial y} \quad \text{at } z=a.$$

$$\therefore \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \quad \text{at } z=a.$$

$\therefore$ ② becomes,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\times 0 \quad \text{at } z=a$$

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{at } z=a$$

i.e, $f_{\bar{z}} = 0$ at $z=a$.

H.P.

Note:- $f(z)$ is differentiable at $z_0 \Rightarrow f_{\bar{z}}=0$ at $z_0$
i.e, $f_{\bar{z}} \neq 0 \Rightarrow f(z)$ is not differentiable at $z_0$

Example: Show that the function $f(z)=|z|^2$ is not differentiable except at origin.

Solution:- $f(z) = |z|^2$
$= z\cdot\bar{z}$

$$\therefore \frac{\partial f}{\partial \bar{z}} = z$$

i.e, $f_{\bar{z}} = z$

Hence $f_{\bar{z}} \neq 0 \Leftrightarrow z \neq 0$

Thus $f_{\bar{z}} \neq 0 \Rightarrow f(z)$ is not diff. for all $z\neq 0$

$f(z)=|z|^2$ is not differentiable except at origin

polygonal line

Date 06/10/18

[illegible] RM

[illegible] 10M of Defin. [illegible]

**Theorem:** If $f'(z)=0$ on domain $D$, then P.T. $f(z)$ is constant in $D$

**Proof:-** given $f'(z)$ is identically equal to zero i.e., $f'(z)=0$ in domain $D$.

$$f'(z)=0 \quad \forall\ z\in D.$$

Where $f(z)=u(x,y)+iv(x,y)$

clearly $f(z)$ is differentiable at every point of $D$.

$\therefore$ We have $f'(z)=\dfrac{\partial f}{\partial x} \quad \forall\ z\in D$

$$0=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial x}=0 \ \& \ \frac{\partial v}{\partial x}=0$$

Thus, $u(x,y)$ & $v(x,y)$ are independent of $x$.

$\therefore$ $u(x,y)$ & $v(x,y)$ are constant functions with respect to $x$.

i.e, $u(x,y)$ & $v(x,y)$ are constants along every horizontal line. . . . ①

Now we have $f'(z)=-i\dfrac{\partial f}{\partial y}$

$$0=-i\left[\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right]$$

$$0+0i=-i\frac{\partial u}{\partial y}-i^2\frac{\partial v}{\partial y}$$

$$\Rightarrow 0+0i=-i\frac{\partial u}{\partial y}+1\,\frac{\partial v}{\partial y}$$

$$\Rightarrow 0+0i=\frac{\partial v}{\partial y}+i\left(-\frac{\partial u}{\partial y}\right) \qquad \because -i^2=1$$

$$\therefore \frac{\partial v}{\partial y}=0 \ \& \ \frac{\partial u}{\partial y}=0$$

$\Rightarrow$ $u(x,y)$ & $v(x,y)$ are independent of $y$.

$\Rightarrow$ $u(x,y)$ & $v(x,y)$ are constant function along every vertical line ... (2)

Hence by statement (1) & (2) we have $u(x,y)$ & $v(x,y)$ are constants along every horizontal & vertical line in D. ... (3)

Suppose $z_1, z_2 \in D$ are arbitrary points.

As D is domain, $z_1$ & $z_2$ can be joined by using polygonal line in D, as shown in fig.

Now at the point $z_1$, value of $f(z)$ is $f(z_1)$ & at the point $z_2$, value of $f(z)$ is $f(z_2)$

We have, $f(z) = u(x,y) + iv(x,y)$

By statement (3), u & v are constants along every horizontal & vertical line.

$\therefore$ $f(z)$ is also constant along every vertical & horizontal line in D.

$\therefore$ $f(z)$ is constant on every line segment of polygonal line joining $z_1$ & $z_2$

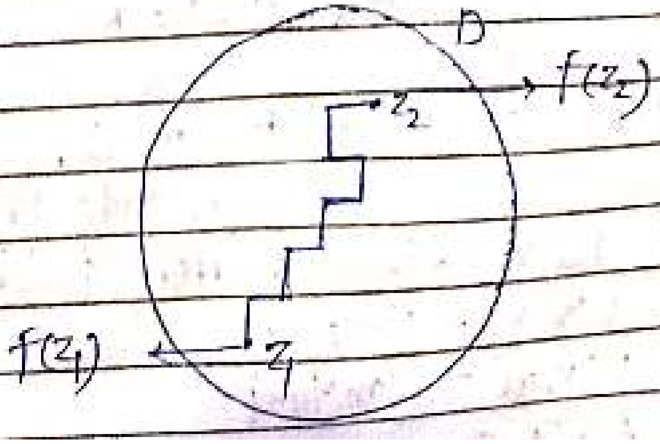
Thus, we have, $f(z_1) = f(z_2)$

But $z_1$ & $z_2$ are arbitrary points in D.

$\therefore$ Value of $f(z)$ is same at all points of D.

$\therefore$ $f(z)$ is constant in D.

H.P.



Imp.

**Corollary:** If $f(z)$ is ~~constant along every horizontal~~ a real valued function define in domain $D$ & $f(z)$ is differentiable on $D$ then P.T. $f(z)$ is constant in $D$.

**Proof:** Let $f(z) = u + iv \quad \forall z \in D$

Given $f(z)$ is real valued function defined in $D$.

$\therefore$ We have $v = 0 \quad \forall z \in D$

i.e, $f(z) = u \quad \forall z \in D$.

Also given that $f(z)$ is differentiable for all $z$ in $D$.

$\therefore$ By theorem, C-R equations holds for all $z$ in $D$.

i.e, We have, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \quad \forall z \in D \quad \ldots (1)$

But, $v = 0 \quad \forall z \in D$.

$\Rightarrow \dfrac{\partial v}{\partial x} = 0$ & $\dfrac{\partial v}{\partial y} = 0 \quad \forall z \in D$.

$\therefore$ By (1) $\quad \dfrac{\partial u}{\partial x} = 0 \quad \dfrac{\partial u}{\partial y} = 0$

$\Rightarrow u(x,y)$ is independent of $x$ & $y$.

$\Rightarrow u(x,y)$ is constant function in $D$.

Suppose $u(x,y) = K$ on $D$.

Hence we have $\quad f(z) = u + iv$

$\Rightarrow f(z) = K + i0$

$\Rightarrow f(z) = K \quad \forall z \in D$.

$\therefore f(z)$ is constant in $D$.

**Note:** If $f(z) = u + iv$ is differentiable on $D$ then and if

(i) $u(x,y) = \text{constant}$

(ii) $v(x,y) = \text{constant}$

(iii) $Arg(f(z)) = \text{constant}$, then $f(z)$ is constant in $D$

Unit(2): 2016 2M

## * Analytic function:

Definition: A function $f(z)$ is said to be analytic at the pt $z_0$ if it is differentiable everywhere in the neighbourhood of $z_0$

Definition (Function analytic in a domain D):

If $f(z)$ is analytic at every point of domain D then $f(z)$ is said to be analytic throughout domain D.

Definition ( Entire function):

If $f(z)$ is analytic at every point of complex plane then $f(z)$ is said to be an entire function

## * Examples of entire function:

1. $f(z)=K$, K constant
2. $f(z)=z^n$ $n \in N$

Entire ⇒ Analytic ⇒ differentiable ⇒ continuous ⇒ well define
not defined ⇒ not cont. ⇒ not diff. ⇒ not analytic ⇒ not entire.

Date

③ $f(z) =$ polynomial Function
④ $f(z) = e^z$
⑤ $f(z) = \sin z$
⑥ $f(z) = \cos z$.
⑦ $f(z) = \dfrac{\sin z}{e^z}$

⑧ $f(z) = \dfrac{\cos z}{e^z}$

⑨ $f(z) = \dfrac{z^n}{e^z}$

⑩ $f(z) = \dfrac{K}{e^z}$

⑪ $f(z) = \dfrac{P(z)}{e^z}$, $P(z)$ = polynomial.

* Some other Examples :

① let $D = \{ z \in \mathbb{C} \mid z \neq 0 \}$

$f(z) = \dfrac{1}{z}$ $\quad z \neq 0$

$f(z)$ is not defined at $z = 0$

$\therefore f(z)$ is not analytic at $z = 0$

But $f(z) = 1/z$ is analytic for all $z$ in $D$.

$\therefore f(z) = 1/z$ is analytic in $D$.

② $f(z) = |z|^2$ $\forall z \in D$
is differentiable only at origin.
But $f(z)$ is not differentiable in the neighbourhood of $z = 0$.

$\therefore f(z)$ is not analytic at $z = 0$

**Note:-** Th$^m$ A set of all points at which function $f(z)$ is analytic is an open set.

If $f(z)$ is analytic on the closed set then there exist an open set containing that the closed set such that $f(z)$ is analytic on that open set.

**Note:-** If $f(z)$ & $g(z)$ are analytic at $z=z_0$ then

(i) $f(z) \pm g(z)$

(ii) $f(z) \cdot g(z)$

(iii) $\dfrac{f(z)}{g(z)}$, $g(z) \neq 0$

are analytic functions.

**Note:-** If $f(z)$ is analytic at $z_0$ & $g(z)$ is analytic at $w_0 = f(z_0)$ then $(g \circ f)(z)$ is analytic at $z_0$

If $f(z)$ & $g(z)$ are entire functions then

(i) $f \pm g$ (iii) $f \circ g$

(ii) $f \cdot g$ (iv) $g \circ f$

are entire functions

Also $\dfrac{f}{g}$ is entire function provided $g(z) \neq 0 \ \forall z \in \mathbb{C}$.

Th$^m$ $f(z) = \dfrac{z}{1-z}$ is analytic in the domain ~~D=~~

$D = \{ z \in \mathbb{C} \mid z \neq 1 \}$

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Theorem : [Lagrange's Mean Value theorem]:

If the real valued function $f(x)$ is con
(i) continuous on $[a,b]$
(ii) differentiable in $(a,b)$
then $\exists$ $c \in (a,b)$, such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

imp 10M
Theorem: If $f(x,y)$ is define in domain D with $f(x,y)$ & $f_x$ & $f_y$ continuous at points in D then for given $(x,y) \in D$. choose $\delta > 0$ such that,

$(x+h, y+k) \in D$ for all point satisfying

$$\sqrt{h^2+k^2} < \delta$$

then Prove that, $f(x+h, y+k) - f(x,y) = f_x(x,y)h + f_y(x,y)k + \epsilon_1 h + \epsilon_2 k$

Where $\epsilon_1 \to 0$, $\epsilon_2 \to 0$ as $h \to 0$ & $k \to 0$

Proof :- Given $f(x,y)$ is define in domain D with $f_x(x,y) = f_1(x,y) = \frac{\partial f}{\partial x}$ and $f_y(x,y) = f_2(x,y) = \frac{\partial f}{\partial y}$ $\forall (x,y) \in D$.

Also given that,

$f_1(x,y)$ & $f_2(x,y)$ are continuous for all point in D.
i.e, $f(x,y)$ is continuous and differentiable for all $(x,y)$ in D.
Given $(x,y) \in D$ and $\delta > 0$ is so choosen that,
$(x+h, y+k) \in D$ where, $\sqrt{h^2+k^2} < \delta$.

Now consider,

$$f(x+h, y+K) - f(x,y) = [f(x+h, y+K) - f(x, y+K)] + [f(x,y+K) - f(x,y)] \quad \cdots (1)$$

Again consider,

$$[f(x+h, y+K) - f(x, y+K)] = \frac{h[f(x+h, y+K) - f(x, y+K)]}{h}$$

$$= \left[\frac{f(x+h, y+K) - f(x, y+K)}{x+h-x}\right] h \quad \cdots (2)$$

Clearly, $f(\xi, y+K)$ is ~~sin~~ function of ~~stmp~~ single variable. $\xi$ where $\xi \in [x, x+h]$

Clearly $[x, x+h]$ lie completely in D.
$\therefore$ $f(\xi, y+K)$ is continuous in D
Moreover, $\frac{\partial f}{\partial x} = f_1(\xi, y+K)$ is continuous in D.

Hence, the ~~sin~~ $f(\xi, K)$ is continuous in $[x, x+h]$.
Moreover differentiable in $[x, x+h]$

$\therefore$ By theorem, (Lagrange's theorem)
there exist $\theta_1 > 0$ such that,

$$\frac{f(x+h, y+K) - f(x, y+K)}{x+h-x} = \left.\frac{\partial f}{\partial x}\right|_{at\ (x+\theta h,\ y+K)} \quad (0<\theta<1)$$

$$\therefore \frac{f(x+h, y+K) - f(x, y+K)}{h} = f_1(x+\theta h, y+K) \quad (0<\theta<1)$$

put this value in eqn (2) then, we get

$f(x+h, y+k) - f(x, y+k) = f_x(x+\theta h)$

$f(x+h, y+k) - f(x, y+k) = f_x(x+\theta h, y+k) \cdot h$

$\therefore f(x+h, y+k) - f(x, y+k) = f_x(x+\theta h, y+k) h \quad \cdots (3)$

Now consider,

$$f(x, y+k) - f(x, y) = \left[\frac{f(x, y+k) - f(x, y)}{k}\right] k$$

$$= \left[\frac{f(x, y+k) - f(x, y)}{y+k-y}\right] \times k \quad (4)$$

~~$f(x, y+k) - f(x,y)$~~ Clearly $f(x, u)$ is the function of single variable $u$ where $u \in [y, y+k]$.
The interval $[y, y+k]$ lie completely in D

$\therefore f(x, u)$ is continuous function in D.
Moreover $\frac{\partial f}{\partial y}$ exist and is continuous on $[y, y+k]$.
Hence $f(x, u)$ is continuous & differentiable on $[y, y+k]$
Hence by Lagrange's theorem, there exist $0<\theta'<1$ such that,

$$\frac{f(x, y+k) - f(x, y)}{y+k-y} = \frac{\partial f}{\partial y}\bigg|_{at\ (x, y+\theta' k)} \quad 0<\theta'<1$$

$= f_y(x, y+\theta' K) \quad (0<\theta'<1)$

put this value in eqn ④ then we get,

$$f(x, y+K) - f(x,y) = f_y(x, y+\theta' K) \cdot K \quad (0<\theta'<1) \quad \cdots ⑤$$

Hence by eqn ①, ③ & ⑤ we get

$$f(x+h, y+K) - f(x,y) = [f_x(x+\theta h, y+K)]h + [f_y(x, y+\theta' K)]K \quad \cdots ⑥$$

Where $0<\theta<1$ & $0<\theta'<1$

We know that, the functions $f_x$ & $f_y$ are continuous for all points in D.

$\therefore$ By definition of continuity, we have for every $\epsilon>0$ & $\epsilon'>0$ ~~for~~ $\exists$ $\delta>0$ ($\sqrt{h^2+K^2}<\delta$) such that,

$$|f_x(x+\theta h, y+K) - f_x(x,y)| < \epsilon$$

$$\therefore |f_y(x, y+\theta' K) - f_y(x,y)| < \epsilon'$$

Suppose, $f_x(x+\theta h, y+K) - f_x(x,y) = \epsilon_1 \quad (\epsilon_1 < \epsilon)$

$$\therefore f_x(x+\theta h, y+K) = f_x(x,y) + \epsilon_1 \quad \cdots ⑦$$

and $f_y(x, y+\theta' K) - f_y(x,y) = \epsilon_2. \quad (\epsilon_2 < \epsilon)$

$$\therefore f_y(x, y+\theta' K) = f_y(x,y) + \epsilon_2 \quad \cdots ⑧$$

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Where $\epsilon_1 \to 0$, & $\epsilon_2 \to 0$ as $\delta \to 0$ i.e. as $h \to 0$ & $k \to 0$

Hence by eqn (6) (7) & (8) we have,

$$f(x+h, y+k) - f(x,y) = (f_x(x,y) + \epsilon_1)h + (f_y(x,y) + \epsilon_2)k$$

$$\therefore f(x+h, y+k) - f(x,y) = h \cdot f_x(x,y) + k \cdot f_y(x,y) + \epsilon_1 h + \epsilon_2 k$$

Where $\epsilon_1 \to 0$ & $\epsilon_2 \to 0$ as $h \to 0$ & $k \to 0$.

H.P.

2016 10M

**Theorem:** If $f(z) = u + iv$ is defined in domain D. such that u & v are continuous & has continuous partial derivatives that satisfies C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \forall z \text{ in } D.$$

then prove that $f(z)$ is analytic in D.

**Proof:-** Given $f(z) = u + iv$ is define in domain D.

Moreover u & v are continuous for all z in D.

Hence $f(z)$ is continuous for all z in D

Also given that, the first ordered partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ exist & are continuous for all z in D.

Hence by theorem,

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \&$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

are continuous in D

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put $\frac{\partial f(x,y)}{\partial x} = f_x(x,y)$

$\frac{\partial f(x,y)}{\partial y} = f_y(x,y)$

$\therefore f_x(x,y)$ & $f_y(x,y)$ are continuous in D.

Also given that, partials of u & v satisfies C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \forall z \in D \quad \text{--- (1)}$$

Claim: TPT f(z) is analytic in D.

i.e, $\lim_{\Delta h \to 0} \frac{f(z+\Delta h) - f(z)}{\Delta h}$ exist for z in D.

Now let $z = (x,y) \in D$ be an arbitrary point choose $\Delta h = h + ik$ so that,

$$\sqrt{h^2+k^2} < \delta$$

i.e, the point $z + \Delta h \in D$.

We have $z = x + iy = (x,y)$

$\Delta h = h + ik = (h,k)$

$z + \Delta h = x + iy + h + ik = (x+h) + i(y+k)$

$= (x+h, y+k)$

$\therefore f(z) = f(x,y)$

$f(z+\Delta h) = f(x+h, y+k)$

Clearly $f(x,y)$ is defined in D.
$f_x$ & $f_y$ are ~~continious~~ continueous in D.

Moreover $\delta>0$ so choosing that,

$(x+h, y+k) \in D$ & $\sqrt{h^2+k^2} < \delta$

$\therefore$ By theorem we have,

$$f(x+h, y+k) - f(x,y) = f_x(x,y)\cdot h + f_y(x,y)\cdot k + \epsilon_1 h + \epsilon_2 \cdot k \quad \cdots (2)$$

Where $\epsilon_1 \to 0$ & $\epsilon_2 \to 0$ as $h \to 0$ & $k \to 0$
Now consider,

$$\frac{f(z+\Delta h) - f(z)}{\Delta h} = \frac{f(x+h, y+k) - f(x,y)}{\Delta h}$$

$$\therefore \frac{f(z+\Delta h) - f(z)}{\Delta h} = \frac{f_x(x,y)\cdot h + f_y(x,y)\cdot k + \epsilon_1 h + \epsilon_2 k}{\Delta h} \quad \cdots (3)$$

($\because$ by (2))

Now we have, $\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$

$$= \frac{-\partial v}{\partial x} + i\left(\frac{\partial u}{\partial x}\right)$$

$$= i\left[\frac{-1}{i}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]$$

$$= i\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

$$\therefore \frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$$

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$\therefore f_y(x,y) = i f_x(x,y)$

put this value in ③ then we get,

$$\frac{f(z+\Delta h)-f(z)}{\Delta h} = \frac{f_x(x,y)\cdot h + f_x(x,y)\cdot k + \epsilon_1 h + \epsilon_2 k}{\Delta h}$$

$$\therefore \lim_{\Delta h\to 0} \frac{f(z+\Delta h)-f(z)}{\Delta h} = \lim_{\Delta h\to 0} \frac{f_x(x,y)\cdot(h+ik)}{\Delta h} + \lim_{\Delta h\to 0} \epsilon_1 \cdot \frac{h}{\Delta h}$$

$$+ \lim_{\Delta h\to 0} \epsilon_2 \cdot \frac{k}{\Delta h}.$$

$$= \lim_{(h+ik)\to(0+ik)} \frac{f_x(x,y)(h+ik)}{(h+ik)} + \lim_{(h+ik)\to(0+ik)} \epsilon_1 \cdot \frac{h}{(h+ik)} + \lim_{(h+ik)\to(0+ik)} \epsilon_2 \frac{k}{(h+ik)}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} f_x(x,y) + \lim_{\substack{h\to 0\\ k\to 0}} \epsilon_1 \frac{h}{h+ik} + \lim_{\substack{h\to 0\\ k\to 0}} \epsilon_2 \frac{k}{h+ik} \quad \cdots ④$$

Now, we have

$$h^2 \le h^2+k^2 \quad \& \quad k^2 \le h^2+k^2$$

$$\therefore \frac{h^2}{h^2+k^2} \le 1 \quad \& \quad \frac{k^2}{h^2+k^2} \le 1$$

$$\frac{|h|}{\sqrt{h^2+k^2}} \le 1 \quad \& \quad \frac{|k|}{\sqrt{h^2+k^2}} \le 1$$

$$\left|\frac{h}{h+ik}\right| \le 1 \quad \& \quad \left|\frac{k}{h+ik}\right| \le 1$$

Moreover $\epsilon_1 \to 0$ & $\epsilon_2 \to 0$ as $h\to 0$, $k \to 0$

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$\therefore$ Both the limits $\lim\limits_{\substack{h\to 0\\ k\to 0}} \epsilon_1\left(\frac{h}{h+ik}\right)$ & $\lim\limits_{\substack{h\to 0\\ k\to 0}} \epsilon_2\left(\frac{k}{h+ik}\right)$

on R.H.S of ④ exist & are equal to zero.

$\therefore$ By ① limit on L.H.S of ④ exist & it is given by

$$\lim_{\Delta h\to 0} \frac{f(z+\Delta h)-f(z)}{\Delta h} = f_x(x,y)$$ ($\because$ here $\Delta h\to 0$ in any direction)

$f'(z)$ exist at $z=(x,y)\in D$.

But $(x,y)\in D$ is arbitary.

$\therefore$ $f'(z)$ exist for all z in D.

$\therefore$ $f(z)$ is differentiable for all z in D.

$\therefore$ $f(z)$ is analytic in D.

Hence proved

Note:- If $f(x)$ is bounded function & $\lim\limits_{x\to 0} g(x)=0$ then

$$\lim_{x\to 0} f(x)\cdot g(x) = 0.$$

Note:- ① $f(z)=u+iv$ where u & v are continuous with partials are continuous.

$\Rightarrow$ $f(z)$ is analytic in D.

Conversely, $f(z)$ is analytic in D $\Rightarrow$ $f(z)$ is differentiable for all z in D.

$\Rightarrow$ C-R equation holds $\forall$ z in D.

② Converse of above is also true.

$\therefore$ We can rewrite the theorem as $f(z) = u + iv$ where $u$ & $v$ are continuous with continuous partial derivatives then C-R eqns hold $\forall$ $z$ in D iff $f(z)$ is analytic in D.

Example: $f(z) = e^z$ then find $f'(z)$

Solution:- Given $f(z) = e^z$

We know that $e^z$ is an entire fun$^n$.

$\therefore$ $e^z$ is analytic $\forall$ $z$ in $\mathbb{C}$

$\therefore$ By theorem,

C-R eqns holds for all $z$ in $\mathbb{C}$

Now here,

$$f(z) = e^z = e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x (\cos y + i \sin y)$$

$$(u+iv) = e^x \cos y + i \sin y$$

$\therefore$ $u = e^x \cos y$ & $v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \& \quad \frac{\partial v}{\partial x} = e^x \sin y$$

As $f(z)$ is differential for all $z$ in $\mathbb{C}$

$\therefore$ We have,

$$f'(z) = \frac{\partial f}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$= e^x\cos y + ie^x \sin y$

$\therefore f'(z) = e^x(\cos y + i\sin y)$

$= e^x e^{iy}$

$= e^{x+iy}$

$\boxed{f'(z) = e^z}$

**Example:** Find all the values of a, b, & c for which f(z) an entire function

2015 8M

2016 4M i] $f(z) = (x+ay) - i(bx+cy) \rightarrow a=b,\ c=-1$ or

ii] $f(z) = ax^2 - by^2 + icxy \rightarrow a=b=c/2$

iii] $f(z) = e^x\cos y + ie^x\sin(y+b) + c \rightarrow b = 2n\pi$ c-arbitary

iv] $f(z) = a(x^2-y^2) + ibxy + c \rightarrow b = 2a$ & c-arbitary

**Solution:** ii] $f(z) = ax^2 - by^2 + icxy$

$\therefore u = ax^2 - by^2$ & $v = cxy$

$\frac{\partial u}{\partial x} = 2ax$ & $\frac{\partial v}{\partial x} = cy$

$\frac{\partial u}{\partial y} = -2by$ & $\frac{\partial v}{\partial y} = cx$

Hence partials of u & v are continuous.

$\Rightarrow$ f(z) is an entire function.

$\Leftrightarrow$ f(z) is analytic for all z in $\mathbb{C}$

$\Leftrightarrow$ C-R eqns holds for all z in $\mathbb{C}$

$\Leftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$

$\Leftrightarrow \quad 2ax = cx \quad \& \quad cy = -(-2by)$

$\Leftrightarrow \quad 2a = c \quad \& \quad cy = 2by$

$\Leftrightarrow \quad 2a = c \quad \& \quad c = 2b$

$\Leftrightarrow \quad a = b = c/2.$

i] $f(z) = (x+ay) - i(bx+cy)$

$\therefore u = x+ay \quad \& \quad v = -(bx+cy) = -bx-cy$

$\frac{\partial u}{\partial x} = 1 \quad \& \quad \frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a \quad \& \quad \frac{\partial v}{\partial y} = -c$

Hence partials of u & v are continuous

$\Rightarrow$ f(z) is an entire function

$\Leftrightarrow$ f(z) is analytic for all z in $\mathbb{C}$

$\Leftrightarrow$ C-R eqns holds for all z in $\mathbb{C}$

$\Leftrightarrow$

$\Leftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Leftrightarrow \quad 1 = -c \quad \& \quad a = -(-b)$

$c = -1 \quad \& \quad a = b$

IV] let $f(z) = a(x^2-y^2) + ibxy + c$

$u = ax^2 - ay^2$ & $v = bxy$ & $c$ - arbitary

$\therefore \frac{\partial u}{\partial x} = 2ax$ & $\frac{\partial v}{\partial x} = by$

$\frac{\partial u}{\partial y} = -2ay$ & $\frac{\partial v}{\partial y} = bx$

Hence partials of u & v are continuous.

$\Rightarrow$ f(z) is an entire function

$\Leftrightarrow$ f(z) is analytic for all z in $\mathbb{C}$

$\Leftrightarrow$ f(z) C-R eqn holds for all z in $\mathbb{C}$

$\Leftrightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Leftrightarrow$ $2ax = bx$ & $-2ay = -by$

$\Leftrightarrow$ $2a = b$ & $2a = b$

$\Leftrightarrow$ $b = 2a$ & $c$ - arbitary.

iii) $f(z) = e^x \cos y + ie^x \sin(y+b) + c.$

$u = e^x \cos y$ $\quad v = e^x \sin(y+b)$ & $c$ - arbitary

$\frac{\partial u}{\partial x} = e^x \cos y$ & $\frac{\partial v}{\partial x} = e^x \sin(y+b)$

$\frac{\partial u}{\partial y} = \cancel{-e^x \cos y}$ $-e^x \sin y$ $\quad \frac{\partial v}{\partial y} = e^x \cos(y+b)$

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Thus $f(z)$ is an Entire function iff $f(z)$ is analytic $\forall z \in \mathbb{C}$.

iff C-R equations holds $\forall z$.

iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

iff $e^x \cos y = e^x \cos(y+b)$ & $e^x \sin(y+b) = -(-e^x \sin y)$

$\Leftrightarrow$ $\cos y = \cos(y+b)$ & $\sin(y+b) = \sin y$

$\Leftrightarrow$ $b = 2n\pi$ , $n = 0, \pm 1, \ldots$ $\forall z \in (x, y)$

Thus $f(z)$ is an entire function if $b = 2n\pi$ & $c$ arbitary.

**Theorem:** If $f(z) = u + iv$ is differentiable with continuous partials at $z = re^{i\theta}$, $r \neq 0$, then prove that partials of $u$ and $v$ with respect to $r$ & $\theta$ satisfies the C-R equation in polar form.

i.e,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

**Proof:-** Given $f(z) = u + iv$ where $u(x, y)$ & $v(x, y)$ is differential at $z = re^{i\theta}$ $r \neq 0$.

$\therefore$ By theorem C-R eqns hold at $z = re^{i\theta}$ $r \neq 0$

ie, We have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \cdots (1) \quad \cancel{z \neq re^{i\theta}}$$

Now, $z = re^{i\theta}$

$\Rightarrow$ $x + iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $x + iy = r\cos\theta + ir\sin\theta$.

$\therefore\ x = r\cos\theta$ & $y = r\sin\theta$

We have, $\frac{\partial x}{\partial r} = \cos\theta$ & $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ & $\frac{\partial y}{\partial \theta} = r\cos\theta$

Consider,

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta \quad \cdots (2)$$

Now,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$= -r\sin\theta\cdot\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = -\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \sin\theta\frac{\partial u}{\partial x} - \cos\theta\frac{\partial u}{\partial y} \quad \cdots (3)$$

Also we have,

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta \qquad \{\because \text{ by } ①$$

$$\frac{\partial v}{\partial r} = \sin\theta\frac{\partial u}{\partial x} - \cos\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ④ \qquad \{\because \text{ by } ②$$

Now,

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}.$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}r\cos\theta$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = -\sin\theta\frac{\partial v}{\partial x} + \cos\theta\cdot\frac{\partial v}{\partial y}$$

$$= -\sin\theta\left(-\frac{\partial u}{\partial y}\right) + \cos\theta\cdot\left(\frac{\partial u}{\partial x}\right)$$

$$= \sin\theta\frac{\partial u}{\partial y} + \cos\theta\frac{\partial u}{\partial x}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r} \quad ⑤$$

Thus $f(z)$ is differentiable at $z = re^{i\theta}$, $r \neq 0$.

$\rightarrow$ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

& $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Note:- $f(z)$ is differentiable at $z = re^{i\theta}$ $r \neq 0$ iff partials of $u$ and $v$ with respect to $r$ & $\theta$ continuous and satisfies C.R equations.

$$u_r = \frac{1}{r} v_\theta \quad \& \quad v_r = -\frac{1}{r} u_\theta$$

Note:- Let $f(z) = \log z$ $\quad z \neq 0$.

put $z = re^{i\theta}$ $\quad r \neq 0$.

$$\therefore f(z) = \log(re^{i\theta})$$
$$f(z) = \log r + \log e^{i\theta}$$

$$u + iv = \log r + i\theta$$
$$u = \log r \quad \& \quad v = \theta$$

Clearly $u$ and $v$ are continuous function $\forall z \neq 0$.

Now,

$$\frac{\partial u}{\partial r} = \frac{1}{r} \quad \& \quad \frac{\partial v}{\partial r} = 0$$

$$\frac{\partial u}{\partial \theta} = 0 \quad \& \quad \frac{\partial v}{\partial \theta} = 1$$

Clearly $\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}, \frac{\partial u}{\partial \theta}$ & $\frac{\partial v}{\partial \theta}$ are continuous $\forall z \neq 0$.

Consider,

$$\frac{\partial u}{\partial r} = \frac{1}{r} = \frac{1}{r} \cdot 1$$

ie,

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

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Similarly $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ C-R equations in polar form holds $\forall z \neq 0$

$\therefore$ By theorem,

$f(z) = \log z$ is differentiable $\forall z \neq 0$.

$\therefore f(z) = \log z$ is analytic for all $z \neq 0$.

Note:- $\log z$ function is not defined at $z = 0$ only.

Note:- If $z = -1$ then

$$\log z = \log|z| + i\arg z + 2n\pi i$$
$$= \log|-1| + i\arg(-1) + 2n\pi i$$
$$= \log 1 + i\pi + 2n\pi i$$

$$\therefore \log z = (2n+1)\pi i$$

Note:- In polar form we have

$$f'(z) = e^{-i\theta} f_r$$

ie, $f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}$

Problem:- If $f(z) = \log z$ then find $f'(z)$ $\forall z \neq 0$.

Solution:- Here $f(z) = \log z$ $\forall z \neq 0$

ie, $f(z) = u + iv = \log(re^{i\theta})$

ie, $u + iv = \log r + \log e^{i\theta}$

ie, $u + iv = \log r + i\theta$.

$u = \log r$ & $v = \theta$

$\frac{\partial u}{\partial r} = \frac{1}{r}$ & $\frac{\partial v}{\partial r} = 0$

$\therefore f'(z) = e^{-i\theta} \frac{\partial f}{\partial r}$

$= e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$

$= e^{-i\theta}\left[\frac{1}{r} + i(0)\right]$

$= \frac{1}{r} e^{-i\theta}$

$= \frac{1}{re^{i\theta}}$

$\therefore f'(z) = \frac{1}{z} \quad z \neq 0$

Homowork:

Problem: Show that the function $f(z) = 1/z$ $\forall$ $z \neq 0$ is analytic for all $z \neq 0$. Also find $f'(z) = ?$

Solution: Here $f(z) = 1/z$ $\forall$ $z \neq 0$ is analytic.

i.e, $f(z) = u + iv = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$

$= \frac{1}{r}(\cos\theta - i\sin\theta)$

$\Rightarrow u = \frac{1}{r}\cos\theta \quad v = \frac{-1}{r}\sin\theta$

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$$\frac{\partial u}{\partial r} = \frac{-1}{r^2}\cos\theta \qquad \frac{\partial v}{\partial r} = \frac{1}{r^2}\sin\theta.$$

$$f'(z) = e^{-i\theta}\cdot f_r$$

$$= e^{-i\theta}\cdot\frac{\partial f}{\partial r}$$

$$= e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= +e^{-i\theta}\left[\frac{-1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta\right]$$

$$= -e^{-i\theta}\left[+\frac{1}{r^2}\cos\theta - i\frac{1}{r^2}\sin\theta\right]$$

$$= \frac{-e^{-i\theta}}{r^2}\left[\cos\theta + (-i\sin\theta)\right]$$

$$= \frac{-e^{-i\theta}}{r^2}\left[\cos\theta - i\sin\theta\right]$$

$$= -\frac{e^{-i\theta}}{r^2}\left[e^{-i\theta}\right]$$

$$= \frac{-e^{-2i\theta}}{r^2}$$

$$= \frac{-1}{r^2(e^{i\theta})^2}$$

$$= \frac{-1}{(re^{i\theta})^2} = \frac{-1}{z^2}$$

$$\therefore f'(z) = \frac{-1}{z^2}.$$

Imp

**Theorem:** If $f(z)$ & $g(z)$ are continuous & differentiable at $z_0$ with $f(z_0)=g(z_0)=0$ & $g'(z_0)\neq 0$ then P.T. $\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$

**Proof:-** Given $f(z)$ & $g(z)$ are differentiable at $z_0$, $f(z_0)=g(z_0)=0$

Moreover $g'(z_0)\neq 0$

We have,

$$f'(z_0)=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

$$\therefore f'(z_0)=\lim_{z\to z_0}\frac{f(z)}{z-z_0} \quad \text{---} (1) \qquad \because f(z_0)=0$$

Similarly we have,

$$g'(z_0)=\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0} \quad \text{---} (2)$$

$$g'(z_0)=\lim_{z\to z_0}\frac{g(z)}{z-z_0} \qquad \because g(z_0)=0$$

As $g'(z_0)\neq 0$,

Consider,

$$\frac{f'(z_0)}{g'(z_0)}=\frac{\lim_{z\to z_0}\frac{f(z)}{z-z_0}}{\lim_{z\to z_0}\frac{g(z)}{z-z_0}}$$

$$=\lim_{z\to z_0}\frac{f(z)/(z-z_0)}{g(z)/(z-z_0)}$$

$$\frac{f'(z_0)}{g'(z_0)}=\lim_{z\to z_0}\frac{f(z)}{g(z)}. \qquad \therefore \lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}$$

Problem: If $f(z) = |z|^2$ and $g(z) = z$ then find $\lim\limits_{z\to 0} \frac{|z|^2}{z}$.

Solution:- We have,

$$f(z) = |z|^2 \quad g(z) = z$$

Clearly $f(z)$ & $g(z)$ are differentiable at $z=0$.

$\therefore$ Here we have $z_0 = 0$.

Now, $f(z_0) = f(0) = |0|^2 = 0$.

& $g(z_0) = g(0) = 0$.

Moreover we have

$$g'(z) = 1 \quad \forall z$$

$$\therefore g'(z_0) = g'(0) = 1 \neq 0$$

$\therefore$ By theorem, we have,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

$$\lim_{z\to 0} \frac{f(z)}{g(z)} = \frac{f'(0)}{g'(0)}$$

$$\lim_{z\to 0} \frac{|z|^2}{z} = \frac{f'(0)}{1} = f'(0) \quad \cdots (1)$$

We have,

$f(z) = |z|^2$ is differentiable at $z=0$ only,

We have,

$$f'(z) = \frac{\partial f}{\partial x} \quad \text{at } z=0.$$

Now, $f(z) = |z|^2$

$$= (x^2+y^2) + 0i$$

$\Rightarrow \quad u = x^2+y^2$ & $v=0$.

$\therefore \dfrac{\partial u}{\partial x} = 2x$ & $\dfrac{\partial v}{\partial x} = 0$.

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= 2x + i0.$$
$$= 2x$$

at $z=0$, $x=0$ & $y=0$

$\therefore$ at $z=0$

$$\frac{\partial f}{\partial x} = 2\times 0 = 0.$$

We have,

$$f'(z) = \frac{\partial f}{\partial x} = 0 \quad \text{at } z=0$$

i.e, $f'(0) = 0$

$\therefore$ ① becomes,

$$\lim_{z\to 0} \frac{|z|^2}{z} = 0.$$

2014 3M

**Problem:** If $f(z) = \sin z$ then find $\displaystyle\lim_{z\to 0} \frac{\sin az}{\sin z}$.

**Solution:** Given $f(z) = \sin z$ & $g(z)$ = put $g(z) = \sin az$.

Clearly, $f(z)$ & $g(z)$ are differentiable at $\forall z$.

Moreover $f(0) = \sin 0 = 0$

& $g(0) = \sin(a \cdot 0) = \sin 0 = 0$.

$\therefore$ Here we have $z_0 = 0$

$\therefore$ $f(z)$ & $g(z)$ are differentiable at $z=0$ &

$f(z_0) = g(z_0) = 0$

Moreover $f(z) = \sin z$

$f'(z) = \cos z$

$f'(0) = \cos 0$

$f'(0) = 1 \neq 0$.

$\therefore$ By theorem we have

$$\lim_{z \to z_0} \frac{g(z)}{f(z)} = \frac{g'(z_0)}{f'(z_0)}$$

$$\lim_{z \to 0} \frac{g(\cdot)}{f(\cdot)} = \lim_{z \to 0} \frac{\sin az}{\sin z} = \frac{g'(0)}{f'(0)}$$

$$= \frac{g'(0)}{1}$$

Now, $g(z) = \sin az$

$g'(z) = a \cos az$

$g'(0) = a\cos(a \cdot 0) = a \cos 0$

$g'(0) = a$

$$\therefore \lim_{z \to 0} \frac{\sin az}{\sin z} = a$$

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H.W. 2019 3M

**Problem:** If $f(z) = 1-e^z$ & $g(z) = 3z$ then find the value of

$$\lim_{z\to 0} \frac{f(z)}{g(z)}$$

H.W.

**Problem:** If $f(z) = 1-\cos z$ & $g(z) = \sin^2 z$ then find $\lim_{z\to 0} \frac{f(z)}{g(z)}$

**Solution:-** (1) Given $f(z) = 1-e^z$ & $g(z) = 3z$

* Clearly $f(z)$ & $g(z)$ are differentiable for all $z$.

More over, $f(0) = 1-e^0 = 1-1 = 0$

& $g(0) = 3.0 = 0$

Here we have $z_0 = 0$

$\therefore f(z)$ & $g(z)$ are differentiable at $z=0$ & $f(z_0) = g(z_0) = 0$

Moreover $f(z) = 1-e^z$ $\therefore f'(z) = -e^z$

i.e. $f'(0) = -1 \neq 0$

$\therefore$ By theorem we have,

$$\lim_{z\to 0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

$$\lim_{z\to 0} \frac{1-e^z}{3z} = \frac{f'(z_0)}{g'(z_0)}$$

$$\lim_{z\to 0} \frac{1-e^z}{3z} = \frac{-1}{g'(z_0)}$$

Now $g(z) = 3z$

$g'(z) = 3$

$g'(0) = 3$

$$\therefore \lim_{z\to 0} \frac{1-e^z}{3z} = \frac{-1}{3}.$$

(2) Given $f(z) = 1-\cos z$ & $g(z) = \sin^2 z$

Clearly $f(z)$ & $g(z)$ are differentiable for all $z$.
Moreover,

$$f(0) = 1-\cos 0 = 1-1 = 0$$

$$\& \quad g(0) = \sin^2(0) = 0$$

Here $g'(z_0) = 2\sin z \cdot \cos z$.

$g'(0) = 0$ ∴ } But $\lim g'(z_0) \neq 0$

By Theorem,

$$\lim_{z \to 0} \frac{1-\cos z}{\sin^2 z} = \lim_{z \to 0} \frac{f(z)}{g(z)}$$

Using L-Hospital rule

$$\lim_{z \to 0} \frac{1-\cos z}{\sin^2 z} = \lim_{z \to 0} \frac{\sin z}{2\sin z \cdot \cos z}$$

$$= \lim_{z \to 0} \frac{1}{2\cos z}$$

$$= \frac{1}{2\cos 0}$$

$$= \frac{1}{2}$$

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2016 2M

## * Harmonic function:

**Definition:** A real valued continuous function $u(x,y)$ is said to be ~~har~~ define in domain D is said to be harmonic in D if it has continuous 1st & 2nd order partial derivatives which satisfies the eqn Laplace eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

imp 2015 2M

**Theorem:** If $f(z) = u+iv$ is analytic function in domain D then prove that both real & imaginary parts i.e, u & v are harmonic functions in domain D.

**Proof:-** Given $f(z) = u(x,y) + iv(x,y)$ is analytic $\forall z$ in D.

$\therefore$ f(z) is continuous in D.

$\therefore$ By theorem,

$u(x,y)$ & $v(x,y)$ are continuous functions in D.

$\therefore$ $u(x,y)$ & $v(x,y)$ are continuous real valued functions in domain D.

As f(z) is analytic in D.

$\therefore$ By theorem,

I & II order partials of u & v are continuous in D & they satisfy the C-R equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \dots (1) \quad \forall z \text{ in } D$$

$\therefore$ $u(x,y)$ & $v(x,y)$ are continuous real valued functions in D. with continuous first & second order partials in D.

Moreover we have,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) \quad \because \text{by } ①$$

$$= \frac{\partial^2 v}{\partial x \cdot \partial y} + \left(-\frac{\partial^2 v}{\partial x \cdot \partial y}\right)$$

$$= \frac{\partial^2 v}{\partial x \cdot \partial y} - \frac{\partial^2 v}{\partial x \cdot \partial y}$$

$$= 0. \quad \forall z \in D.$$

ie, $u(x,y)$ satisfies the laplace eqn.
Similarly we can prove that,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \forall z \text{ in } D.$$

$\therefore v(x,y)$ also satisfies the laplace equation.

$\therefore u(x,y)$ & $v(x,y)$ are harmonic functions in D.
$\therefore$ Real & Imaginary parts of an analytic functions are harmonic in domain D.

Hence proved.

Note:- If $f(z) = u + iv$ is analytic in D then $v$ is said to be harmonic conjugate of $u$.

Theorem: If $f(z) = u + iv$ is analytic function in domain D. then P.T. $u$ is harmonic conjugate of $-v$

Proof:- Given $f(z) = u + iv$ is analytic function in domain D.
$\therefore$ By theorem,

$u$ & $v$ are harmonic functions in D.

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Moreover, $v$ is harmonic conjugate of $u$.
Now define,

$$g(z) = i f(z)$$

Clearly $g(z)$ is analytic function in D.
Moreover,

$$\begin{aligned} g(z) &= i f(z) \\ &= i(u+iv) \\ &= iu + i^2 v \\ &= iu - v \\ &= (-v) + iu \end{aligned}$$

$\Rightarrow \quad g(z) = A + iB \qquad (\because A = -v, \; B = u)$

As $g(z)$ is analytic in D.
$\therefore$ By theorem
A & B are harmonic in D.
Moreover, B is ~~hcm~~ harmonic conjugate of A.
Hence $u$ is harmonic conjugate of $(-v)$
H.P.

Note:- If $u$ is harmonic function in D then we can find another harmonic function $v$ such that $f(z) = u+iv$ is analytic function in D.

Example: If $u(x,y) = x^2 + y$ then find an analytic function whose real part is $u(x,y)$

Solution:- Given $u(x,y) = x^2 + y$
Clearly $u(x,y)$ is continuous function for all $(x,y)$

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Now, $\frac{\partial u}{\partial x} = 2x$ $\frac{\partial u}{\partial y} = 1$

$\frac{\partial^2 u}{\partial x^2} = 2$ $\frac{\partial^2 u}{\partial y^2} = 0$

$\frac{\partial^2 u}{\partial x \cdot \partial y} = 0$

Here first & 2nd order partials of $u(x,y)$ are continuous for all $(x,y)$

Now consider, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 \neq 0$ $\forall (x,y)$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$

Hence $u(x,y)$ does not satisfy Laplace equation

$\therefore$ $u(x,y)$ is not harmonic function.

$\therefore$ there does not exist any analytic function $f(z)$ whose real part $u(x,y)$.

Problem: show that the function $u(x,y) = x + e^{-x}\cos y$ is harmonic. Also find harmonic conjugate of $u$. Moreover find an analytic function $f(z)$ whose real part is $u(x,y)$.

Solution: Given $u(x,y) = x + e^{-x}\cos y$

Clearly $u(x,y)$ is continuous function.

Now,

$\frac{\partial u}{\partial x} = 1 - e^{-x}\cos y$ & $\frac{\partial u}{\partial y} = -e^{-x}\sin y$

$\frac{\partial^2 u}{\partial x^2} = e^{-x}\cos y$ & $\frac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$

Clearly partials of $u$ are continuous.

Moreover,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}\cos y + (-e^{-x}\cos y)$$

$$= e^{-x}\cos y - e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \quad \forall\ (x,y)$$

$\therefore$ $u(x,y)$ is harmonic function.

Thus $u(x,y)$ is the real part of some analytic function $f(z)$

suppose $f(z) = u + iv$ is the required analytic function

$\therefore$ partials of $u$ & $v$ satisfies the C-R eqns.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{--- (1)}$$

Thus,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\rightarrow \quad \frac{\partial v}{\partial x} = -(-e^{-x}\sin y)$$

$$\frac{\partial v}{\partial x} = e^{-x}\sin y$$

Integrating with resp. to $x$ :

$$v(x,y) = \frac{e^{-x}}{-1}\sin y + \phi(y)$$

$\phi(y)$ is arbitrary function of $y$ which is constant with resp. to $x$

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$$\therefore v(x,y) = -e^{-x}\sin y + \phi(y) \cdots \quad ②$$

Now differentiate $v(x,y)$ with respect to $y$ partially, then we get,

$$\frac{\partial v}{\partial y} = -e^{-x}\cos y + \phi'(y) \cdots \quad ③$$

Now by ① we have,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 1 - e^{-x}\cos y \cdots \quad ④$$

$\therefore$ By ③ & ④ we have,

$$\phi'(y) = 1$$

integrating on both sides,

$\therefore \phi(y) = y + c$     c - arbitary constant.

put $\phi(y) = y + c$ in ② then we get,

$$v(x,y) = -e^{-x}\sin y + y + c$$

This is required harmonic conjugate of $u(x,y)$

Now, $f(z) = u + iv$

$$\Rightarrow \quad f(z) = x + e^{-x}\cos y + i(y - e^{-x}\sin y + c)$$

$$= x + e^{-x}\cos y + iy - ie^{-x}\sin y + ic$$

$$= (x+iy) + e^{-x}(\cos y - i\sin y) + K$$

$$= z + e^{-x}e^{-iy} + K \qquad K = ic.$$

$$= z + e^{-(x+iy)} + K$$

$f(z) = z + e^{-z} + K$     where K - arbitary constant.

is the required analytic function whose real part is $u(x,y)$.

Unknown

2016. 7M Q. $u = (ax^3 + xy^2 + x)$ is harmonic in C & find analytic fun $f(z)$ whose real part is u.

H.W. Problem: Show that the function $u(x,y) = ax+by$ is harmonic. Also find its harmonic conjugate. Moreover find all the analytic function whose real part is $u(x,y)$.

Solution:- Given, $u(x,y) = ax+by$

Clearly $u(x,y)$ is continuous function.

Now,

$$\frac{\partial u}{\partial x} = a \quad \& \quad \frac{\partial u}{\partial y} = b$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \& \quad \frac{\partial^2 u}{\partial y^2} = 0$$

Clearly partials of u are continuous

Moreover,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0+0 = 0 \quad \forall (x,y)$$

$u(x,y)$ is harmonic function.

Thus $u(x,y)$ is the real part of some analytic function $f(z)$.

Suppose $f(z) = u+iv$ is the required analytic function

$\therefore$ partials of u & v satisfies the C-R eq?

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (1)$$

Thus,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} = -b$$

Integrating bo w.r.to x.

$v(x,y) = -bx + \phi(y)$.

$\phi(y)$ is arbitary function of $y$.
Which is constant with respect to $x$.

$\therefore\ v(x,y) = -bx + \phi(y) \quad \cdots ②$

Now, differentiate $v(x,y)$ with respect to $y$ partially then we get,

$$\frac{\partial v}{\partial y} = 0 + \phi'(y) \quad \cdots ③$$

Now by eqn ① we get

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = a \quad \cdots ④$$

By eqn ③ & ④ we have,

$\phi'(y) = a$

$\phi(y) = ay + c$

where $c$ - arbitary

put $\phi(y) = ay + c$ in ② then we get

$$v(x,y) = -bx + ay + c$$

This is required harmonic conjugate of $u(x,y)$

Now $f(z) = u + iv$

$$\Rightarrow \quad f(z) = ax + by + i(-bx + ay + c)$$
$$= ax + by - ibx + iay + ic$$
$$= a(x + iy) + by - ibx + ic$$
$$= az - ib\left(x + \frac{1}{i}y\right) + ic$$
$$= az - ib(x + iy) + ic$$
$$= az - ibz + ic$$

$$f(z) = (a - ib)z + K.$$

2015 GM

Problem: Show that the following function is harmonic & determine its harmonic conjugate. also find f(z)

$$u(x,y) = x^3 - 3xy^2$$

Solution: Given $u(x,y) = x^3 - 3xy^2$

Clearly $u(x,y)$ is continuous real valued function.
Now,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad ; \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \qquad \frac{\partial^2 u}{\partial y^2} = -6x$$

Clearly partials of $u(x,y)$ are continuous.
Moreover,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$$

$\therefore$ $u(x,y)$ is harmonic in plane.
Now we will find $v(x,y)$.
Suppose
$f(z) = u + iv$ is an analytic function whose real part is $u(x,y)$
$\therefore$ Partials of $u$ & $v$ satisfies C-R equations.
i.e., we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Hence we get,

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \cdots (1)$$

$$\frac{\partial v}{\partial x} = -(-6xy) = 6xy \quad \cdots (2)$$

Now integrate $\frac{\partial v}{\partial x}$ with resp to $x$ then we get

$$v(x,y) = 6y\frac{x^2}{2} + \phi(y)$$

Where $\phi(y)$ is arbitary function of $y$.

$$v(x,y) = 3x^2y + \phi(y) \quad \cdots (3)$$

diff. (3) with resp to $y$ partially,

$$\frac{\partial v}{\partial y} = 3x^2 + \phi'(y) \quad \cdots (4)$$

By eqn (1) & (4) we get,

$$\phi'(y) = -3y^2$$

$$\phi(y) = -\frac{3y^3}{3} + c$$

$$\phi(y) = -y^3 + c$$

Where $c$ is arbitary constant.

put $\phi(y) = -y^3 + c$ in (3) we get

$$v(x,y) = 3x^2y + (-y^3 + c)$$

$$v(x,y) = 3x^2y - y^3 + c$$

This is required harmonic function conjugate of $u(x,y)$

Now $f(z) = u + iv$

$(-1) = i^2$
$(-1) = i^3$

$\Rightarrow f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$
$= x^3 - 3xy^2 + i3x^2y - iy^3 + ic$
$= x^3 + 3x(i^2y^2) + 3x^2(iy) + i^3y^3 + ic$

$\because i^2 = -1 \quad i^3 = -i$

$= x^3 + 3x^2(iy) + 3x(i^2y^2) + (iy)^3 + ic$
$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + c'$
$= (x+iy)^3 + c'$

$\because c' = ic$

$f(z) = z^3 + c'$

These are the required analytic functions whose real part is $u(x,y)$.

v.imp

**Problem:** Find the value of a such that the function $u(x,y) = ax^2y - y^3 + xy$ is harmonic. Then find harmonic conjugate of u. Also find all the analytic functions whose real part is $u(x,y)$.

**Solution:-** Given $u(x,y) = ax^2y - y^3 + xy$ is harmonic function.
Clearly $u(x,y)$
$\therefore$ We have,
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{--- (1)}$$

Here $\frac{\partial u}{\partial x} = 2axy + y \qquad \frac{\partial u}{\partial y} = ax^2 - 3y^2 + x$

$\frac{\partial^2 u}{\partial x^2} = 2ay \qquad \frac{\partial^2 u}{\partial y^2} = -6y$

From (1) we have,

$$2ay - 6y = 0$$

$2ay = 6y$

$a = 3.$

Thus, $u(x,y) = 3x^2y - y^3 + xy$.

$\therefore \frac{\partial u}{\partial x} = 6xy + y$ & $\frac{\partial u}{\partial y} = 3x^2 - 3y^2 + x$

Suppose $v(x,y)$ is harmonic conjugate of $u$.
& $f(z) = u + iv$ is an analytic function whose real part is $u(x,y)$.
$\therefore$ partials of $u$ & $v$ satisfies C-R equations.
$\therefore$ We have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Hence we have,

$$\frac{\partial v}{\partial y} = 6xy + y \quad \cdots (2)$$

$$\& \quad \frac{\partial v}{\partial x} = -(3x^2 - 3y^2 + x)$$

$$\frac{\partial v}{\partial x} = -3x^2 + 3y^2 - x \quad \cdots (3)$$

integrating with respect to $x$.

$$v(x,y) = -\frac{3x^3}{3} + 3y^2x - \frac{x^2}{2} + \phi(y)$$

Where $\phi(y)$ is an arbitrary function

$$\therefore v(x,y) = -x^3 + 3xy^2 - \frac{x^2}{2} + \phi(y) \quad \cdots (4)$$

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differentiating ④ w.r.to $y$ partially then we get,

$$\frac{\partial v}{\partial y} = 6xy + \phi'(y) \quad \cdots ⑤$$

from eqn ② & ⑤ we get

$$\phi'(y) = y$$

$$\phi(y) = y^2/2 + c$$

where $c$ is arbitrary constant.

put $\phi(y) = \frac{y^2}{2} + c$ in eqn ④ we get

$$v(x,y) = -x^3 + 3xy^2 - x^2/2 + y^2/2 + c$$

This is required harmonic conjugate of $u(x,y)$

Now $f(z) = u + iv$

$$f(z) = 3x^2y - y^3 + xy + i(-x^3 + 3xy^2 - x^2/2 + y^2/2 + ic)$$

$$= 3x^2y - y^3 + xy + (-ix^3) + 3ixy^2 - ix^2/2 + iy^2/2 + ic$$

$$= 3x^2y - y^3 + xy - ix^3 + 3ixy^2 - ix^2/2 + iy^2/2 + ic$$

$$= 3x^2y - ix^3 + 3x^2y - \frac{ix^2}{2} + 3ixy^2 + \frac{iy^2}{2} - y^3 + xy + c'$$

$c' = ic$

$$= (-ix^3 + 3x^2y + 3ixy^2 - y^3) + \left(-\frac{ix^2}{2} + \frac{iy^2}{2} + xy\right) + c'$$

$$= -i\left[x^3 + \frac{1}{-i}3x^2y - 3xy^2 - \frac{1}{(-i)}y^3\right] - i\left[\frac{x^2}{2} + \frac{1}{(-i)}xy - \frac{y^2}{2}\right] + c'$$

$$= -i\left[x^3 + i3x^2y + i^2 3xy^2 + (-i)y^3\right] - \frac{i}{2}\left[x^2 + i2xy - y^2\right] + c'$$

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$\because \frac{1}{i} = -i \,,\ \frac{1}{-i} = i \,,\ i^2 = -1$

$\therefore f(z) = -i[x^3+3x^2(iy)+3x(iy)^2+(iy)^3] - \frac{i}{2}[x^2+2x(iy)+i^2y^2] + c'$

$\because i^3 = -i \,,\ i^2 = -1$

$\therefore f(z) = -i[x^3+3x^2(iy)+3x(iy)^2+(iy)^3] - \frac{i}{2}[x^2+2x(iy)+(iy)^2] + c'$

$f(z) = -i(x+iy)^3 - \frac{i}{2}(x+iy)^2 + c'$

$\therefore f(z) = -iz^3 - \frac{i}{2}z^2 + c'$

H.W.

Problem: $u(x,y) = x^4+y^4-6x^2y^2-4xy$

imp:

Theorem: If $f(z)$ is analytic in D & $|f(z)|$ is constant in D. then P.T. $f(z)$ is constant in D.

Proof:- Given, $f(z)$ is analytic in D.

Let $f(z) = u+iv$

Also given that,

$|f(z)| = K$ K-constant

$|u+iv| = K$

$\sqrt{u^2+v^2} = K$

$u^2+v^2 = K^2 \quad \ldots (1)$

diff. (1) w.r.to x partially then we get,

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$

i.e, $\frac{u\partial u}{\partial x} + \frac{v\partial u}{\partial x} = 0$ ---- (2)

Similarly diff (1) w.r.to y partially then we get,

$$\frac{2u\partial u}{\partial y} + \frac{2v\partial v}{\partial y} = 0$$

i.e, $\frac{u\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ ---- (3)

As f(z) is analytic in D.

∴ partials of u & v satisfies C-R equations.

ie, we have,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ ---- (4)}$$

Now by (2) & (4) we have,

$$\frac{u\partial v}{\partial y} + v\left(\frac{-\partial u}{\partial y}\right) = 0$$

$$\frac{u\partial v}{\partial y} - \frac{v\partial u}{\partial y} = 0 \text{ ---- (5)}$$

Multiply (3) by v & (5) by u, we get,

$$\frac{uv\partial u}{\partial y} + v^2\frac{\partial v}{\partial y} = 0 \text{ ---- (6)}$$

& $\frac{u^2\partial v}{\partial y} - \frac{uv\partial u}{\partial y} = 0$ ---- (7)

adding eqs (6) & (7) then we have,

$$\frac{v^2\partial v}{\partial y} + \frac{u^2\partial v}{\partial y} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial y} = 0$$

$$K^2 \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial v}{\partial y} = 0$$

$\therefore$ v is independent of y.

By (7) & (9) we have,

$$u\left(\cdot\frac{\partial v}{\partial x}\right) + v\frac{\partial u}{\partial x} = 0 \quad \text{--- (8)}$$

Multiply (6) by (−u) & (8) by v then we get

$$u^2\frac{\partial v}{\partial x} - uv\frac{\partial u}{\partial x} = 0 \quad \text{--- (9)}$$

$$\& \quad uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} = 0 \quad \text{--- (10)}$$

adding eqn (9) & (10) we get,

$$u^2\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial x} = 0$$

$$(u^2+v^2)\frac{\partial v}{\partial x} = 0$$

$$K^2\frac{\partial v}{\partial x} = 0$$

$K^2$ – constant.

$$\Rightarrow \frac{\partial v}{\partial x} = 0$$

$\Rightarrow$ v is independent of x.

$\therefore$ V is independent of both x & y.

$\therefore$ $v(x,y)$ = constant

Similarly we can prove that,
$u(x,y)$ is constant.

let $v = c_2$ & $u = c_1$. $\therefore f(z) = c_1 + ic_2$

$\therefore f(z)$ is constant in D. H.P.

## Introduction:

We know the following property of Riemann integral.

(1) $$\int_a^b [c_1 f_1(t) + c_2 f_2(t)]\,dt = c_1 \int_a^b f_1(t)\,dt + c_2 \int_a^b f_2(t)\,dt$$

(2) $$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt.$$

Where $t \in [a, b]$ & $f(t)$ is real valued function of real variable $t$.

Suppose $f$ is function $f : [a, b] \rightarrow \mathbb{C}$ is defined as

$$f(t) = u(t) + i v(t)$$

where, $u(t)$ & $v(t)$ are real valued functions of real variable $t$.

Now we define the integral of $f(t)$ as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt$$

Thus we have $\operatorname{Re}\left[\int_a^b f(t)\,dt\right] = \int_a^b u(t)\,dt$

i.e, $$\operatorname{Re}\left[\int_a^b f(t)\,dt\right] = \int_a^b [\operatorname{Re} f(t)]\,dt.$$

Similarly, we have,

$$\operatorname{Im}\left[\int_a^b f(t)\,dt\right] = \int_a^b [\operatorname{Im} f(t)]\,dt$$

Thus, $Re\left[\int_a^b f(t)dt\right]$ & $Im\left[\int_a^b f(t)dt\right]$ are real integrals

$\therefore$ We can apply properties (1) & (2) to these integral

i.e, to the integral of $f(t)$.

**Definition (Curve (or) Continuous Curve (or) arc):-**

A continuous curve in a complex plane is define paramatrically as $c: z(t) = x(t) + iy(t)$, $t \in [a,b]$ where $x(t)$ & $y(t)$ are real valued functions & $z(t)$ is complex valued continuous function of real variable $t$.

**Note:-** (1) Curve is always denoted by $C$

(2) Curve is connected set of points

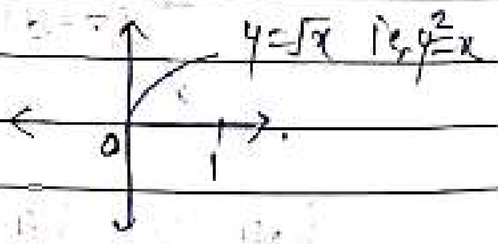
**Example:** $z(t) = t^2 + it$. $t \in [0,1]$

Here

$y(t) = t$, $x(t) = t^2$

$\therefore$ We have $x = y^2$

$y = \sqrt{x}$




**Definition (Initial & Terminal point of the curve):**

Let $C$ be the curve defined by

$$c: z(t) = x(t) + iy(t) \quad t \in [a,b]$$

Then,

$z(a) = x(a) + iy(a)$ is called as initial point of $c$

& $z(b) = x(b) + iy(b)$ is called as terminal point of $c$.

Example: If $c: z(t) = t + it^2 \quad t \in [0,1]$
then
$z(0) = 0 + i0 = (0,0)$ is initial point of c &
$z(1) = 1 + 1 \cdot i = (1,1)$ is terminal point of c.

Definition (Closed curve):-

The curve c is said to be closed curve if initial & terminal points of c are same.

ie, $c: z(t) = x(t) + iy(t) \quad ; t \in [a,b]$

is closed curve if & only if, $z(a) = z(b)$.

Example: $c: z(t) = \cos t + i \sin t = e^{it} \quad t \in [0, 2\pi]$

Clearly z(t) represents a unit circle because

$|z(t)| = |e^{it}| = 1$

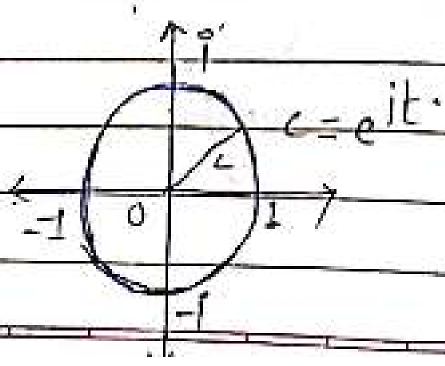
i.e. $|z| = 1$

Moreover

$z(0) = \cos 0 + i \sin 0 = 1$

$z(2\pi) = \cos 2\pi + i \sin 2\pi = 1$

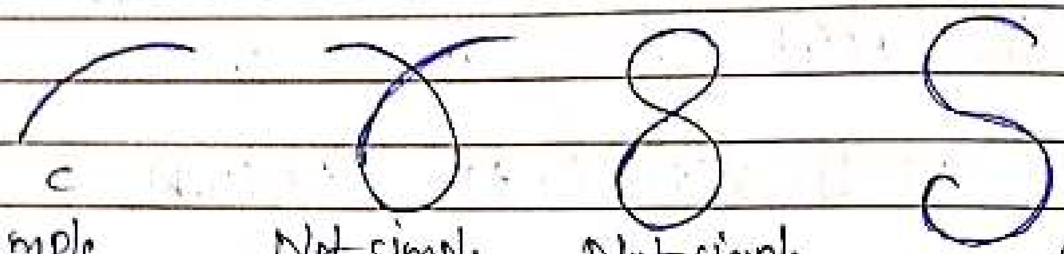
$\therefore z(0) = z(2\pi) = 1 \Rightarrow$ c is closed curve as shown in fig.



**Definition (Simple Curve):-**

The curve $C$ is defined as $z(t) = x(t) + iy(t)$ $t \in [a,b]$ is called as simple curve if whenever $t_1 \neq t_2$
$\Rightarrow$ $z(t_1) \neq z(t_2)$

ie, The curve $C$ does not intersect to itself.



simple    Not simple    Not simple    simple

**Example:-** $C : z(t) = t + it^2$ $t \in [0,1]$ is simple curve.

**Definition (Simple closed Curve (or) Jordan Curve):-**

A closed curve $C : z(t) = x(t) + iy(t)$ $t \in [a,b]$ is called as simple closed curve if the curve is simple in $(a,b)$.
i.e, if $z(t_1) \neq z(t_2)$ whenever $t_1 \neq t_2$ with one possible exception $z(a) = z(b)$ then $z(t)$ is said to be simple closed curve (or) Jordan curve.

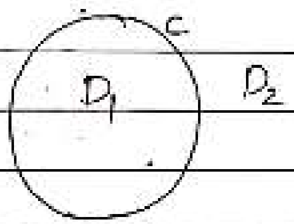
Not simple not closed    simple not closed    simple closed    Not simple closed

**Example:** $c : z(t) = \cos t + i\sin t = e^{it}$ $\quad t \in [0, 2\pi]$

$\ell(z(t)) =$

## Theorem (Jordan curve theorem):-

**State:-** If $c$ is simple closed curve then complement of $c$ consists of two disjoint domains one is bounded domain & other is unbounded domain which has common boundary $c$ as shown in fig.



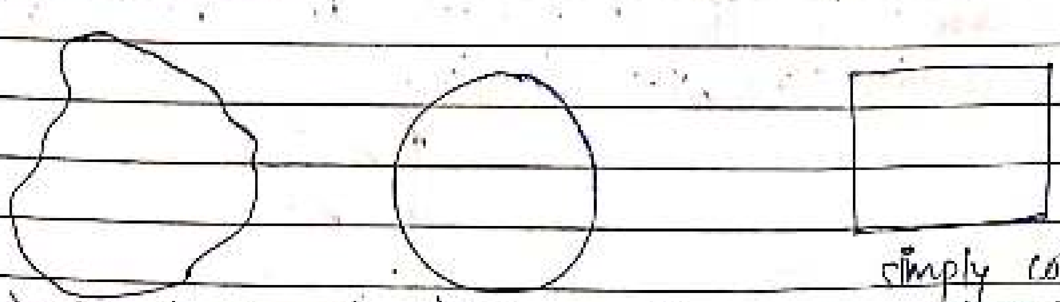

$D_1$ - bounded domain
$D_2$ - unbounded domain
$c$ - Jordan curve.

2016 2M

## Definition (Simply connected domain):-

A region $R$ or the domain $D$ is said to be simply connected if each simple closed curve contained in $R$ or $D$ contains the points of $R$ inside only.

**Note:-** Toupologically a simply connected domain can shrink to a point, geometrically a simply connected domain has no wholes.

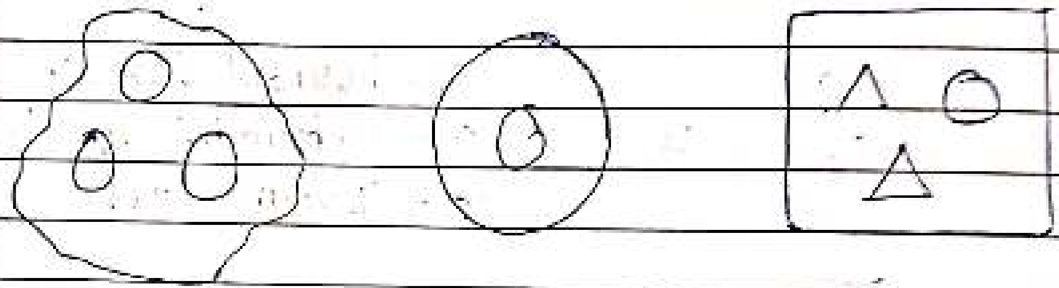


simply connected domain     simply connected domain     simply connected domain

Note :- The punctured plane or punctured disc is not simply connected domain.

Definition (Multiply Connected domain) :

The domain D which is not simply connected is called as multiply connected domain.
i.e, the multiply connected domain has wholes inside it.

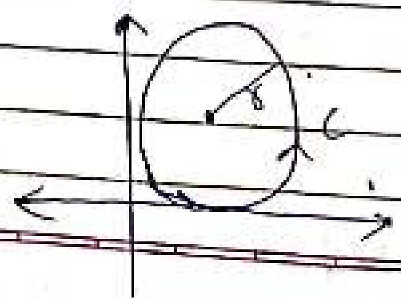


Multiply connected domain.

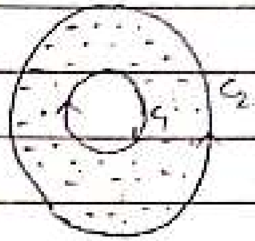
Definition (Possitive orientation) of domain D

The boundary c is said to have positive orientation or is said to traverse in the positive sense if a person walking on c always have the domain to ~~his~~ ~~left~~ his left.

Example :- The boundary of disk $|z-z_0| \leq r$ has positive orientation if it traverses in anticlock wise or counter clock wise direction.



**Example:** Consider the annulus betn the two circles $C_1$ & $C_2$ as shown in fig.




Then $C_2$ has positive orientation in counter clockwise direction & $C_1$ has positive orientation in clockwise direction.

**Note:-**
$C_1 : z_1(t) = e^{it} = \cos t + i\sin t \quad t \in [0, 2\pi]$
$C_2 : z_2(t) = e^{-it} = \cos t - i\sin t \quad t \in [0, 2\pi]$
$C_3 : z_3(t) = -e^{it} = -\cos t - i\sin t \quad t \in [0, 2\pi]$
$C_4 : z_3(t) = -e^{-it} = -\cos t + i\sin t \quad t \in [0, 2\pi]$

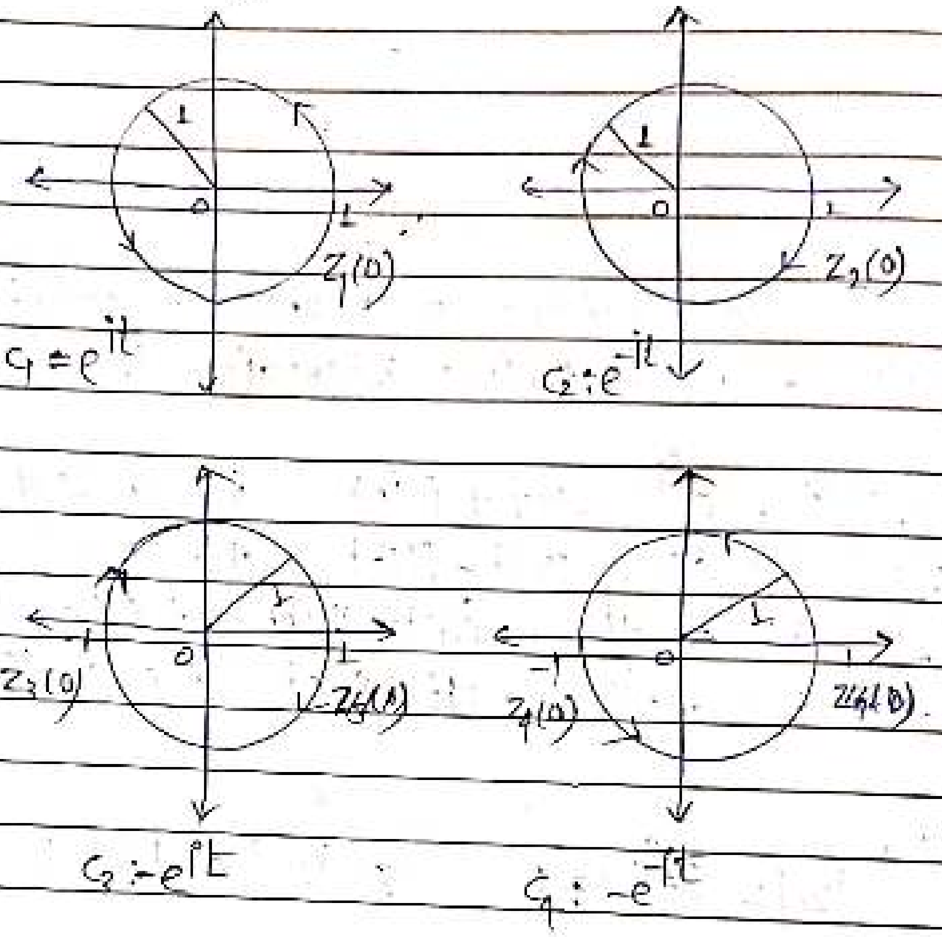
Here,
$$|z_1| = |e^{it}| = 1, \quad |z_2| = |e^{-it}| = 1$$
$$|z_3| = |-e^{it}| = 1 \ \& \ |z_4| = |-e^{-it}| = 1$$

i.e. $C_1, C_2, C_3$ & $C_4$ represents the unit circles but they are the different curves.

$\therefore$ Initial points of 1] $C_1$ is $z_1(0) = e^{i0} = \cos 0 + i\sin 0 = 1$
2] $C_2$ is $z_2(0) = e^{-i0} = \cos 0 - i\sin 0 = 1$
3] $C_3$ is $z_3(0) = -e^{i0} = -\cos 0 - i\sin 0 = -1$
4] $C_4$ is $z_4(0) = -e^{-i0} = -\cos 0 + i\sin 0 = -1$

We can observe that $C_1$ & $C_3$ has counter clockwise direction i.e. positive orientation but $C_2$ & $C_4$ has clockwise direction i.e. negative orientation.

i.e, four circles are different from each other as shown in fig.




2016 2-M

**Definition (Smooth Curve):**

A curve c having continuous derivative is called a smooth curve.

i.e $c : z(t) = x(t) + iy(t)$, $t \in [a, b]$ is said to be smooth curve if $x'(t)$ & $y'(t)$ are continuous function.

**Note:-** If c is smooth curve & $f(z)$ is continuous fun? on c that $f(z(t))$ & $z'(t)$ are continuous on c. where $z(t) = x(t) + iy(t)$ $t \in [a, b]$
If c is smooth $z'(t) = x'(t) + iy'(t)$ exist & is continuous.

Moreover,

$$z = z(t)$$

$$\Rightarrow \frac{dz}{dt} = z'(t)$$

$$\Rightarrow dz = z'(t)\,dt$$

& $f(z) = f(z(t))$

$$\therefore \int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Example: The curve $C: z(t) = e^{it} = \cos t + i\sin t \quad t \in [0,1]$ is a smooth curve.

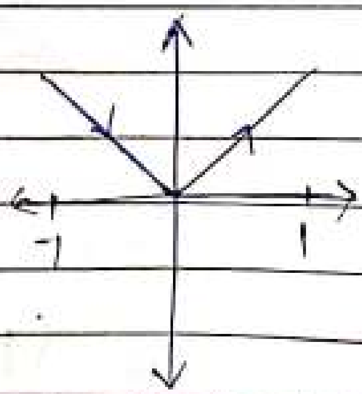
$\because x(t) = \cos t \qquad y(t) = \sin t$
$\quad\; x'(t) = -\sin t \qquad y'(t) = \cos t$

As $x'(t)$ & $y'(t)$ are continuous functions, $C: z(t)$ is smooth curve.

Example: $C: t + i|t| \quad t \in [-1, 1]$ is not a smooth curve because, here $y(t) = |t|$

$$= t \qquad t \in [0, 1]$$
$$= -t \qquad t \in [-1, 0]$$

$\therefore y'(t) = 1 \quad t \in [0, 1]$
$\quad\; y'(t) = -1 \quad t \in [-1, 0]$




$C: z(t): t + i|t|$

$\therefore y'(t)$ is not continuous.

Hence $C$ is not smooth curve.

Here $y(t) = |t|$ & $x(t) = t$

$\therefore y = |x| \quad t \in [-1, 1]$

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**Problem:** Evaluate $\int_c |z|^2 dz$ where c is defined as

2014 GM
(i) $c: c_1 : z_1(t) = t + it \quad 0 \le t \le 1$
(ii) $c_2 : z_2(t) = 1 + it^2 \quad 0 \le t \le 1$

**Solution:** Given $f(z) = |z|^2$

1] $c: c_1 : z_1(t) = t + it \quad t \in [0,1]$

$\therefore f(z) = f(z_1(t)) = |z_1(t)|^2 = |t + it|^2$

$= \left[\sqrt{t^2 + t^2}\right]^2 = 2t^2$

Now $dz = z'(t)dt$

Here, $z'(t) = z_1'(t) = 1 + i$

$\therefore \int_c f(z)dz = \int_{c_1} f(z)dz$

$= \int_0^1 f(z_1(t)) \cdot z_1'(t)dt$

$= \int_0^1 (2t^2)(1+i)dt$

$= (1+i)2 \int_0^1 t^2 dt$

$= (2+2i)\left(\frac{t^3}{3}\right)_0^1$

$= \left(\frac{2}{3} + 2i\right)\left[\frac{1}{3} - \frac{0}{3}\right]$

$\therefore \int_c f(z)dz = \frac{2}{3} + \frac{2}{3}i$

Now, $c : c_2 : z_2(t) = t + it^2 \qquad 0 \le t \le 1$

$$f(z) = f(z_2(t)) = |z_2(t)|^2 = |t + it^2|^2$$

$$= \left(\sqrt{t^2 + t^4}\right)^2$$

$$f(z) = t^2 + t^4$$

Now $dz = z'(t)\,dt \qquad z(t) = z_2(t) = t + it^2$

Here, $z'(t) = z_2'(t) = 1 + 2ti$

$$\int_c f(z)\,dz = \int_{c_2} f(z)\,dz$$

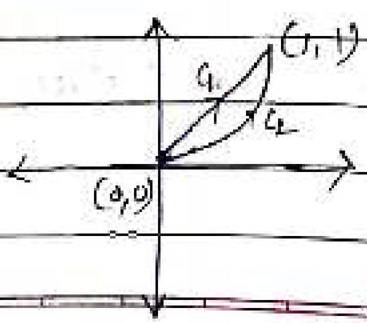
$$= \int_0^1 (t^2 + t^4)(1 + 2ti)\,dt$$

$$= \int_0^1 [t^2 + t^4 + i2t^3 + i2t^5]\,dt$$

$$= \left[\frac{t^3}{3} + \frac{t^5}{5} + \frac{i2t^4}{4} + \frac{i2t^6}{6}\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{1}{5} + \frac{i2}{4} + \frac{i2}{6}\right]$$

$$\int_c f(z)\,dz = \frac{8}{15} + i\frac{5}{6}$$




$$\int_c f(z)\,dz = \frac{8}{5} + \frac{5}{6}i$$

Note:- Here the curve $C_1$ & $C_2$ represent the straight line $y=x$ & parabola $y=x^2$ respectively.
Both the curves have same initial & terminal point but.

$$\int_{C_1} |z|^2 dz \neq \int_{C_2} |z|^2 dz$$

Problem: Evaluate $\int_C z^2 dz$ where,

2016 CM } 1] $C: C_1: z(t) = t+it$, $t \in [0,1]$

2015 3M → 2] $C: C_2: z(t) = t+it^2$, $t \in [0,1]$

Solution:- Given $f(z) = z^2$

$\therefore f(z(t)) = [z(t)]^2$

1] $C: C_1: z(t) = t+it$ $\quad t \in [0,1]$

$\therefore$ Here

$$f(z) = f(z_1(t))$$
$$= [z_1(t)]^2$$
$$= (t+it)^2$$
$$= t^2+2it^2+i^2t^2$$
$$= t^2+2it^2-t^2$$

$$f(z) = 2it^2$$

Now,

$$dz = z'(t)dt$$

$\therefore$ Here

$$dz = z_1'(t)dt$$
$$dz = (1+i)dt$$

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$\therefore \int_c f(z)dz = \int_0^1 f(z(t)) \cdot z'(t)dt$

$= \int_0^1 (2it^2)(1+i)dt$

$= 2i(1+i)\int_0^1 t^2 dt$

$= (2i+2i^2)\left(\frac{t^3}{3}\right)_0^1$

$= (2i-2)\left(\frac{1}{3}-0\right)$

$\int_c f(z)dz = \frac{2}{3}i - \frac{2}{3}$ ie, $\cancel{\int_c f(z)dz = -\frac{2}{3}+}$

ie, $\boxed{\int_c f(z)dz = -\frac{2}{3}+\frac{2}{3}i}$

2] $c: c_2: z_2(t) = t+t^2 i \quad t \in [0,1]$

$\therefore f(z) = f(z_2(t))$

$= (z_2(t))^2$

$= (t+it^2)^2$

$= t^2+2it^3+i^2t^4$

$\therefore f(z) = t^2+2it^3-t^4$

& $dz = z_2'(t)dt$

$dz = (1+2it)dt$

$\therefore \int_c f(z)dz = \int_0^1 f(z_2(t)) \cdot z'(t)dt$

$$\int_C f(z)\,dz = \int_0^1 (t^2 + 2it^3 - t^4)(1+2it)\,dt$$

$$= \int_0^1 \left[t^2 + 2it^3 - t^4 + 2it^3 + 4i^2t^4 - 2it^5\right] dt$$

$$= \left[\frac{t^3}{3} + \frac{2it^4}{4} - \frac{t^5}{5} + \frac{2it^4}{4} + \frac{4i^2t^5}{5} - \frac{2it^6}{6}\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{2i}{4} - \frac{1}{5} + \frac{2i}{4} + \frac{4i^2}{5} - \frac{2i}{6}\right]$$

$$= \left[\frac{1}{3} + \frac{2i}{4} - \frac{1}{5} + \frac{2i}{4} - \frac{4}{5} - \frac{2i}{6}\right]$$

$$= \left(\frac{1}{3} - \frac{1}{5} - \frac{4}{5}\right) + i\left[\frac{1}{2} + \frac{1}{2} - \frac{1}{3}\right]$$

$$= \left[\cancel{\frac{1}{3}} - \cancel{\frac{5}{5}}\right] \left[\frac{1}{3} - 1\right] + i\left[1 - \frac{1}{3}\right]$$

$$\int_C f(z)\,dz = -\frac{2}{3} + \frac{2}{3}i$$

$$\boxed{\int_C f(z)\,dz = -\frac{2}{3} + \frac{2}{3}i}$$

Problem :- Evaluate $\int_C \bar{z}\,dz$ along the curves

* 1] $z(t) = e^{it}$ $\quad t \in [-\pi, \pi]$ $\quad$ Ans. $\to 2\pi i$

2015 3M $\cdot$ 2] $z(t) = e^{2it}$ $\quad t \in [-\pi, \pi]$ $\quad$ Ans $\to 4\pi i$

3] $z(t) = e^{it} - 1 \quad t \in [-\pi, \pi]$ → Ans → $2\pi i$

4] $z(t) = t + it \quad t \in [0, 2]$

Oct'15 8M * 5] $z(t) = 5e^{it} + 3 \quad t \in [0, \pi]$

Solution:-

4] Here, $f(z) = \bar{z}$

$$\therefore f(z(t)) = \overline{z(t)}$$

Given, $c : z(t) = t + it, \quad t \in [0, 2]$

$$\therefore \overline{z(t)} = t - it$$

$$z'(t) = 1 + i$$

$$f(t) = f(z(t)) = \overline{z(t)} = t - it$$

$$\therefore \cancel{\int_c f(z)\,dz} \quad \int_c f(z)\,dz = \int_0^2 f(z(t)) \cdot z'(t)\,dt$$

$$\therefore \int_c f(z)\,dz = \int_0^2 (t - it)(1+i)\,dt$$

$$= (1+i) \int_0^2 (t - it)\,dt$$

$$= (1+i) \left( \frac{t^2}{2} - \frac{it^2}{2} \right)_0^2$$

$$= (1+i) \left( \frac{2^2}{2} - \frac{i2^2}{2} \right)$$

$$= (1+i)(2 - 2i)$$

$$= 2 + 2$$

$$= 4$$

$$\therefore \int_c f(z)\,dz = 4.$$

5] Here, $z(t) = 5e^{it}+3 \quad t \in [0,\pi]$

$$z'(t) = 5e^{it}\cdot i$$

$$z'(t) = 5ie^{it}$$

$$\overline{z(t)} = 5e^{-it}+3$$

$$\therefore \quad f(z) = f(z(t)) = \overline{z(t)} = 5e^{-it}+3$$

$$\therefore \quad \int_c f(z)\,dz = \int_0^{\pi} f(z(t))\, z'(t)\,dt$$

$$= \int_0^{\pi} (5e^{-it}+3)(5ie^{it})\,dt$$

$$= 5i \int_0^{\pi} (5e^{-it}+3)e^{it}\,dt$$

$$\int_c f(z)\,dz = 5i \int_0^{\pi} (5+3e^{it})\,dt$$

$$= 5i \left[5t + \frac{3e^{it}}{i}\right]_0^{\pi}$$

$$= \left[25it + \frac{15\cancel{i}e^{it}}{\cancel{i}}\right]_0^{\pi}$$

$$= \left[25i\pi + 15e^{i\pi} - 0 - 15e^{0}\right]$$

$$= \left[25i\pi + 15e^{i\pi} - 15\right]$$

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$\therefore \int_C f(z)\,dz = \cancel{25\pi i - 15 - 15}$ $\quad \because\ e^{i\pi} = \cos\pi + i\sin\pi$

$e^{i\pi} = -1$

$\int_C f(z)\,dz = 25\pi i - 30.$

$\therefore \int_C f(z)\,dz = -30 + 25\pi i.$

Problem: Along the curve $c : z(t) = e^{it}$ $(-\pi \le t \le \pi)$ Evaluate $\int_C f(z)\,dz$. for the following functions

1] $f(z) = z^2$

2] $f(z) = 1/z$

3] $f(z) = 1/z^2$

2014 4M 4] $f(z) = 2\bar{z} - i\left(z + 1/z\right)$

Solution:- Here $c : z(t) = e^{it}$ $\quad t \in [-\pi, \pi]$

$\therefore z'(t) = ie^{it}$

[3] $f(z) = 1/z^2$

$f(z(t)) = \dfrac{1}{[z(t)]^2}$

$= \dfrac{1}{(e^{it})^2} = \dfrac{1}{e^{2it}} = e^{-2it}$

$f(z(t)) = e^{-2it}$

$\therefore \int_c f(z)\,dz = \int_a^b \cancel{f(z(t))}\,d \quad \int_a^b f(z(t))\cancel{dt}\, z'(t)\,dt.$

$= \int_{-\pi}^{\pi} e^{-2it}\, i e^{it}\, dt$

$= i\int_{-\pi}^{\pi} e^{-2it} \cdot e^{it}\, dt$

$= i\int_{-\pi}^{\pi} e^{-it}\, dt$

$= i\left[\frac{e^{-it}}{-i}\right]_{-\pi}^{\pi} = -\left[e^{-it}\right]_{-\pi}^{\pi}$

$= -\left[e^{-\pi i} - e^{i(-\pi)}\right]$

$= -\left[e^{-i\pi} - e^{i\pi}\right]$

$-e^{-\pi i} + e^{\pi i}$

$-(-1)+(-1)$

$1-1$

$0.$

$e^{-\pi i} = \cos(-\pi) + i\sin(-\pi)$

$= \cos\pi - i\sin\pi = -1$

$e^{\pi i} = \cos\pi + i\sin\pi = -1$

$\therefore \int_c f(z)\,dz = -\left[(-1)+(-1)\right]$

$\therefore \int_c f(z)\,dz = 0.$

1] Given $f(z) = z^2$

$$f(z(t)) = [z(t)]^2$$

$$f(z(t)) = [e^{it}]^2 = e^{2it}$$

$$\therefore \int_c f(z)dz = \int_a^b f(z(t))z'(t)dt$$

$$= \int_{-\pi}^{\pi} e^{2it}\, ie^{it} dt$$

$$= i\int_{-\pi}^{\pi} e^{2it} e^{it} dt$$

$$= i\int_{-\pi}^{\pi} e^{3it} dt$$

$$= i\left[\frac{e^{3it}}{3i}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3}\left[e^{3i\pi} - e^{-3i\pi}\right]$$

$$e^{3i\pi} = \cos 3\pi + i\sin 3\pi = -1$$

$$= \frac{1}{3}\left[(-1) - (-1)\right]$$

$$e^{-3\pi i} = \cos 3\pi - i\sin 3\pi = -1$$

$$= \frac{1}{3} \times 0$$

$$\int_c f(z)dz = 0$$

[4] Here, $f(z) = 2\bar{z} - i(z + 1/z)$

Given $z(t) = e^{it}$

$\bar{z} = \overline{z(t)} = e^{-it}$.

$\therefore$ $f(z(t)) = 2\overline{z(t)} - i\left[z(t) + \frac{1}{z(t)}\right]$

$= 2e^{-it} - i\left[e^{it} + \frac{1}{e^{it}}\right]$

$= 2e^{-it} - i\left[e^{it} + e^{-it}\right]$

$\therefore \int_C f(z)dz = \int_a^b f(z(t))\, z'(t)\, dt$

$= \int_{-\pi}^{\pi} [2e^{-it} - i(e^{it} + e^{-it})]ie^{it}\,dt$

$= i\int_{-\pi}^{\pi} [2 - ie^{2it} - i]\,dt$

$= i\left[2t - \frac{ie^{2it}}{2i} - it\right]_{-\pi}^{\pi}$

$= i\left[2t - \frac{e^{2it}}{2} - it\right]_{-\pi}^{\pi}$

$\int_C f(z)dz = i\left\{\left[2\pi - \frac{e^{2\pi i}}{2} - i\pi\right] - \left[-2\pi - \frac{e^{-2\pi i}}{2} + i\pi\right]\right\}$

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$$\int_C f(z)\,dz = i\left\{2\pi - \frac{1}{2} - i\pi + 2\pi + \frac{1}{2} - \pi i\right\}$$

$$\because e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$$
$$e^{-2\pi i} = \cos 2\pi - i\sin 2\pi = 1$$

$$\therefore \int_C f(z)\,dz = i[4\pi - 2\pi i]$$

$$\therefore \int_C f(z)\,dz = 4\pi i - 2\pi i^2$$

$$\int_C f(z)\,dz = 4\pi i + 2\pi$$

$\boxed{2}$ Here $f(z) = 1/z$

Given $z(t) = e^{it}$

$$f(z(t)) = \frac{1}{[z(t)]}$$

$$= \frac{1}{[e^{it}]}$$

$$f(z(t)) = e^{-it}$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\cdot z'(t)\,dt$$

$$= \int_{\pi}^{\pi} e^{-it}\cdot ie^{it}\,dt$$

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$$\int_C f(z)\,dz = i\int_{-\pi}^{\pi} e^{-it} \cdot e^{it}\,dt$$

$$= i\int_{-\pi}^{\pi} e^{0}\,dt$$

$$= i\int_{-\pi}^{\pi} dt$$

$$= i\left[t\right]_{-\pi}^{\pi}$$

$$= i\left[\pi - (-\pi)\right]$$

$$= i\left[2\pi\right]$$

$$= 2\pi i$$

$$\therefore \int_C f(z)\,dt = 2\pi i$$

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* Paramatrizations

Introduction:-

Let $c$ be a curve paramatrized as $c: z(t) = x(t) + iy(t)$

Suppose $c$ is smoth curve, if we divide interval $[a, b]$ in two parts $[a, c]$ & $[c, b]$ then by using $z(t)$ we obtain two curves $c_1$ & $c_2$ as

$$c_1: z_1(t) = x(t) + iy(t) \qquad t \in [a, c]$$
$$c_2: z_2(t) = x(t) + iy(t) \qquad t \in [c, b]$$

& Suppose if $f(z)$ is continuous function $c$ then we have $c = c_1 + c_2$. Now we have;

$$\int_c f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^c f(z(t))\,z'(t)\,dt + \int_c^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^c f(z_1(t)) \cdot z_1'(t)\,dt + \int_c^b f(z_2(t)) \cdot z_2'(t)\,dt$$

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

Hence we have,

$$\int_{c_1+c_2} f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz.$$

In general if,

$$c = c_1 + c_2 + \cdots + c_n$$

then

$$\int_{c_1+c_2+\cdots+c_n} f(z)\,dz = \int_{c_1} f(z)\,dz + \cdots + \int_{c_n} f(z)\,dz$$

where, each $c_i$ is smooth curve.

**Definition** (Sectionally continuous function):

The function $f(z)$ is said to be sectionally continuous function on $[a,b]$ if it has at most finite number of discontinuities with right & left hand limit exist at each point of discontinuity.

The function $f(x) = [x]$ $x \in [-1, 1]$ is sectionally continuous function.

**Definition** (Contour):-

A curve $c$ having sectionally continuous derivative is called as contour.

i.e, $c : z(t) = x(t) + i\,y(t)$ $t \in [a,b]$

is said to be a contour, if $x'(t)$ & $y'(t)$ are sectionally continuous functions.

**Note:-** Every contour $c$ is the finite sum of smooth curves i.e, $c = c_1 + c_2 + \cdots + c_n$ where $c$ - contour & $c_i$ - smooth !!

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Example: $C: z(t) = t + i|t| \quad t \in [-1, 1]$

is contour

because here $x(t) = t$ & $y(t) = |t|$ are continuous on $[-1, 1]$

Also,

$x'(t) = 1$ is continuously on $[-1, 1]$

but,

$$y'(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t \leq 1 \end{cases}$$

is sectionally continuous on $[-1, 1]$

~~Curve~~ Curve

[ $C: z(t) = x(t) + iy(t)$
$x(t), y(t)$ - continuous ]

Smooth curve
{ $x'(t), y'(t)$ are continuous }

Contour
{ $x'(t), y'(t)$ are sectionally continuous }
Sectionally smooth.

Note: Every smooth curve is contour but every contour is not smooth.

Problem: Evaluate $\int_C z\,dz$ along the contour $C: z(t) = \begin{cases} 2t & 0 \leq t \leq 1 \\ 2 + i(t-1) & 1 \leq t \leq 2 \end{cases}$

Solution: Given contour is

$$C: z(t) = \begin{cases} 2t & 0 \leq t \leq 1 \\ 2 + i(t-1) & 1 \leq t \leq 2 \end{cases}$$

Suppose $\quad c_1 : z(t) = 2t \qquad 0 \le t \le 1$
$\qquad\qquad c_2 : z(t) = 2 + i(t-1) \qquad 1 \le t \le 2$

$\therefore\ c = c_1 + c_2$

$$\therefore \int_c f(z)dz = \int_{c_1+c_2} f(z)dz$$

$$= \int_{c_1} f(z)dz + \int_{c_2} f(z)dz \quad \cdots \text{(1)}$$

on $c_1$, $f(z) = z$
$\therefore\ f(z(t)) = z(t) = 2t \qquad 0 \le t \le 1$
& $\quad z'(t) = 2$

On $c_2$: $\ f(z) = z$
$\therefore\ f(z(t)) = z(t) = 2 + i(t-1) \qquad 1 \le t \le 2$

$\therefore\ z'(t) = i$

$\therefore$ (1) becomes,

$$\int_c z\,dz = \int_0^1 f(z(t))\cdot z'(t)\,dt + \int_1^2 f(z(t))\cdot z'(t)\,dt$$

$$= \int_0^1 2t(2)\,dt + \int_1^2 (2+i(t-1))\,i\,dt$$

$$= 4\int_0^1 t\,dt + i\int_1^2 (2+i(t-1))\,dt$$

$$\int_C z\,dz = \int_0 4\left[\frac{t^2}{2}\right]_0^1 + i\left[2t + i\left(\frac{t^2}{2} - t\right)\right]_1^2$$

$$= 4\left[\frac{1}{2} - 0\right] + i\left\{\left[2\times 2 + i\left(\frac{2^2}{2} - 2\right)\right] - \left[2\times 1 + i\left(\frac{1}{2} - 1\right)\right]\right\}$$

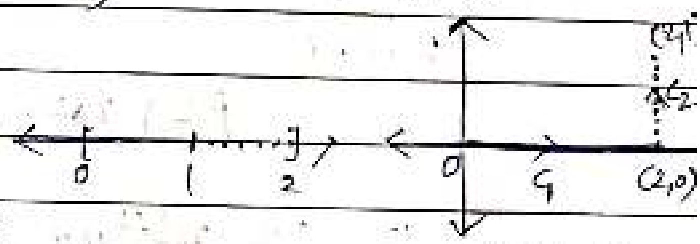
$$= \frac{4}{2} + i\left\{(4 + i(0)) - \left(2 + i\left(-\tfrac{1}{2}\right)\right)\right\}$$

$$= 2 + i\left\{4 - 2 + \frac{i}{2}\right\}$$

$z(t) : [0,2] \to \mathbb{C}$

$$= 2 + i\left(2 + \frac{i}{2}\right)$$

$$= 2 + 2i + \frac{i^2}{2}$$




$C = C_1 + C_2$

$$= 2 + 2i - \frac{1}{2}$$

$$\int_C z\,dz = \frac{3}{2} + 2i.$$

**Definition (Length of the Curve):**

If $c$ is a smooth curve or contour defined parametrically as $c : z(t) = x(t) + iy(t)$ $\quad t \in [a,b]$ then the length of the curve $c$ is denoted by $L$ & it is define as

$$L = \int_C |dz| = \int_a^b |z'(t)|\,dt .$$

Note: If $c$ is a contour then, the length of the $c$ is finite

**Note:-** If the curve is not contous, then the length of the curve may or may not be finite.

**Note:-** If $L<\infty$ i.e, the length of the curve is finite then the curve C is said tobe rectifiable curve.

**Formula:-** If c is an arbitrary curve defined on $[a,b]$ then the length of the curve is defined as

$$L = \sup_{P}\left\{\sum_{k=1}^{n}|z(t_k)-z(t_{k-1})|\right\} \quad P\text{-partition}$$

where

$$P=\{t_0=a, t_1, t_2, \ldots t_{n-1}, t_n=b\}$$

is an arbitary partition of $[a,b]$.

**Problem:-** Find the length of the contours

1] $z(t)=3e^{2it}+2 \qquad -\pi\leq t\leq\pi$

2016 4M 2] $z(t)=e^t\cos t+ie^t\sin t \qquad -\pi\leq t\leq\pi$

3] $z(t)=2e^{it} \qquad 0\leq t\leq\pi/2 \quad \rightarrow \pi$

4] $z(t)=e^{it} \qquad 0\leq t\leq\pi \quad \rightarrow \pi$

**Solution:-** 1] Given, $c: z(t)=3e^{2it}+2 \qquad -\pi\leq t\leq\pi$

$$\therefore z'(t)=3e^{2it}\cdot 2i+0$$

$$z'(t)=6ie^{2it}$$

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$$|z'(t)| = |6ie^{2it}|$$
$$= 6 \cdot |i| \cdot |e^{2it}|$$
$$= 6$$

$\therefore$ length of the curve is

$$\int_c |dz| = \int_a^b |z'(t)|\,dt$$

$$L = \int_{-\pi}^{\pi} 6\,dt$$

$$= 6\int_{-\pi}^{\pi} dt$$

$$= 6\left[t\right]_{-\pi}^{\pi}$$

$$= 6\left[\pi - (-\pi)\right] = 6\left[\pi + \pi\right] = 6\left[2\pi\right]$$

$$L = 12\pi \qquad \therefore \boxed{L = 12\pi}$$

2] Given $c : z(t) = e^t \cos t + ie^t \sin t \quad -\pi \le t \le \pi$

$$z(t) = e^t(\cos t + i\sin t) \qquad -\pi \le t \le \pi$$
$$z(t) = e^t \cdot e^{it} \qquad -\pi \le t \le \pi$$
$$\therefore z(t) = e$$

$$\therefore c : z(t) = e^{t+it} \qquad -\pi \le t \le \pi$$
$$c : z(t) = e^{t(1+i)} \qquad -\pi \le t \le \pi$$

$$z'(t) = e^{t(1+i)}(1+i)$$

$$|z'(t)| = |e^{t(1+i)} \cdot (1+i)|$$

$$= |1+i| \cdot |e^{t(1+i)}|$$

$$= |1+i| \cdot |e^{t} \cdot e^{it}|$$

$$= \sqrt{2} \cdot e^{t}$$

$$|z'(t)| = \sqrt{2} e^{t}$$

$\therefore$ length of the curve is

$$L = \int_c |dz|$$

$$= \int_a^b |z'(t)| dt$$

$$= \int_{-\pi}^{\pi} \sqrt{2} e^{t}$$

$$= \sqrt{2} \int_{-\pi}^{\pi} e^{t} dt \qquad = \sqrt{2} \left[ e^{t} \right]_{-\pi}^{\pi}$$

$$= \sqrt{2} \left[ e^{\pi} - (e^{-\pi}) \right]$$

$$L = \sqrt{2} \left[ e^{\pi} - e^{-\pi} \right]$$

3] $z(t) = 2e^{it} \qquad 0 \le t \le \pi/2$

$z'(t) = 2ie^{it}$

$|z'(t)| = |2ie^{it}| = 2$

length of the curve is $\int_c |dz| = \int_a^b |z'(t)| dt$

$$L = \int_0^{\pi/2} 2\,dt = 2\int_0^{\pi/2} dt = 2[t]_0^{\pi/2}$$

$$= 2\left[\pi/2\right]$$

$$L = \pi$$

$\therefore$ length of the curve $L = \pi$

4] Given $c : z(t) = e^{it} \quad 0 \leq t \leq \pi$

$$z(t) = e^{it} \qquad t\,[0, \pi]$$

$$z'(t) = ie^{it}$$

$$|z'(t)| = |ie^{it}| = |i| \cdot |e^{it}|$$

$$|z'(t)| = 1$$

$\therefore$ length of the curve is

$$\int_c |dz| = \int_a^b |z'(t)|\,dt = \int_0^{\pi} 1\,dt$$

$$= [t]_0^{\pi}$$

$$= [\pi - 0]$$

$$= \pi.$$

$$\therefore \boxed{L = \pi}$$

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# Theorem: M.L. Inequality :

**Statement:** Suppose $f(z)$ is continuous func on a contour $C$ such that $|f(z)| \le M$ on $C$ then,

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le M\int_C |dz| = ML.$$

where $L$ - length of the curve $C$.

**Proof :-**

Given, $C$ is a contour having length $L$.

$$\therefore L = \int_C |dz| \quad \cdots (1)$$

Also given that $f(z)$ is continuous function $C$ with $|f(z)| \le M$ on $C$ $\cdots (2)$

Suppose, the curve $C$ is parametrised as

$$C: z(t) = x(t) + iy(t) \quad t \in [a,b]$$

$\therefore$ We have,

$$\int_C f(z)dz = \int_a^b f(z(t)) \cdot z'(t)dt$$

$$\therefore \left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t)) \cdot z'(t)dt\right|$$

$$\le \int_a^b |f(z(t))| \, |z'(t)|dt$$

$$\le \int_C |f(z)| \cdot |dz| \quad \cdots (3)$$

$$\le \int_C M|dz| \quad (\because \text{by } (2))$$

$$\left|\int_C f(z)dz\right| \leq M\int_C |dz| \quad \ldots (4)$$

$\therefore$ by (3) & (4), we get

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|\cdot|dz|$$

$$\leq M\int_C |dz| = ML$$

$(\because$ by $(1'))$

ie, $\left|\int_C f(z)dz\right| \leq \int_C |f(z)||dz| \leq M\int_C |dz| = ML$

This is M.L inequality

**Problem:** Find the bound of the integral $\int_C \frac{dz}{z^2+10}$.

where $C$ is the circle parametrised as

$C: z(t) = e^{it} \quad -\pi \leq t \leq \pi$

(OR)

Show that $\left|\int_C \frac{dz}{z^2+10}\right| \leq \frac{2\pi}{9}$

**Solution:** Given $f(z) = \frac{1}{z^2+10}$

Clearly, $f(z)$ is continuous everywhere in complex plane except at $z = \pm\sqrt{10}\,i$ & $C$ is given by

$$C: z(t) = e^{it} \quad -\pi \leq t \leq \pi$$

$\therefore$ On $C$,

$$|z| = |z(t)| = |e^{it}| = 1 \quad \ldots (1)$$

Clearly c represents a unit circle & hence $f(z)$ is continuous on c.

We know by triangle inequality

$$||z_1|-|z_2|| \le |z_1 - z_2| \quad \forall z_1, z_2$$

$$-|z_1-z_2| \le ||z_1|-|z_2|| \le |z_1-z_2|$$

Hence $\forall z_1, z_2$ we have.

Now put $z_1 = -10$ & $z_2 = z^2$

then, we get

$$|-10| - |z^2| \le |-10-z^2|$$

$$10 - |z|^2 \le |-(z^2+10)|$$

$$10 - |z|^2 \le |z^2+10|$$

$$|z^2+10| \geq 10 - |z|^2$$

$$\frac{1}{|z^2+10|} \le \frac{1}{10-|z|^2} \quad \cdots ②$$

Now, consider,

$$|f(z)| = \left|\frac{1}{z^2+10}\right|$$

$$\le \frac{1}{10-|z|^2} \quad \because \text{by } ②$$

$$\le \frac{1}{10-1} \quad \because \text{by } ①$$

$$\therefore |f(z)| \le 1/9 \text{ on } c \quad \text{by} \cdots ③$$

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Now on $c$, $z(t) = e^{it}$ $t \in (-\pi, \pi]$

$\therefore z'(t) = ie^{it}$

$\therefore |z'(t)| = |i| \cdot |e^{it}|$

$= 1$

$\therefore$ length of the curve $c$ is

$$L = \int_c |dz|$$

$$= \int_a^b |z'(t)| dt$$

$$= \int_{-\pi}^{\pi} 1 \, dt$$

$$= [t]_{-\pi}^{\pi}$$

$$= [+\pi - (-\pi)]$$

$$L = 2\pi$$

Here, $f(z)$ is continuous on contour $c$ with $|f(z)| \le 1/3$ on $c$ & $L = 2\pi$

$\therefore$ by M.L inequality

$$\left| \int_c f(z) dz \right| \le M \cdot L$$

$$\therefore \left| \int_c \frac{dz}{z^2+10} \right| \leq \frac{1}{9} \cdot 2\pi$$

$$\therefore \left| \int_c \frac{dz}{z^2+10} \right| \leq \frac{2\pi}{9}$$

$\therefore$ bound of $\int_c \frac{dz}{z^2+10}$ is $\frac{2\pi}{9}$

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**Example:** Show that $\left| \int_c \frac{dz}{z^2+10} \right| \leq \frac{2\pi}{3}$

where $c: z(t) = 2e^{it}$ $t \in [-\pi, \pi]$.

**Solution:-** Given $f(z) = \frac{1}{z^2+10}$

Clearly $f(z)$ is continuous everywhere in complex plane except at

$$z = \pm \sqrt{10}\, i \quad \&$$

$c: z(t) = 2e^{it}$ $t \in [-\pi, \pi]$

On $c$,

$$|z| = |z(t)| = |2e^{it}| = 2|e^{it}|$$

$$|z| = 2 \quad \cdots (1)$$

Clearly $c$ represents a circle with radius 2

Hence $f(z)$ is continuous on $c$.

We know by triangle inequality

$$\big| |z_1| - |z_2| \big| \leq |z_1 - z_2|$$

$$-|z_1-z_2| \le \big||z_1|-|z_2|\big| \le |z_1-z_2|$$

Here, $z_1, z_2$ we have,
Now put $z_1=-10$ & $z_2=z^2$
then we get,

$$|-10|-|z^2| \le |-10-z^2|$$

$$10-|z|^2 \le |-(z^2+10)|$$

$$10-|z|^2 \le |z^2+10|$$

$$|z^2+10| \ge 10-|z|^2$$

$$\frac{1}{|z^2+10|} \le \frac{1}{10-|z|^2} \quad \text{--- (2)}$$

Now consider,

$$|f(z)| = \left|\frac{1}{z^2+10}\right|$$

$$\le \frac{1}{10-|z|^2} \quad \because \text{by (2)}$$

$$\le \frac{1}{10-4} \quad \because \text{by (1)}$$

$$\therefore |f(z)| \le \frac{1}{6} \quad \text{on } c \quad \text{--- (3)}$$

Now, on c, $z(t)=2e^{it}$. $t \in [-\pi,\pi]$

$$z'(t) = 2ie^{it}$$

$$|z'(t)| = |2|\,|i|\,|e^{it}|$$

$$= 2.$$

$$|z'(t)| = 2 \dots$$

∴ length of the curve 'c' is,

$$L = \int_c |dz|$$

$$= \int_a^b |z'(t)dt| \, dt$$

$$= \int_{-\pi}^{\pi} 2 \, dt$$

$$= 2\int_{-\pi}^{\pi} 1 \cdot dt$$

$$= 2\left[t\right]_{-\pi}^{\pi}$$

$$= 2[\pi - (-\pi)]$$

$$= 2[2\pi]$$

$$= 4\pi$$

Here $f(z)$ is continuous on contour c with $|f(z)| < 1/6$ on c. & $L = 4\pi$

$$\left|\int_c f(z)dz\right| \leq M \cdot L$$

$$\therefore \left|\int_c \frac{dz}{z^2+10}\right| \leq \frac{1}{6} \cdot 4\pi$$

$$\leq \frac{4\pi}{6}$$

$$\leq \frac{2\pi}{3}$$

$$\left|\int_c \frac{dz}{z^2+10}\right| \leq 2\pi/3$$

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Note:- If $c: z(t) = x(t) + iy(t)$, $t \in [a,b]$ then
$-c: z(-t) = x(-t) + iy(-t)$ $t \in [a,b]$.

$$\int_{-c} f(z)dz = -\int_c f(z)dz.$$

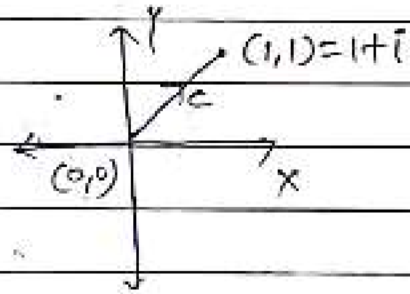
Problem:- Evaluate, $\int_c |z|dz$, $\int_c x\,dz$ & $\int_c y\,dz$

where c is the ~~curve~~ joining origin to the point 1+i straight line.

Solution:- Given c is the straight line joining origin to the pt 1+i

∴ c can be parametrised as

$c: z(t) = t + it$ $0 \le t \le 1$




∴ Here, $x(t) = t$ & $y(t) = t$

1) Now $f(z) = |z|$

∴ $f(z(t)) = |z(t)|$

∴ $f(z(t)) = |t+it| = \sqrt{t^2+t^2} = \sqrt{2t^2} = \sqrt{2}\,t$

& $z'(t) = 1+i$

$$\therefore \int_c f(z)dz = \int_a^b f(z(t))z'(t)dt$$

$$\therefore \int_c |z|dz = \int_0^1 (\sqrt{2})t\,(1+i)$$

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$$\int_c |z|\,dt = \int_0^1 (|z|)\,t\,(1+i)\,dt$$

$$= \sqrt{2}\,(1+i)\int_0^1 t\,dt$$

$$= \sqrt{2}\,(1+i)\left(\frac{t^2}{2}\right)_0^1$$

$$= \sqrt{2}\,(1+i)\left(\frac{1}{2}-0\right)$$

$$= \frac{\sqrt{2}\,(1+i)}{2}$$

$$\int_c |z|\,dz = \frac{(1+i)}{\sqrt{2}}$$

2) $\int_c x\,dz = ?$

$$\int_c x\,dz = \int_0^1 x(t)\,z'(t)\,dt$$

$$= \int_0^1 t\,(1+i)\,dz$$

$$= (1+i)\int_0^1 t\,dt$$

$$= (1+i)\left[\frac{t^2}{2}\right]_0^1$$

$$= (1+i)\left(\tfrac{1}{2}-0\right)$$

$$\int_c x\,dz = \frac{1+i}{2}$$

3] $\int_c y\,dz = \int_0^1 y(t)\cdot z'(t)\,dt$

$= \int_0^1 t\cdot(1+i)\,dz$

$= (1+i)\int_0^1 t\,dz$

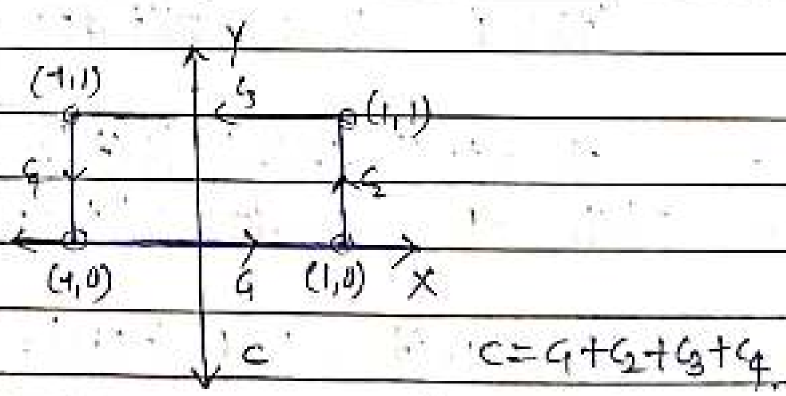
$= (1+i)\left[\frac{t^2}{2}\right]_0^1$

$= (1+i)\left(\frac{1}{2}-0\right)$

$\int_c y\,dz = \frac{(1+i)}{2}$

Problem: Evaluate $\int_c |z|\,dz$ along the rectangle $c$ having corners : $-1, +1, 1+i, -1+i$

Solution:-




Given $c$ is a rectangle having corners $-1, 1, 1+i, -1+i$.

$\therefore$ $C$ is the sum of the 4 curves $c_1, c_2, c_3$ & $c_4$ as shown in fig.

i.e. $C = c_1+c_2+c_3+c_4$

Where $c_i$'s are parametrised as,

$$C_1 : z(t) = t + i\cdot 0 \qquad -1 \le t \le 1$$
$$C_2 : z(t) = 1 + it \qquad 0 \le t \le 1$$
$$C_3 : z(t) = -t + i \qquad -1 \le t \le 1$$
$$C_4 : z(t) = -1 - it \qquad 0 \le t \le 1$$

$C_1$ { $y$ is const ∴ 0
& $x$ change ∴ $t$

Now,

$$\int_C |z|dz = \int_{C_1+C_2+C_3+C_4} |z|dz.$$

$$\int_C |z|dz = \int_{C_1} |z|dz + \int_{C_2} |z|dz + \int_{C_3} |z|dz + \int_{C_4} |z|dz. \quad \ldots (1)$$

On $C_1$, $z(t) = t \quad \therefore |z| = |z(t)| = |t|$ , $t \in [-1,1]$
$z'(t) = 1$ , $\therefore dz = z'(t)\,dt = dt$.

On $C_2$, $z(t) = 1+it \quad \therefore |z| = |z(t)| = |1+it| = \sqrt{1+t^2}$
$z'(t) = i \quad \therefore dz = z'(t)dt = i\,dt$ , $t \in [0,1]$

On $C_3$, $z(t) = -t+i \quad \therefore |z| = |z(t)| = |-t+i| = \sqrt{t^2+1}$
$z'(t) = -1 \quad \therefore dz = z'(t)dt = -dt$ , $t \in [-1,1]$

On $C_4$, $z(t) = -1-it \quad \therefore |z| = |z(t)| = |-1-it| = \sqrt{1+t^2}$
$z'(t) = -i \quad \therefore dz = z'(t)dt = -i\,dt$ , $t \in [0,1]$

∴ (1) becomes,

$$\int_C |z|dz = \int_{-1}^{1} |t|dt + \int_0^1 \sqrt{1+t^2}\, i\,dt + \int_{-1}^{1} \sqrt{t^2+1}\,(-dt) + \int_0^1 \sqrt{1+t^2}\,(-i\,dt)$$

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If $f(x) = f(-x)$ then $f$ is even fun e.g. $|x|$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(x)$ is even & $\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd fun.

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$$\therefore \int_C |z|\,dz = \int_{-1}^{1} |t|\,dt + i\int_0^1 \sqrt{1+t^2}\,dt - \int_{-1}^{1} \sqrt{1+t^2}\,dt - i\int_0^1 \sqrt{1+t^2}\,dt$$

$$= \int_{-1}^{1} |t|\,dt - \int_{-1}^{1} \sqrt{1+t^2}\,dt + i\left[\int_0^1 \left(\sqrt{1+t^2} - \sqrt{1+t^2}\right)dt\right]$$

$$= \int_{-1}^{1} |t|\,dt - \int_{-1}^{1} \sqrt{1+t^2}\,dt + 0.$$

$$= 2\int_0^1 |t|\,dt - 2\int_0^1 \sqrt{1+t^2}\,dt \qquad \because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$f(-x) = f(x)$ even

$\because |t|$ & $\sqrt{1+t^2}$ are even functions.

In first integral on R.H.S we have

$t \geq 0 \quad \therefore |t| = t$

$\therefore$ We get,

$$\int_C |z|\,dz = 2\int_0^1 t\,dt - 2\int_0^1 \sqrt{1+t^2}\,dt$$

$$= 2\left[\frac{t^2}{2}\right]_0^1 - 2\left\{\frac{t}{2}\sqrt{t^2+1} - \frac{1}{2}\log\left(t + \sqrt{t^2+1}\right)\right\}_0^1$$

$$= \left(t^2\right)_0^1 - \left\{t\sqrt{t^2+1} - \log\left(t + \sqrt{t^2+1}\right)\right\}_0^1$$

$$= (1-0) - \left\{\left[1\cdot\sqrt{1+1} - \log\left(1+\sqrt{1+1}\right)\right] - \left[0 - \log\left(0+\sqrt{0+1}\right)\right]\right\}$$

$$\int_C |z|dz = 1 - \{\sqrt{2} - \log(1+\sqrt{2}) + \log 1\}$$

$$= 1 - \sqrt{2} + \log(1+\sqrt{2}) \qquad \because \log 1 = 0$$

$$\therefore \int_C |z|dz = 1 - \sqrt{2} + \log(1+\sqrt{2}).$$

H.W.

Problem: $\int_C |z| dz$ , $c: re^{it}$ $0 \le t \le 2\pi$ Ans - 0.

Solution:-

given $\int_C |z|dz$ &

$c: re^{it}$ $0 \le t \le 2\pi$

$f(z) = |z|$

$c: z(t) = re^{it}$

$f(z(t)) = |z(t)|$

$= |re^{it}|$

$= |r| \cdot |e^{it}|$

$= (\sqrt{x^2+y^2})^{1/2} \cdot 1$

$f(z(t)) = x^2+y^2 \quad \cdots (1)$

But $re^{it} = r\cos t + i\sin t$

Here,

$x(t) = r\cos t$ & $y(t) = r\sin t$.

thus we get

$$x^2+y^2 = r^2\cos^2 t + r^2\sin^2 t$$
$$= r^2(\cos^2 t + \sin^2 t)$$
$$= r^2$$

put in eqn ①, we get,

$$f(z(t)) = r^2 \quad \cdots ②$$

$$z'(t) = rie^{it}$$

We know that,

$$\int_c f(z)dz = \int_a^b f(z(t))\cdot z'(t)dt$$

$$\int_c |z|dz = \int_0^{2\pi} r^2 \cdot rie^{it}dt$$

$$= r^3 i\int_0^{2\pi} e^{it}dt$$

$$= r^3 \cancel{i}\left[\frac{e^{it}}{\cancel{i}}\right]_0^{2\pi}$$

$$= r^3\left[e^{2\pi i} - e^0\right]$$

$$= r^3\left[(\cos 2\pi + i\sin 2\pi) - 1\right] \quad \because e^{2k\pi i} = 1$$

$$= r^3[1-1]$$

$$= r^3[0]$$

$$\therefore \int_c |z|dz = 0$$

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