① $W = Z^2$

Given $W = Z^2$

i.e, $u + iv = (x+iy)^2$

$u + iv = x^2 + 2ixy + i^2y^2$

$\therefore u + iv = x^2 + 2ixy - y^2$

$\therefore u + iv = (x^2 - y^2) + 2ixy$

$u = x^2 - y^2 \quad v = 2xy$

* Observations :

[i] Consider the point $(a, a) = z$

$\therefore$ Here we have $x = a,\ y = a$

$\therefore u = x^2 - y^2 \quad v = 2xy$

$\Rightarrow u = a^2 - a^2 \quad v = 2 \cdot a \cdot a$

$\Rightarrow u = 0 \ , \ v = 2a^2$

$w = (u, v) = (0, 2a^2)$ here $v \geq 0 \ \forall a$

Thus, the image of point $z = (a, a)$ is the point $w = (0, 2a^2)$

Now consider the point $z' = (-a, -a)$

$\therefore$ Here we have $x = -a \ , \ y = -a$

$\therefore u = x^2 - y^2$ & $v = 2xy$

$\Rightarrow u = (-a)^2 - (-a)^2$ and $v = 2(-a)(-a)$

$\Rightarrow u = 0$ and $v = 2a^2 \geq 0$

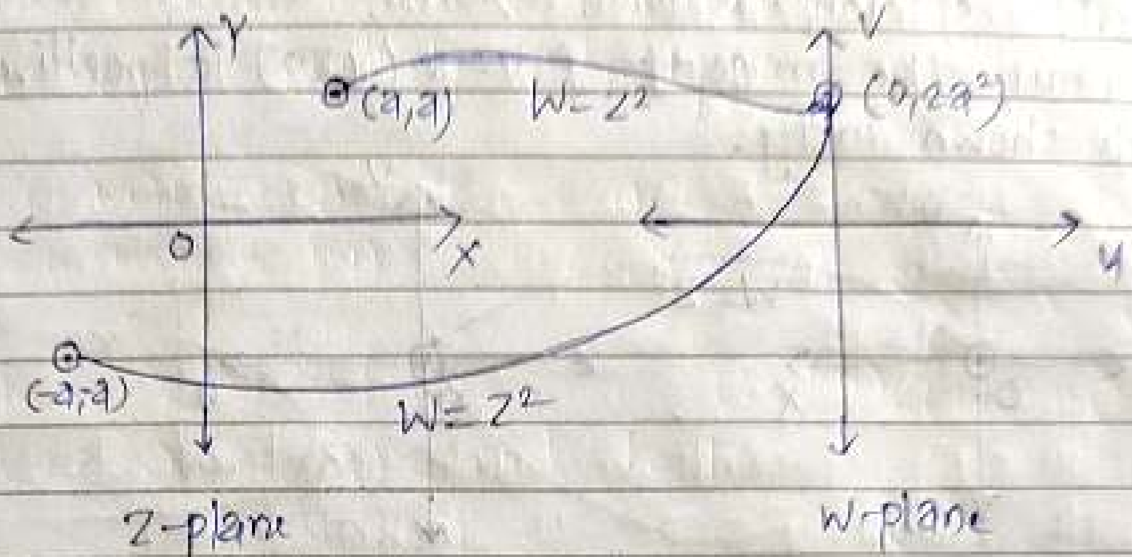
$\therefore W = (u, v) = (0, 2a^2)$

$\therefore$ Image of point $z' = (-a, -a)$ is the point $w = (0, 2a^2)$

Scanned by CamScanner

Thus, $z \neq z' \Rightarrow f(z) = f(z')$

Thus the function $f(z) = z^2$ is not one-one.



[II] The line $y = x$.

Case(I) Let $y = x = 0$ $\therefore z = (0,0) = 0$

then $u = x^2 - y^2 = 0$ & $v = 2xy = 0$

$\therefore W = (u,v) = (0,0) = 0$

Thus, the origin $z = 0$ is mapped onto the origin $w = 0$

Case(II): Let $y = x$ & $x > 0$

$\therefore u = x^2 - y^2 = x^2 - x^2 = 0$

& $v = 2xy = 2 \cdot x \cdot x = 2x^2 > 0$ i.e. $v > 0$

$\therefore W = (u,v) = (0, 2x^2)$ where $v > 0$

This represent positive v-axis.

Thus, image of ray $y = x$ $(x > 0)$ is the positive v-axis

Case (III): Let $y = x$ & $x < 0$

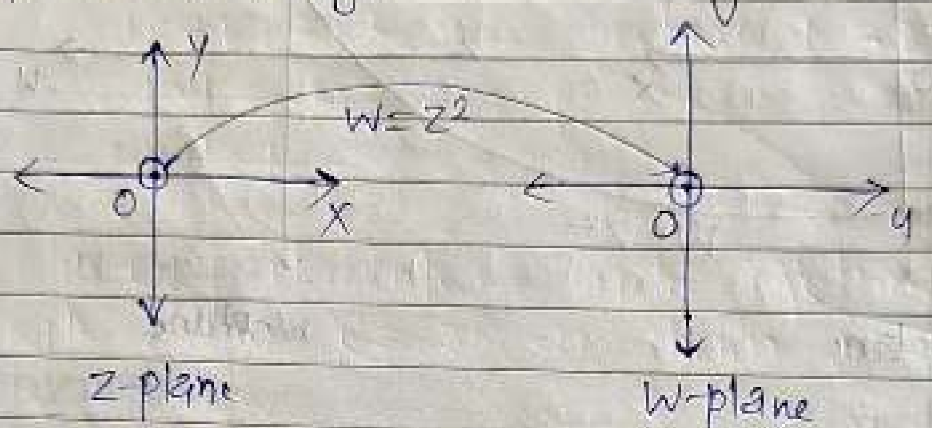
$\therefore u = x^2 - y^2 = x^2 - x^2 = 0$

& $v = 2xy = 2x \cdot x = 2x^2 > 0$

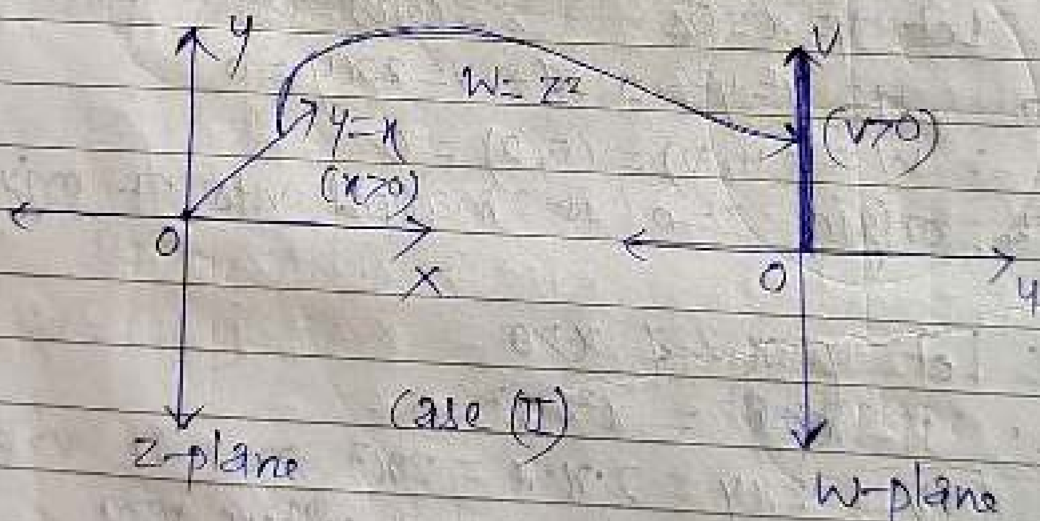
$\therefore \quad w = (u, v) = (0, 2x^2)$, $v > 0$

It represent positive v-axis.

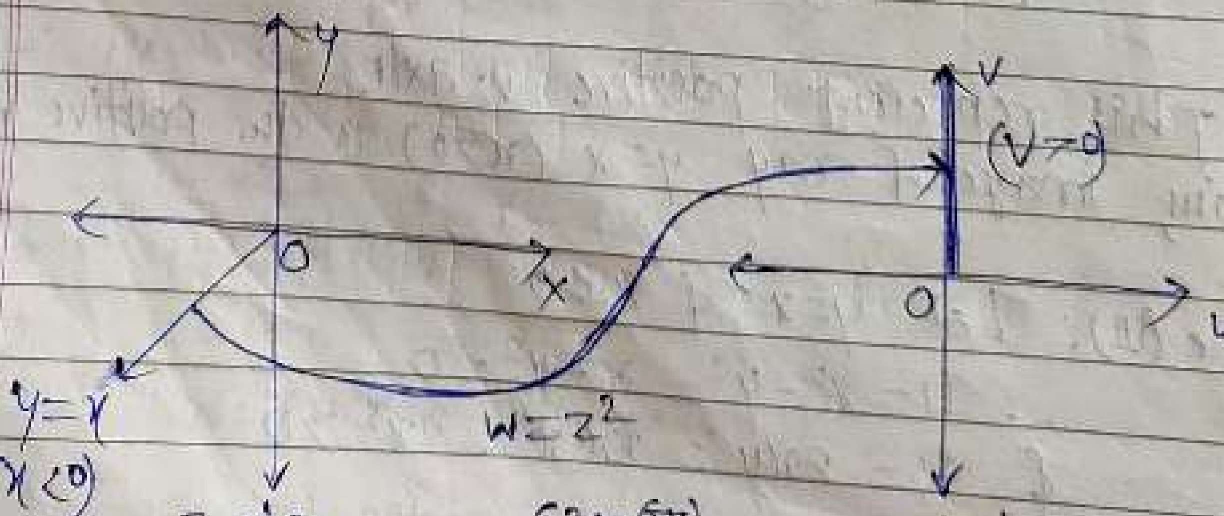
Thus, the image of $y = x$ $(x < 0)$ is positive v-axis as shown in fig.



Case (I)



Case (II)



Case (III)

[iii] line $y = mx$ $(m \neq \pm 1)$

Given $y = mx$, $m \neq \pm 1$

then $u = x^2 - y^2$. & $v = 2xy$

$= x^2 - m^2 x^2$ $\quad = 2xmx$

$= (1-m^2)x^2$ $\quad = 2mx^2$

$\Rightarrow x^2 = \frac{u}{1-m^2}$ $\quad \therefore v = \frac{2m \cdot u}{(1-m^2)}$

Hence $v = \left(\frac{2m}{1-m^2}\right) u$

$v = Ku$ where $K = \frac{2m}{1-m^2}$

This eqn represents the straight line passing through origin.

Thus, the image of the straight line $y = mx$ under the map $W = Z^2$ is again a straight line $v = K \cdot u$.

[iv] The circle $|z| = r$

Given $|z| = r$

$\therefore |z|^2 = r^2$

$\Rightarrow x^2 + y^2 = r^2 \quad \cdots \; ①$

We have

$u = x^2 - y^2$, $v = 2xy$

$u^2 = (x^2 - y^2)^2$ & $v^2 = 4x^2y^2$

$u^2 = x^4 - 2x^2y^2 + y^4$ & $v^2 = 4x^2y^2$

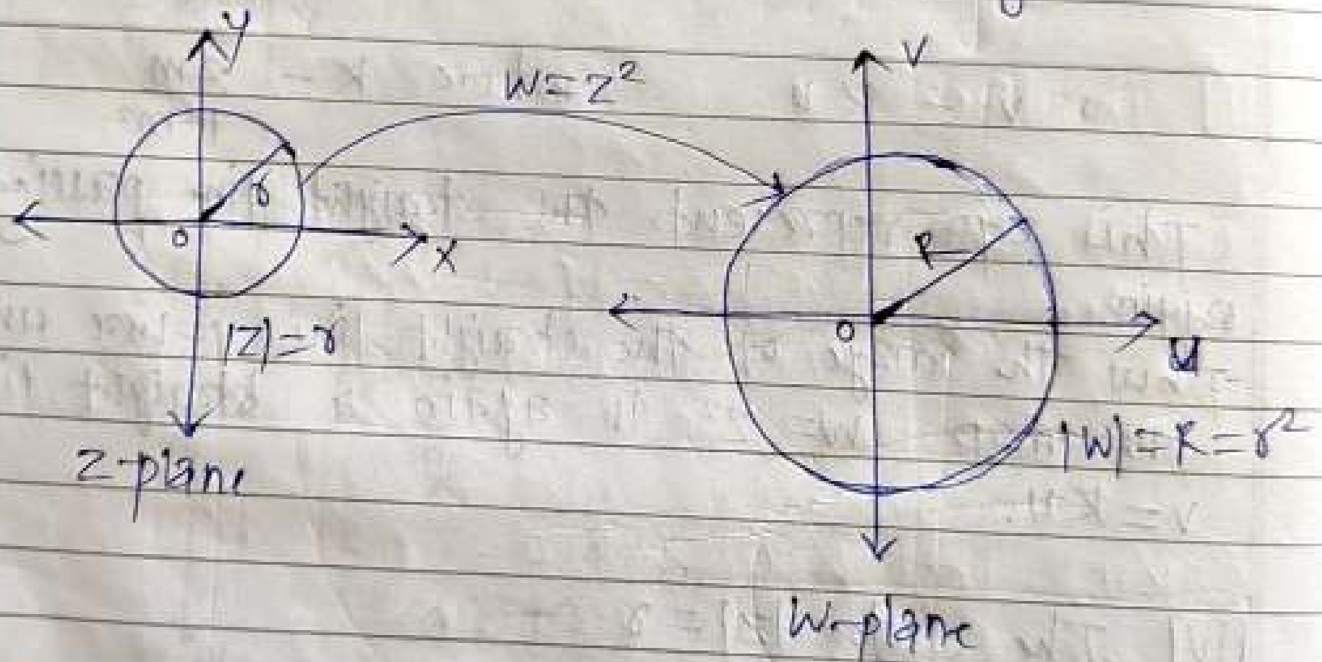
$$\therefore u^2+v^2 = x^4 - 2x^2y^2 + y^4 + 4x^2y^2$$
$$= x^4 + 2x^2y^2 + y^4$$
$$= (x^2+y^2)^2$$
$$= (r^2)^2 \qquad \{\because \text{ by } ①\}$$

$\therefore u^2+v^2 = R$ where $R = r^2$

$\therefore |w|^2 = R^2$

$\therefore |w| = R$

This eqn represents a circle centered at origin with radius $R = r^2$ as shown in fig.



[V] The unit circle $|z|=1$

The image of unit circle $|z|=1$ is again a unit circle $|w|=1$

[VI] The straight lines $y = \pm c$ $(c \neq 0)$

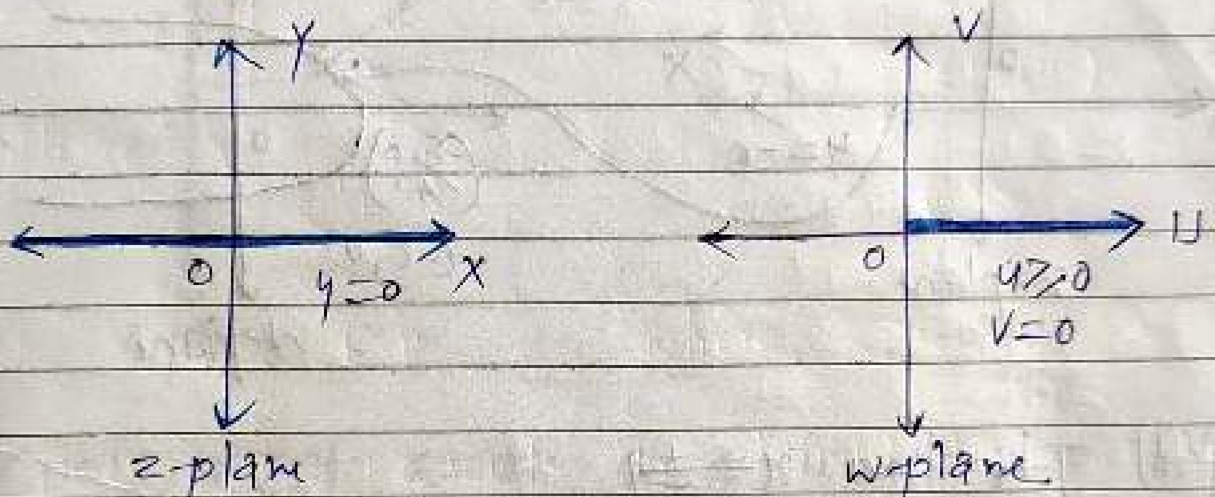
Case(I): If $c=0$ then

$y = \pm c \Rightarrow 0$

which represents the x-axis.
then we have

$$u = x^2 - y^2 = x^2 \quad \& \quad v = 2xy = 0$$

Clearly $u \geq 0$ also we have $v = 0$
This represents the non-negative u-axis.
Hence, the image of x-axis is the non-negative u-axis is shown in figure.



Case (II): $y = \pm c, \quad c \neq 0$

$$\therefore \; u = x^2 - y^2 \quad \& \quad v = 2xy$$
$$= x^2 - c^2 \qquad\qquad = 2cx$$

$$\therefore \; v^2 = 4c^2x^2$$

$$\therefore \; x^2 = \frac{v^2}{4c^2}$$

$$\therefore \; u = x^2 - c^2$$

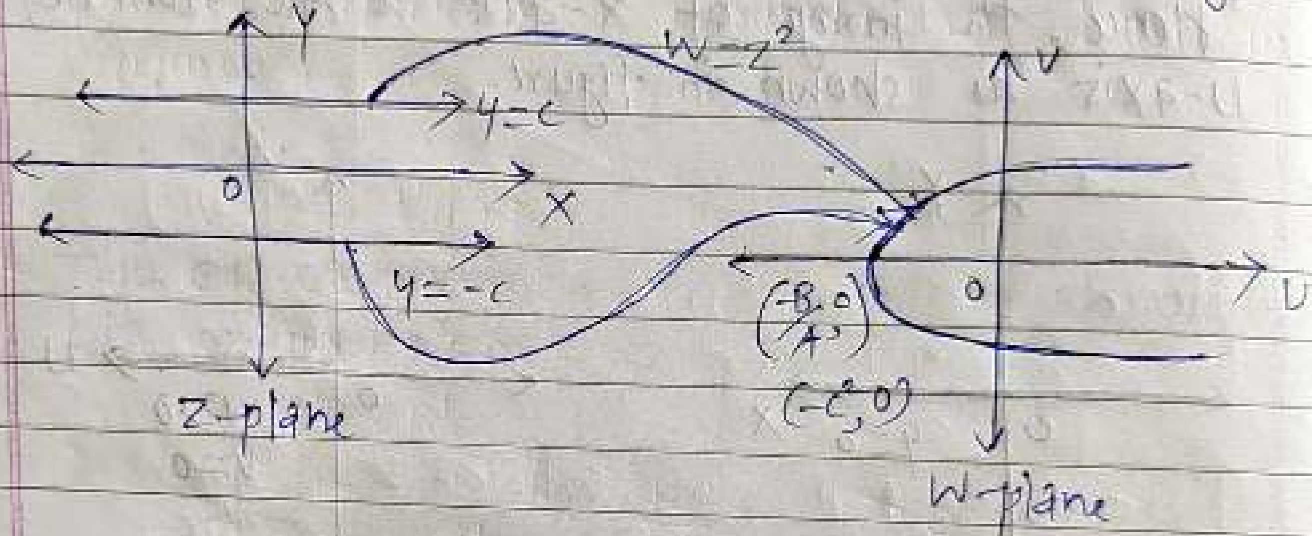
$$\Rightarrow \; u = \frac{v^2}{4c^2} - c^2 = \frac{v^2 - 4c^4}{4c^2} = \frac{v^2 - 4c^4}{4c^2}$$

$$\Rightarrow \; \frac{v^2}{\cancel{4c^2}} = (4c^2)u + 4c^4$$

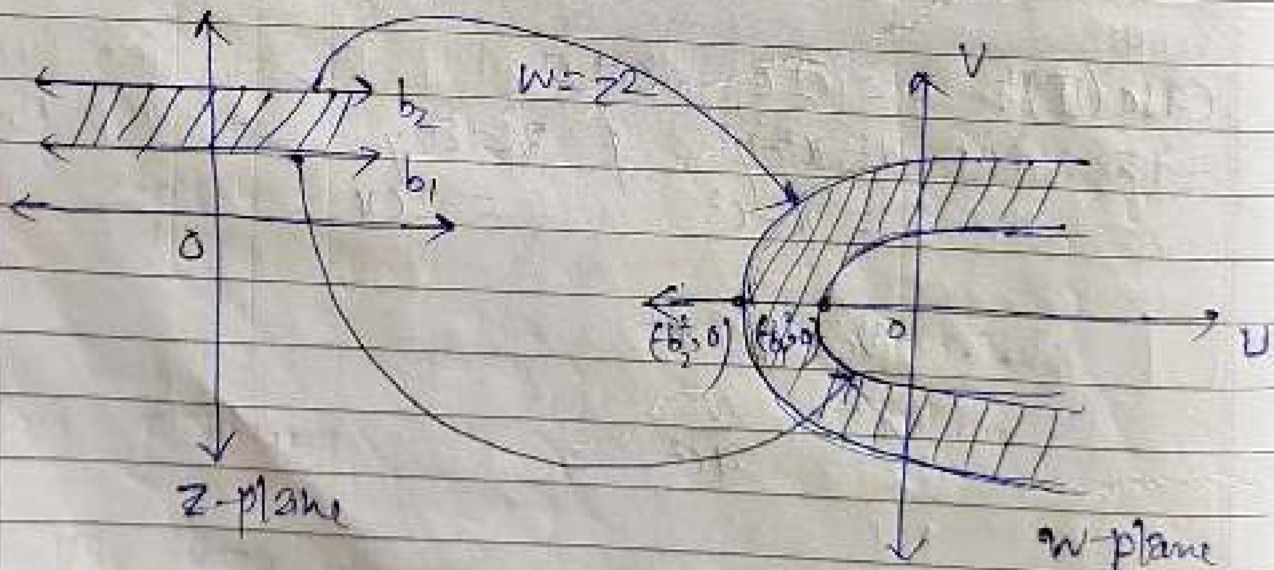
$\Rightarrow \quad V^2 = Au + B$

$A = 4c^2 \quad B = 4c^4$

This eq^n represents a parabola as shown in fig.



[VII] The lines $x = \pm a$



[VII] The lines $x = \pm a$

Case(I): If $x = \pm a$ & $a = 0$
then we have $x = 0$
which represents the y-axis

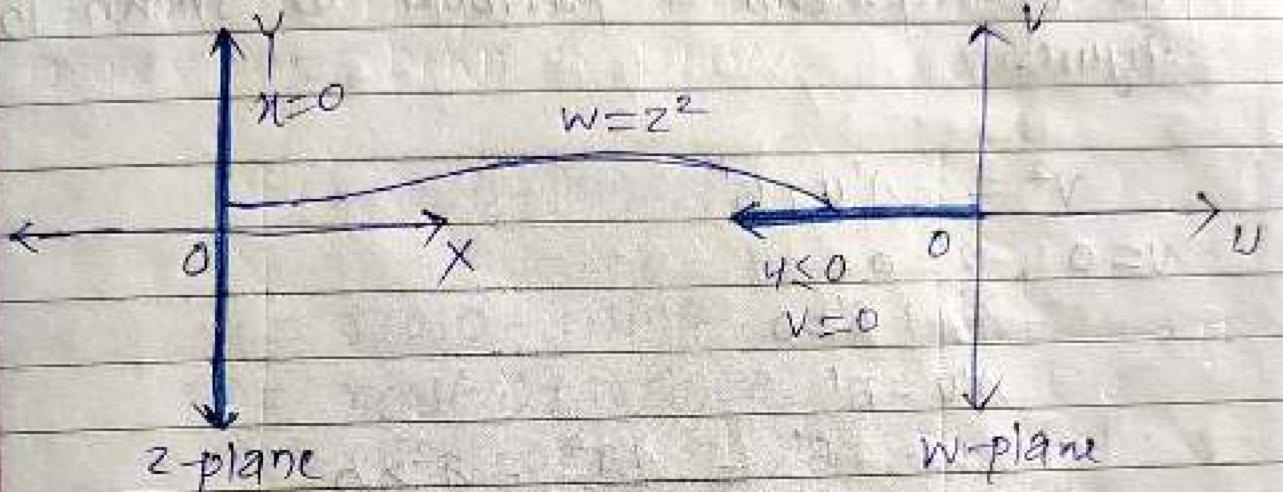
Now, we have

$u = x^2 - y^2$ & $v = 2xy$

$\Rightarrow$ $u = -y^2$ $v = 0$

$\Rightarrow$ $u \le 0$ & $v = 0$

It represent negative u-axis as shown in figure.



Case (II) : If $x = \pm a$ & a is non-zero i.e, $a \neq 0$

then $u = x^2 - y^2$ & $v = 2xy$

$\Rightarrow$ $u = a^2 - y^2$ & $v = 2ay$

Now $v = 2ay \Rightarrow v^2 = 4a^2y^2$

$\therefore$ $y^2 = \dfrac{v^2}{4a^2}$

$\therefore$ $u = a^2 - y^2$

$u = a^2 - \dfrac{v^2}{4a^2}$

$\dfrac{v^2}{4a^2} = a^2 - u$

Scanned by CamScanner

$\therefore\ v^2 = 4a^4 - 4a^2 u$

$v^2 = -A'u + B'$

$A' = 4a^2 \quad B' = 4a^4$

This eqn represents a parabola as shown in figure.

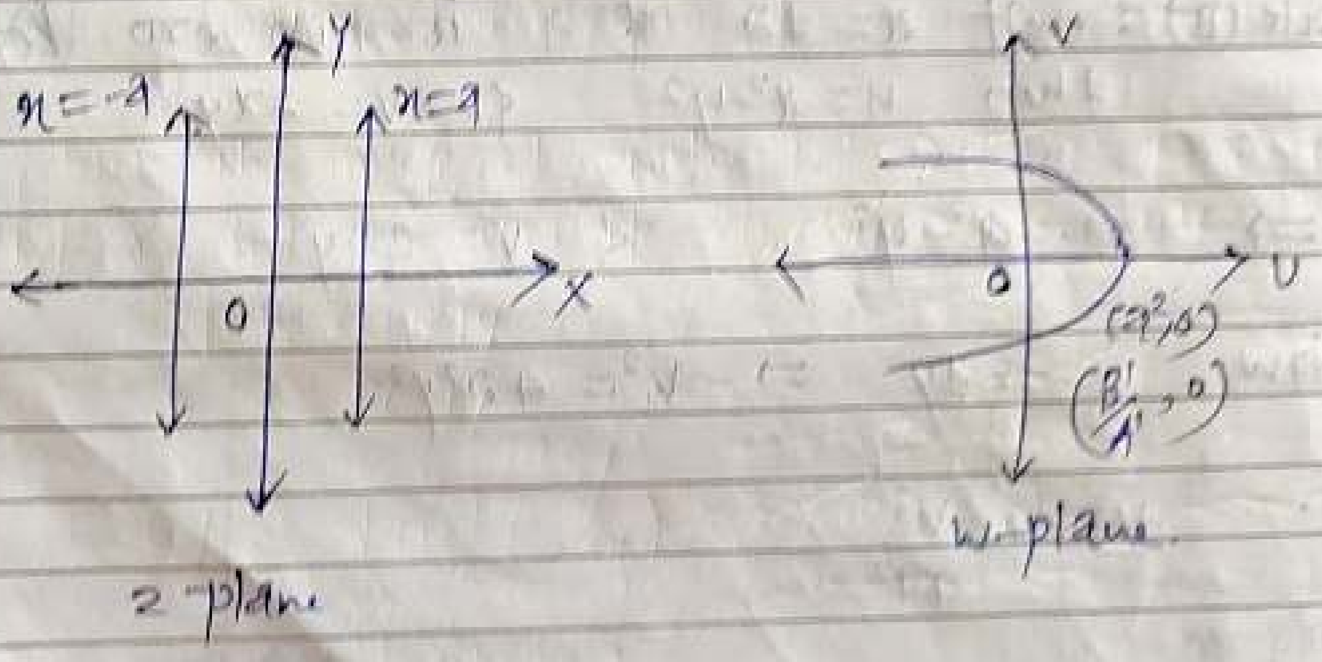
$v^2 = -A'u + B'$

$v = 0,\ v^2 = 0$

$-A'u + B' = 0$

$A'u = B'$

$$u = \frac{B'}{A'} = \frac{4a^4}{4a^2} = a^2 \geq 0$$



The me

② The mapping $w=\bar{z}$

If $z=x+iy$ then in polar co-ordin form we have
$z=r(\cos\theta+i\sin\theta)$.
Thus, polar co-ordinates of z are,
$z=z(r,\theta)$

Now $w=\bar{z}$

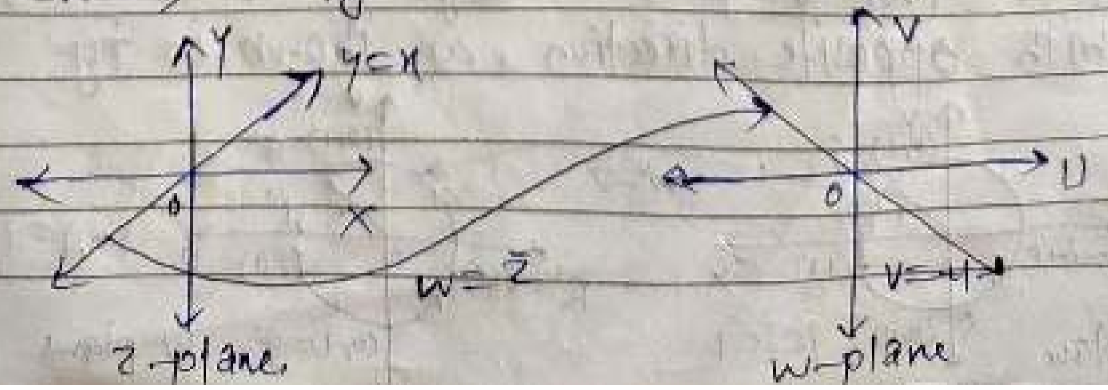
$\Rightarrow \quad w=\overline{[r(\cos\theta+i\sin\theta)]}$

$\Rightarrow \quad w=r\cos\theta-ir\sin\theta$
$w=r\cos(-\theta)+ir\sin(-\theta)$
$\therefore \; w=w(r,-\theta)$

Observations:

[i] The line $y=x$
$w=\bar{z}$.
$\Rightarrow \; u+iv=x-iy$
$\Rightarrow \; u=x \quad v=-y$
Now
$y=x \Rightarrow v=-x$
i.e, $v=-u$

It represents a straight line passing through origin. as shown in fig.



[ii] The circle $|z|=r$

Given $|z|=r \Rightarrow |z|^2=r^2$

$\Rightarrow \quad x+y^2=r^2$

Now $u=x$ & $v=-y$

$\Rightarrow \quad u^2=x^2$ & $v^2=y^2$

$\Rightarrow \quad u^2+v^2=x^2+y^2=r^2$

$u^2+v^2=r^2$

$\Rightarrow \quad |w|^2=r^2$

$\Rightarrow \quad |w|=r$

Thus the image of circle $|z|=r$ is again a circle $|w|=r$ but in opposite direction. Because

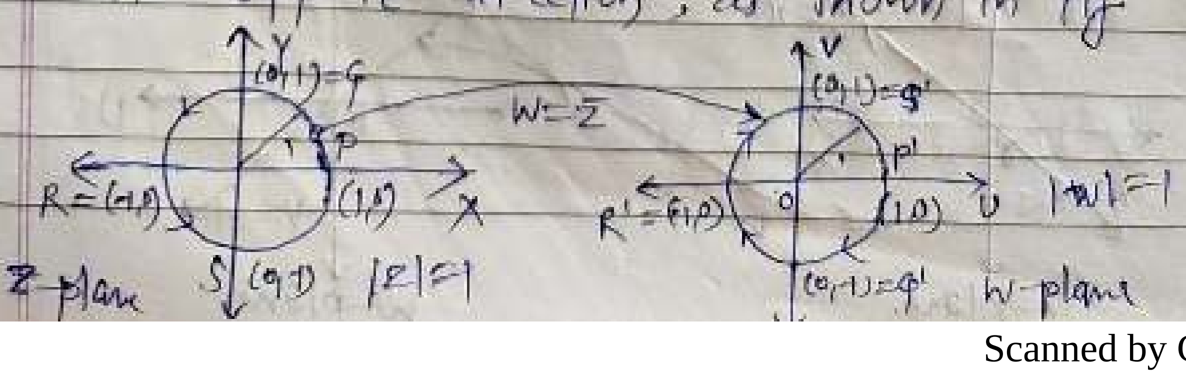
$w=u+iv$

$=x-iy$

$\therefore$ $\arg w=\tan^{-1}(-y/x) = -\tan^{-1}(y/x)$

& $\arg z=\tan^{-1}(y/x)$

Hence $|w|=r$ is the image of circle $|z|=r$ but with opposite direction, as shown in fig



log x cut domain +ve real no.

Page No.
Date

imp

* Elementary functions :-

① Exponential function :

We know that the real valued function $f(x) = e^x$ maps a set of all real numbers one-one and onto the set of positive real numbers i.e, $f: R \to R^+$ define as

$$f(x) = e^x \quad \forall x \in R$$

is one-one and onto. ~~moreover~~

Moreover,

$$e^x \neq 0 \quad \forall x \in R$$

$$\text{and } e^x > 0 \quad \forall x \in R.$$

Moreover, e^x satisfy the law of exponents as

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}$$

Also, we know that e^x is an increasing function (i.e, $x_1 < x_2 \Rightarrow e^{x_1} < e^{x_2}$)

The function e^x has power series representation has

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \{ 0! = 1$$

Now, define e^{iy} as,

$$e^{iy} = 1 + iy + \frac{i^2 y^2}{2!} + \frac{i^3 y^3}{3!} + \cdots + \frac{i^n y^n}{n!} + \cdots$$

Scanned by CamScanner

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \cdots$$

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$ and so on

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right)$$

$e^{iy} = \cos y + i \sin y$ where $y \in \mathbb{R}$ ... ①

$\therefore e^z = e^{x+iy}$

$\therefore e^z = e^x \cdot e^{iy}$

$\therefore |e^z| = |e^x| \cdot |e^{iy}|$

$= |e^x| \cdot |\cos y + i \sin y|$

$= e^x \cdot \sqrt{\cos^2 y + \sin^2 y}$ $\quad \because e^x > 0 \Rightarrow |e^x| = e^x$

$|e^z| = e^x \neq 0 \quad \forall\ z.$

Now $e^{iy} = \cos y + i \sin y$

$\Rightarrow e^{i\pi} = \cos\pi + i \sin\pi$

$\Rightarrow e^{i\pi} = -1 + i\cdot(0)$

$\Rightarrow \boxed{e^{i\pi} + 1 = 0}$ ... ②

Scanned by CamScanner

* Properties of e^z :

1] $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

→ Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$

$\therefore e^{z_1} e^{z_2} = e^{(x_1+iy_1)} \cdot e^{(x_2+iy_2)}$

$= e^{x_1} e^{iy_1} \cdot e^{x_2} e^{iy_2}$

$= e^{(x_1+x_2)} \cdot (\cos y_1 + i\sin y_1) \cdot (\cos y_2 + i\sin y_2)$

$= e^{(x_1+x_2)} (\cos y_1 \cos y_2 + \cos y_1 \, i\sin y_2 + i\sin y_1 \cdot \cos y_2 + i^2 \sin y_1 \cdot \sin y_2)$

$= e^{(x_1+x_2)} [(\cos y_1 \cdot \cos y_2 - \sin y_1 \sin y_2) + i(\cos y_1 \cdot \sin y_2 + \sin y_1 \cdot \cos y_2)]$

$= e^{(x_1+x_2)} [\cos(y_1+y_2) + i\sin(y_1+y_2)]$

$= e^{(x_1+x_2)} \cdot e^{i(y_1+y_2)}$

$= e^{(x_1+x_2)+i(y_1+y_2)}$

$\therefore e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

Thus, the law of exponents hold for the complex valued function e^z.

Scanned by CamScanner

2] e^z is not one-one function.

$\longrightarrow$ Let $e^z = w$ ... (1)

Consider $\forall k \in I$

$$e^{z+2k\pi i} = e^z \cdot e^{2k\pi i}$$
$$= w \cdot e^{2k\pi i}$$

$$= w[\cos 2k\pi + i \sin 2k\pi]$$

$$= w[1 + i(0)]$$

$$e^{z+2k\pi i} = w \quad \dots (2)$$

Thus, $z \neq z + 2k\pi i$

$$e^z = e^{z+2k\pi i} \quad \text{by (1) \& (2)}$$

Hence $f(z) = e^z$ is not one-one function.

Definition (Periodic function):

The function $w = f(z)$ is said tobe periodic function if there exist the complex number c such that,

$$f(z+c) = f(z) \quad \forall z.$$

Here c is said tobe 1 period of function f(z) imp 5M 2016

Theorem: i] Prove that $f(z)=e^z$ is periodic function with period $2\pi i$

ii] P.T. $f(z)=e^{iz}$ is periodic function with period 2π

Proof:- i] Given $f(z)=e^z \ \forall z \quad \cdots ①$

Consider, $f(z+2\pi i)=e^{z+2\pi i}$

$$=e^z \cdot e^{2\pi i} \qquad \because e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}$$

$$=e^z[\cos 2\pi + i\sin 2\pi]$$

$$(\because e^{iy}=\cos y + i\sin y)$$

$$=e^z[1+i(0)]$$

$$f(z+2\pi i)=e^z$$

$\therefore f(z+2\pi i)=f(z) \ \forall z\in\mathbb{C}$ by ①

Hence, $f(z)=e^z$ is periodic function with period $2\pi i$

ii] Given $f(z)=e^{iz} \ \forall z \quad \cdots ②$

Consider $f(z+2\pi)=e^{i(z+2\pi)}$

$$=e^{iz}\cdot e^{2\pi i} \qquad \because e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}$$

$$=e^{iz}[\cos 2\pi + i\sin 2\pi]$$

$$=e^{iz}[1+i(0)]$$

Scanned by CamScanner

$f(z+2\pi) = e^{iz}$

$\therefore \; f(z+2\pi) = f(z) \quad \forall \; z \in \mathbb{C} \quad$ by ②

Hence $f(z) = e^{iz}$ is periodic function with period 2π

Theorem: Prove that,

i] $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2k\pi i$, $k \in I$

ii] e^{az} is periodic function, where $a \neq 0$. Also find its period.

Proof: i] We know that

$$e^{z_1} = e^{z_2} \Leftrightarrow e^{z_1} \cdot e^{-z_2} = e^{z_2} \cdot e^{-z_2}$$

$$\Leftrightarrow e^{z_1 - z_2} = e^{z_2 - z_2}$$

$$\Leftrightarrow e^{z_1 - z_2} = e^{0}$$

$$\Leftrightarrow e^{z_1 - z_2} = 1$$

$$\Leftrightarrow e^{z_1 - z_2} = e^{2k\pi i} \quad \left\{ \because e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi = 1 \right.$$

Hence $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = 2k\pi i$

$$e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2k\pi i$$

ii] Given $f(z) = e^{az}$, $a \neq 0$

consider, $f\left(z+\frac{2\pi i}{a}\right) = e^{a(z+2\pi i/a)}$

$$= e^{az+2\pi i}$$

$$= e^{az} e^{2\pi i}$$

$$= e^{az}[\cos 2\pi + i\sin 2\pi]$$

$$= e^{az}[1+i(0)]$$

$$= e^{az}$$

$$f\left(z+\frac{2\pi i}{a}\right) = f(z) \quad \forall z.$$

Hence, $f(z)$ is periodic function with period $\frac{2\pi i}{a}$.

Q. Find all values of z such that $e^z = a+ib$

Solution: Given $e^z = a+ib$ then we have to find

$$z = x+iy$$

Now

$$e^z = a+ib$$

$$\Rightarrow e^{x+iy} = a+ib$$

$$\Rightarrow e^x \cdot e^{iy} = a+ib$$

$\Rightarrow \quad e^x \cdot (\cos y + i \sin y) = a + ib$

$\Rightarrow \quad e^x \cdot \cos y + i e^x \sin y = a + ib$

$\Rightarrow \quad e^x \cos y = a \quad \cdots (1) \qquad e^x \sin y = b \quad \cdots (2)$

Hence $e^{2x} \cos^2 y = a^2$ & $e^{2x} \sin^2 y = b^2$

$$e^{2x}(\cos^2 y + \sin^2 y) = a^2 + b^2$$

$$e^{2x} = a^2 + b^2$$

$$\therefore \quad 2x = \log(a^2 + b^2)$$

$$x = \frac{1}{2} \log(a^2 + b^2) \quad \cdots (3)$$

Again we have

$$\frac{e^x \cdot \sin y}{e^x \cos y} = \frac{b}{a}$$

$$\tan y = b/a$$

$$\therefore \quad y = \tan^{-1}\left(\frac{b}{a}\right) \quad \cdots (4)$$

Hence, we have

$\cancel{z=}$ $z = x + iy$

$$z = \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1}(b/a)$$

Hence all values of z for which $e^z = a+ib$ are

$\therefore z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right) + 2k\pi i \quad k \in I$

Example: Find all the values of z such that,

i] $e^z = 5-5i$ iv] $e^{3z} = 1$

ii] $e^z = -5+5i$ (2015 ... v] $e^{e^z} = 1$

iii] $e^z = 1$ vi] $e^z = 1$

Solution: i] Given $e^z = 5-5i$

Comparing with $e^z = a+ib$

$\therefore$ We have $a = 5 \quad b = -5$

$\therefore a^2+b^2 = (5)^2 + (-5)^2$
$= 25 + 25$
$= 50$

$\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{-5}{5}\right) = \tan^{-1}(-1) = -\pi/4$ ($\because 5-5i$ lie in 4th quadrant)

$\therefore$ All values of z for which $e^z = 5-5i$ are

$z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right) + 2k\pi i, \quad k \in I$

Scanned by CamScanner

Page No.
Date

$\therefore z = \frac{1}{2}\log(50) + i\left(-\frac{\pi}{4}\right) + 2k\pi i$

$\therefore z = \log(50)^{1/2} - i\pi/4 + 2k\pi i$

$z = \log(25\times 2)^{1/2} - i\left(\pi/4 - 2k\pi\right)$

$z = \log\sqrt{25\times 2} - i\left(\pi/4 - 2k\pi\right)$

$\therefore z = \log(5\sqrt{2}) - i\left(\pi/4 - 2k\pi\right) \quad k\in I$

ii] $e^z = -5+5i$

Compare with $e^z = a+ib$

$a = -5 \quad b = 5$

$a^2+b^2 = (-5)^2 + (5)^2$

$= 25+25$

$= 50$

$\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{5}{-5}\right) = \tan^{-1}\left(\frac{-5}{5}\right) = \frac{3\pi}{4}$

All values of z for which $e^z = -5+5i$ are

$\therefore z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right) + 2k\pi i \quad , k\in I$

$\therefore z = \frac{1}{2}\log(50) + i\left(\frac{3\pi}{4}\right) + 2k\pi i$

Scanned by CamScanner

Add $\pi/2$ | As it is

Add π | -ve of 1st quadrant

$$z = \frac{1}{2}\log(50) + i\left(\frac{3\pi}{4} + 2k\pi\right)$$

$$z = \log(5\sqrt{2}) + i\left(\frac{3\pi}{4} + 2k\pi\right) \quad k \in I$$

2015 कम

V] $e^{e^z} = 1$

Given $e^{e^z} = 1$

$$e^w = 1 \cdots (1)$$

where $w = e^z \cdots (2)$

by (1) we have $e^w = 1 + 0i$

Here $a = 1$ & $b = 0$

$\therefore a^2 = 1$ & $b^2 = 0$

$$a^2 + b^2 = 1$$

$$\tan^{-1}(b/a) = \tan^{-1}(0/1) = \tan^{-1}(0) = 0$$

$$w = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right) + 2k\pi i \quad k \in I$$

$$w = \frac{1}{2}\log 1 + i\tan^{-1}(0) + 2k\pi i$$

$$w = 0 + i(0) + 2\pi k i$$

$$\therefore w = 2\pi k i \quad k \in I$$

Scanned by CamScanner

Page No.
Date

$\therefore$ eqn ② becomes,

$e^z = w$

$\Rightarrow \quad e^z = 2k\pi i$

$e^z = 0 + 2k\pi i$

Here $a = 0$, $b = 2k\pi$

$a^2 = 0 \quad b^2 = 4k^2\pi^2$

$a^2 + b^2 = 4k^2\pi^2$

$\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{2k\pi}{0}\right) = \tan^{-1}(\infty) = \pi/2$

$\therefore z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right) + 2k\pi i \quad \forall k \in I$

$\therefore z = \frac{1}{2}\log[(2k\pi)^2] + i\pi/2 + 2k\pi i$

$z = \log[(2k\pi)^2]^{1/2} + i(\pi/2 + 2k\pi)$

$\therefore z = \log(2k\pi) + i(\pi/2 + 2k\pi) \quad \forall k \in I.$

vi] $e^{z^2} = 1$

Given $e^{z^2} = 1$

ie, $e^w = 1 \quad$ where $w = z^2 \cdots$ ①

Scanned by CamScanner

$\Rightarrow \quad w = 2k\pi i \quad k \in I$

$\therefore$ ① becomes,

$$z^2 = w$$

$\Rightarrow \quad z^2 = w$

$\Rightarrow \quad z^2 = 2k\pi i \quad k \in I \quad$ ②

$\Rightarrow \quad z = (2k\pi i)^{1/2}$

$\Rightarrow \quad z = \sqrt{2k\pi} \cdot (i)^{1/2}$

$$z = \sqrt{2k\pi}\left(\cos\frac{\pi}{2} + i\sin \pi/_2\right)^{1/2}$$

$$\because \quad i = \cos\pi/_2 + i\sin\pi/_2$$

$$z = \sqrt{2k\pi}\left[\cos\pi/_4 + i\sin\pi/_4\right]$$

$$= \sqrt{2}\cdot\sqrt{k\pi}\left[1/\sqrt{2} + i\,1/\sqrt{2}\right]$$

$$z = \sqrt{k\pi}\,(1+i)$$

(OR)

Now, by ② we have.

$$z^2 = 2k\pi i$$

$\Rightarrow \quad (x+iy)^2 = 2k\pi i$

$\Rightarrow \quad x^2 + i^2y^2 + 2ixy = 2k\pi i$

$\Rightarrow \quad (x^2 - y^2) + 2ixy = 0 + 2k\pi i$

$x^2 - y^2 = 0$, $2xy = 2K\pi$

$x^2 - y^2 = 0$ $\quad xy = K\pi \quad \ldots$ ②

$x^2 = y^2$

$x = y$

We have

$$y^2 = K\pi \qquad (\text{from } ②)$$

$$y = \pm\sqrt{K\pi}$$

$$x = y = \pm\sqrt{K\pi}$$

$\therefore z = x + iy$

$\therefore z = \pm\sqrt{K\pi} \pm i\sqrt{K\pi}$

2015 2nd iii] $e^z = 1$

Comparing with $e^z = a + ib$

$\therefore a = 1 \quad b = 0$

$a^2 + b^2 = 1$

$\tan^{-1}(b/a) = \tan^{-1}(0/1) = \tan^{-1}(0) = 0$

All values of z for which $e^z = 1$ are,

$$z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a) + 2K\pi i \qquad K \in I$$

Page No.
Date

$z = \frac{1}{2}\log(1) + i\tan(0) + 2k\pi i$

$z = 0 + i(0) + 2k\pi i$

$z = 2k\pi i \quad k \in I$

iv] $e^{3z} = 1$

comparing with $e^{z} = a+ib$

$Z = 3z$

$a = 1 \quad b = 0$

$a^2 + b^2 = 1$

$\tan^{-1}(b/a) = \tan^{-1}(0/1) = \tan^{-1}(0) = 0$

All values of z for $e^{3z} = 1$ are

Z

$Z = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}(b/a) + 2k\pi i$

$= \frac{1}{2}\log(1) + i\tan^{-1}(0) + 2k\pi i \quad k \in I$

$= \frac{1}{2}\times 0 + i(0) + 2k\pi i \quad k \in I$

$= 0 + 0 + 2k\pi i \quad k \in I$

Scanned by CamScanner

but $z = 3z$

$\therefore 3z = 2k\pi i$

$z = \frac{2}{3}k\pi i \quad , k \in I$

Problem: Prove that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Solution:- Consider,

$$\frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\left[e^{i(x+iy)} - e^{-i(x+iy)}\right]$$

$$= \frac{1}{2i}\left[e^{ix} \cdot e^{i^2 y} - e^{-ix} \cdot e^{-i^2 y}\right]$$

$$= \frac{1}{2i}\left[e^{ix} \cdot e^{i(iy)} - e^{-ix} \cdot e^{-i(iy)}\right]$$

$$= \frac{1}{2i}\left[e^{ix} e^{iy_1} - e^{-ix} e^{-iy_1}\right] \quad \because iy = y_1$$

$$= \frac{1}{2i}\left[(\cos x + i\sin x) \cdot (\cos y_1 + i\sin y_1) - (\cos x - i\sin x)(\cos y_1 - i\sin y_1)\right]$$

$$= \frac{1}{2i}\left[(\cos x \cdot \cos y_1 + i\cos x \cdot \sin y_1 + i\sin x \cdot \cos y_1 + i^2 \sin x \cdot \sin y_1) - (\cos x \cdot \cos y_1 - i\cos x \sin y_1 - i\sin x \cdot \cos y_1 + i^2 \sin x \cdot \sin y_1)\right]$$

$$= \frac{1}{2i}\left[\cos x \cdot \cos y_1 + i\cos x \cdot \sin y_1 + i\sin x \cdot \cos y_1 - \sin x \cdot \sin y_1 - \cos x \cdot \cos y_1 + i\cos x \cdot \sin y_1 + i\sin x \cdot \cos y_1 + \sin x \cdot \sin y_1\right]$$

$$= \frac{1}{2i}\left[2i\left((\cos x\cdot\sin y_1)+(\sin x\cos y_1)\right)\right]$$

$$= \frac{1}{2i}\left[2i\left(\cos x\cdot\sin y_1 + \sin x\cdot\cos y_1\right)\right]$$

$$= \cos x\cdot\sin y_1 + \sin x\cdot\cos y_1$$

$$= \sin(x+y_1)$$

$$= \sin(x+iy)$$

$$= \sin z.$$

2016 8M

Problem: Prove that all the zeros of $\sin z$ & $\cos z$ are reals.

Proof:- We know that,

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

Thus, $\sin z = 0$

$$\Leftrightarrow \frac{e^{iz}-e^{-iz}}{2i} = 0$$

$$\Leftrightarrow e^{iz}-e^{-iz} = 0$$

$$\Leftrightarrow e^{iz}\cdot e^{iz} - e^{-iz}\cdot e^{iz} = 0$$

$$\Leftrightarrow e^{2iz}-1 = 0$$

$\Leftrightarrow \quad e^{2iz} = 1$

$\Leftrightarrow \quad e^{2iz} = e^{2k\pi i}\, e^{0} \qquad [\because e^{2k\pi i} = 1]$

$\Leftrightarrow \quad 2iz = 0 + 2k\pi i \qquad \because e^{z_1} = e^{z_2} \Leftrightarrow z_1 \neq z_2$

where $k \in I$ $\qquad z_1 = z_2 + 2k\pi i$

where,

$$\sin z = 0 \Leftrightarrow 2iz = 2k\pi i \quad , k \in I$$

Hence,

$$\sin z = 0 \Leftrightarrow z = k\pi \quad k \in I$$

$$\therefore \sin z = 0 \Leftrightarrow z = k\pi + 0i$$

$\therefore$ z is purely real number.

$\therefore$ All the zeros of $\sin z$ are real numbers.

Hence prove

imp SM 2016/2014 4M

Q. Prove that : i] $\sin 2z = 2\sin z \cdot \cos z$

ii] $\sin(\pi/2 + z) = \cos z$

solution: i] consider,

$$2\sin z \cdot \cos z = 2\left[\frac{e^{iz} - e^{-iz}}{2i}\right]\left[\frac{e^{iz} + e^{-iz}}{2}\right]$$

$$= \frac{1}{2i}\left[(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})\right]$$

$$2\sin z\cdot\cos z = \frac{1}{2i}\left[e^{iz}\cdot e^{iz} + e^{iz}\cdot e^{-iz} - e^{-iz}\cdot e^{iz} - e^{-iz}\cdot e^{-iz}\right]$$

$$= \frac{1}{2i}\left[e^{2iz} + e^{0} - e^{0} - e^{-2iz}\right]$$

$$= \frac{1}{2i}\left[e^{2iz} - e^{-2iz}\right]$$

$$= \frac{e^{2iz} - e^{-2iz}}{2i}$$

$$\therefore 2\sin z\cdot\cos z = \frac{e^{i(2z)} - e^{-i(2z)}}{2i}$$

$$\therefore \boxed{2\sin z\cdot\cos z = \sin 2z}$$

ii] $\sin(\pi/2 + z) = \cos z$

Consider,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(\pi/2 + z) = \frac{e^{i(\pi/2 + z)} - e^{-i(\pi/2 + z)}}{2i}$$

$$= \frac{e^{i\pi/2}\cdot e^{iz} - e^{-i\pi/2}\cdot e^{-iz}}{2i} \quad \ldots (1)$$

Scanned by CamScanner

$$\therefore \sin(\pi/2+z) = \frac{e^{i\pi/2}\cdot e^{iz} - e^{-i\pi/2}\cdot e^{-iz}}{2i}$$

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2$$

$$e^{i\pi/2} = i \quad \ldots (2)$$

similarly

$$e^{-i\pi/2} = \cos\pi/2 - i\sin\pi/2$$

$$e^{-i\pi/2} = -i \quad \ldots (3)$$

put values of $e^{-i\pi/2}$ & $e^{i\pi/2}$ from (2) & (3) in (1)
we get,

$$\sin(\pi/2+z) = \frac{ie^{iz} - e^{-iz}\cdot(-i)}{2i}$$

$$= \frac{i(e^{iz} + e^{-iz})}{2i}$$

$$= \frac{e^{iz}+e^{-iz}}{2}$$

$$= \cos z$$

$$\therefore \boxed{\sin(\pi/2+z) = \cos z}$$

Scanned by CamScanner

Q. Prove that $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Proof:-

Consider,

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left[e^{iz}+e^{-iz}\right]$$

$$= \frac{1}{2}\left[e^{i(x+iy)} + e^{-i(x+iy)}\right]$$

$$= \frac{1}{2}\left[e^{ix} e^{i^2y} + e^{-ix} \cdot e^{-i^2y}\right]$$

$$= \frac{1}{2}\left[e^{ix} \cdot e^{i(iy)} + e^{-ix} \cdot e^{-i(iy)}\right]$$

$$= \frac{1}{2}\left[e^{ix} \cdot e^{iy_1} + e^{-ix} \cdot e^{-iy_1}\right] \qquad [\because iy = y_1]$$

$$= \frac{1}{2}\left[(\cos x + i\sin x)(\cos y_1 + i\sin y_1) + (\cos x - i\sin x)(\cos y_1 - i\sin y_1)\right]$$

$$= \frac{1}{2}\left[\cos x \cdot \cos y_1 + i\sin y_1 \cdot \cos x + i\sin x \cos y_1 + i^2 \sin x \cdot \sin y_1 + \cos x \cdot \cos y_1 - i\cos x \cdot \sin y_1 - i\sin x \cdot \cos y_1 + i^2 \sin x \sin y_1\right]$$

$$= \frac{1}{2}\left[\cos x \cos y_1 + \cancel{i\sin y_1 \cos x} + \cancel{i\sin x \cos y_1} - \sin x \cdot \sin y_1 + \cos x \cdot \cos y_1 - \cancel{i\cos x \cdot \sin y_1} - \cancel{i\sin x \cos y_1} - \sin x \cdot \sin y_1\right]$$

$$= \frac{1}{2}\left[2\cos x \cdot \cos y_1 - 2\sin x \cdot \sin y_1\right]$$

$$= \cos x \cdot \cos y_1 - \sin x \cdot \sin y_1$$

$$\{\because \cos x \cdot \cos y - \sin x \cdot \sin y = \cos(x+y)$$

$= \cos(x+y_1)$

$= \cos(x+iy)$

$$\frac{e^{iz}+e^{-iz}}{2} = \cos z$$

* Some important formula :-

1] $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$

2] $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$

3] $\tan z = \dfrac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$

4] $\cot z = \dfrac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$

5] $\sec z = \dfrac{2}{e^{iz}+e^{-iz}}$

6] $\operatorname{cosec} z = \dfrac{2i}{e^{iz}-e^{-iz}}$

2016 9M.

Q. Prove that all the zeros of $\cos z$ are real numbers

Proof: We know that $\cos z = \sin(\pi/2 + z)$

Thus, $\cos z = 0 \Leftrightarrow \sin(\pi/2 + z)$

$\Leftrightarrow \sin z_1 = 0 \quad \because z_1 = \pi/2 + z$

$\Leftrightarrow z_1 = k\pi \quad (\because \sin z_1 = 0 \Leftrightarrow z = k\pi \ \forall k \in I)$

$\Leftrightarrow \pi/2 + z = k\pi \ , \ k \in I$

$\Leftrightarrow z = k\pi - \pi/2 \ , \ k \in I$

$\Leftrightarrow z = \frac{2k\pi - \pi}{2} \ , \ k \in I$

Thus,

$\cos z = 0 \Leftrightarrow z = (2k-1)\pi/2 \quad k \in I$

Where, z is purely real number $\forall k \in I$

Hence, all the ~~values~~ zeros of $\cos z$ are purely real numbers.

H. P.

Q. Find all the values of z such that $\cos z = 2$

Solution:- We know that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$\cos z = 2 \Rightarrow \frac{e^{iz}+e^{-iz}}{2} = 2$

$\Rightarrow e^{iz}+e^{-iz} = 4$

$\Rightarrow e^{iz}e^{iz}+e^{iz}\cdot e^{-iz} = 4e^{iz}$

$\Rightarrow e^{2iz}+1 = 4e^{iz}$

$\Rightarrow e^{2iz}-4e^{iz}+1=0$

put $e^{iz}=y \quad \therefore e^{2iz}=y^2$

above eqn becomes,

$y^2-4y+1=0$

This is quadratic eqn in y.
$\therefore$ roots of this eqn are,

$$y = \frac{-(-4) \pm \sqrt{(-4)^2-4\times1\times1}}{2\cdot1}$$

$$= \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm \sqrt{4\times3}}{2}$$

$$= \frac{2(2\pm\sqrt{3})}{2} \qquad \therefore y = 2\pm\sqrt{3}$$

Scanned by CamScanner

$$e^{iz} = 2 \pm \sqrt{3}$$

i.e., $e^{i(x+iy)} = 2 \pm \sqrt{3}$

$$e^{ix} \cdot e^{i^2 y} = 2 \pm \sqrt{3}$$

$$(\cos x + i \sin x)\, e^{-y} = 2 \pm \sqrt{3}$$

$$e^{-y} \cos x + i e^{-y} \sin x = (2 \pm \sqrt{3}) + 0 \cdot i$$

We have,

$$e^{-y} \cos x = 2 \pm \sqrt{3} \quad \cdots \quad (1)$$

$$e^{-y} \sin x = 0 \quad \cdots \quad (2)$$

Now. by (2)

$$e^{-y} \sin x = 0$$

$$\Rightarrow \sin x = 0 \qquad \because e^{-y} \neq 0 \ \forall y$$

$$\Rightarrow x = n\pi \qquad ; n \in I \ \cdots \ (3)$$

put $x = n\pi$ in (1) then we have,

$$e^{-y} \cos n\pi = 2 \pm \sqrt{3}$$

$$(-1)^n e^{-y} = 2 \pm \sqrt{3}$$

but $(2 \pm \sqrt{3}) > 0$

$\therefore \; [(-1)^n e^{-y}] > 0$

$\therefore$ n should an even integer

$\therefore$ We have

$n = 2k \quad , \; k \in I$

by eq^n ③

$x = 2k\pi \; , \; k \in I$

Now put $x = 2k\pi$, $k \in I$ in ① then we get

$$e^{-y} \cos 2k\pi = 2 \pm \sqrt{3}$$

$$e^{-y} = 2 \pm \sqrt{3}$$

$$-y = \log(2 \pm \sqrt{3})$$

$$y = -\log(2 \pm \sqrt{3})$$

$$z = x + iy$$

$$z = 2k\pi - i \log(2 \pm \sqrt{3})$$

Q. $\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

Proof :-

We know that, $\tan x = \dfrac{\sin x}{\cos x}$

We Also,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

& $$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}$$

$$= \frac{(e^{iz} - e^{-iz})/2i}{(e^{iz} + e^{-iz})/2}$$

$$= \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\therefore \boxed{\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}}$$

Q. $\cot z = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

Proof :- We know that,

$$\cot x = \frac{\cos x}{\sin x}$$

We know by formulae of $\sin z$ & $\cos z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

& $$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cot z = \frac{\cos z}{\sin z} = \frac{(e^{iz} + e^{-iz})/2}{(e^{iz} - e^{-iz})/2i}$$

$$= \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$\therefore \boxed{\cot z = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}}$$

& $$\sec z = \frac{2}{e^{iz} + e^{-iz}}$$

Proof:-

We know that $\sec x = \frac{1}{\cos x}$

$$\therefore \sec z = \frac{1}{(e^{iz} + e^{-iz})/2}$$

$$\therefore \boxed{\sec z = \frac{2}{e^{iz} + e^{-iz}}}$$

9. P.T. $\operatorname{cosec} z = \dfrac{2i}{e^{iz} - e^{-iz}}$

Proof :- We know that

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

Also We know,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\therefore \operatorname{cosec} z = \frac{1}{\sin z}$$

$$= \frac{1}{(e^{iz} - e^{-iz})/2i}$$

$$\therefore \boxed{\operatorname{cosec} z = \frac{2i}{(e^{iz} - e^{-iz})}}$$

* Hyperbolic function

The hyperbolic function sinhz & coshz are defined as

$$\sinh z = \frac{e^{z}-e^{-z}}{2} \quad \& \quad \cosh z = \frac{e^{z}+e^{-z}}{2}$$

S17 4M

Q Prove that $\sinh z = -i\sin(iz)$

and $\cosh z = \cos(iz)$

Proof :- ① Consider, $\sin iz$ We know that,

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$\sin(iz) = \frac{e^{i(iz)}-e^{-i(iz)}}{2i}$$

$$= \frac{e^{i^2 z}-e^{-i^2 z}}{2i}$$

$$i\sin iz = \frac{e^{-z}-e^{z}}{2} \quad \because i^2=-1$$

$$-i\sin iz = \frac{e^{z}-e^{-z}}{2}$$

$$-i\sin iz = \sinh z$$

$$\therefore \boxed{\sinh z = -i\sin iz.}$$

Scanned by CamScanner

(2) We know that,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2}$$

$$= \frac{e^{z} + e^{-z}}{2}$$

$\therefore \cos(iz) = \cosh z$

$\therefore \boxed{\cosh z = \cos iz}$

Find the image of the fn - (a) $x=a$
(b) $y=b$

Page No.
Date

* Mapping Properties :-

h_z^2

2015
May

Let $w = e^z$
$\therefore u+iv = e^{(x+iy)}$
$u+iv = e^x . e^{iy}$

in polar form, we have,

$r(\cos\theta + i\sin\theta)$

$w = re^{i\theta} = e^x . e^{iy}$
$\therefore \theta = y \;\; \& \;\; r = e^x$

Where, $r = |w| = e^x$
$\& \; \theta = \arg w = y$

Thus, $\arg w$ i.e, $\arg e^z$ depends upon imaginary part of z i.e, y and $|w|$ or modulus of w is e^x depends upon real part of z i.e, x only.

Observations:

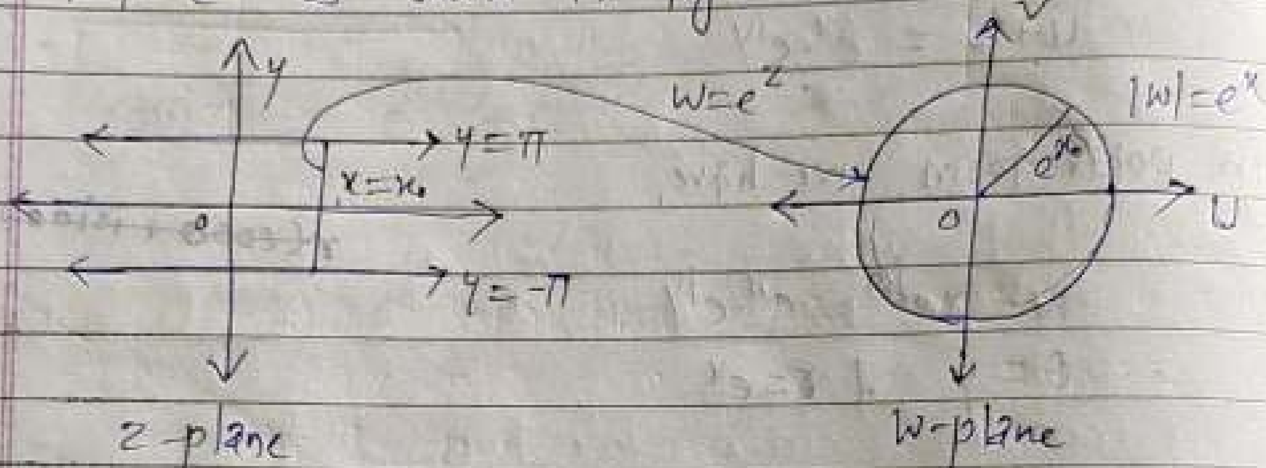
1] $e^{z_1} = e^{z_2} \Leftrightarrow z_1 = z_2 + 2k\pi i \quad \forall \; k \in I$
$\therefore e^z$ is not one-one function.
Moreover it is periodic function with period $2\pi i$ in complex plane.

2] e^z is one-one function in the strip $-\pi < \arg e^z \leq \pi$
$\therefore$ we will define e^z on this strip.

3] line segment $x = x_0$
Consider the line segment $x = x_0$ then we have,

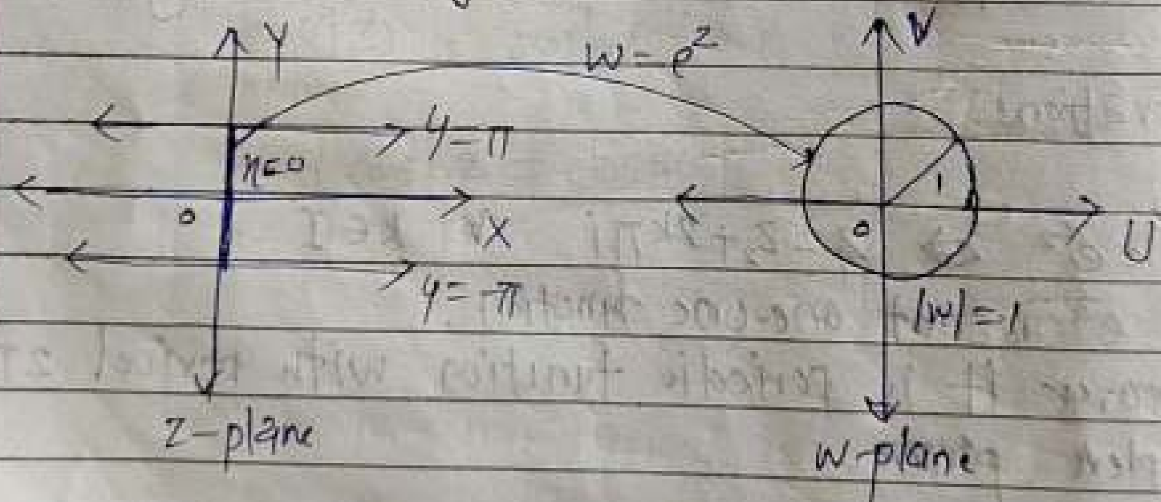
$$|w| = e^{x} = e^{x_0} > 0$$

Thus the image of line segment $x = x_0$ is a circle $|w| = e^{x_0}$ as show in fig.



Note:- i] If $x_0 = 0$ then we have the line segment $x = 0$ in domain, whole image is

$$|w| = e^{x} = e^{0} = 1$$ i.e, the unit circle $|w| = 1$ as shown in fig.

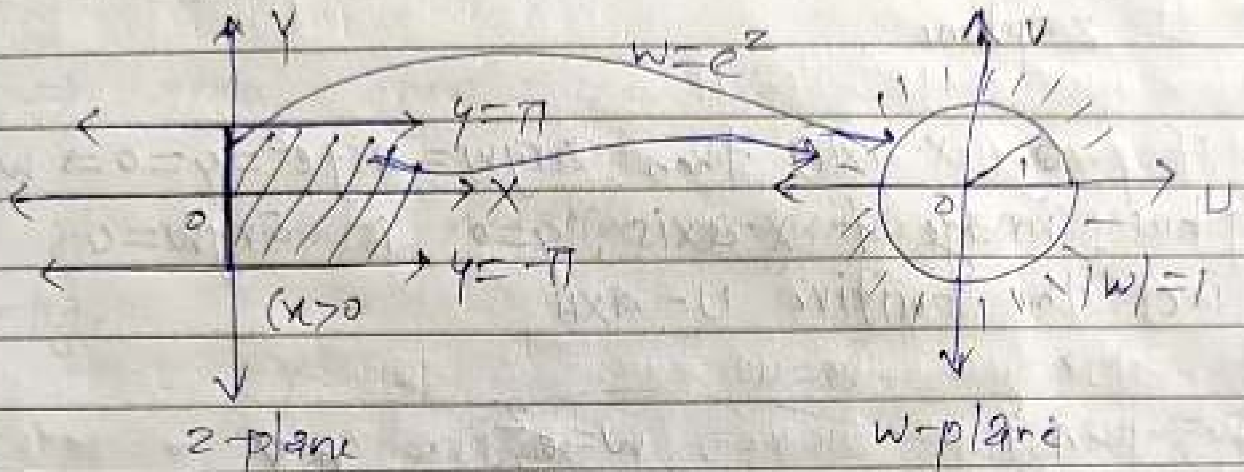


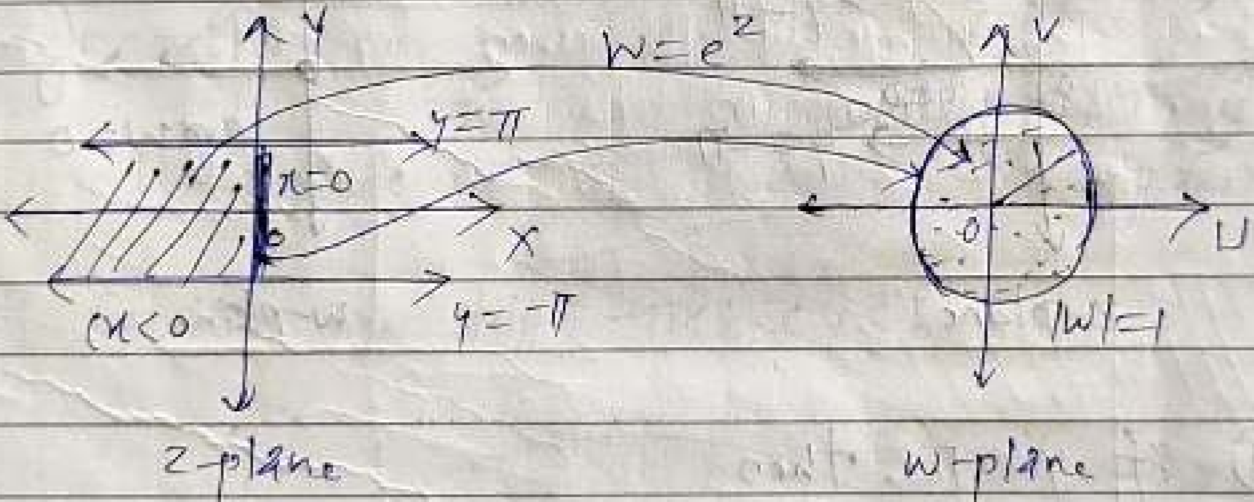
ii] the line segment $x = 0$ is mapped onto a unit circle $|w| = 1$ under $w = e^{z}$.
Moreover the right half strip $x > 0$ is mapped onto the exterior of unit circle $|w| = 1$,

because $x>0 \Rightarrow e^x > e^0$

i.e, $e^x > 1$ i.e, $|w| > 1$

Similarly the left half strip $x<0$ is mapped onto interior of unit circle $|w|=1$ as shown in fig.



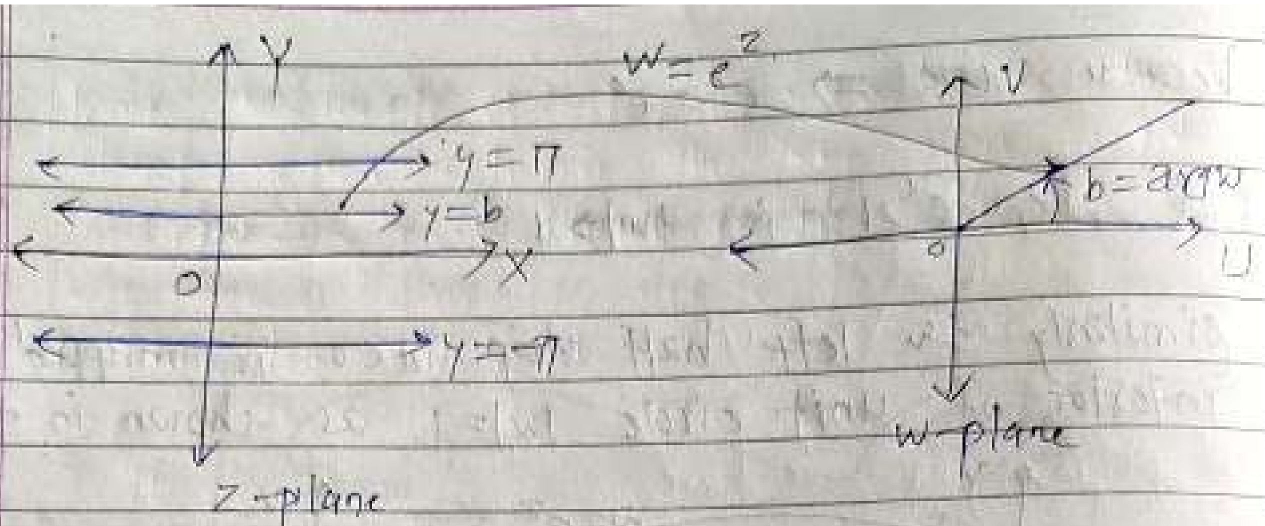


4] The line $y=b$ $(-\pi < b \le \pi)$

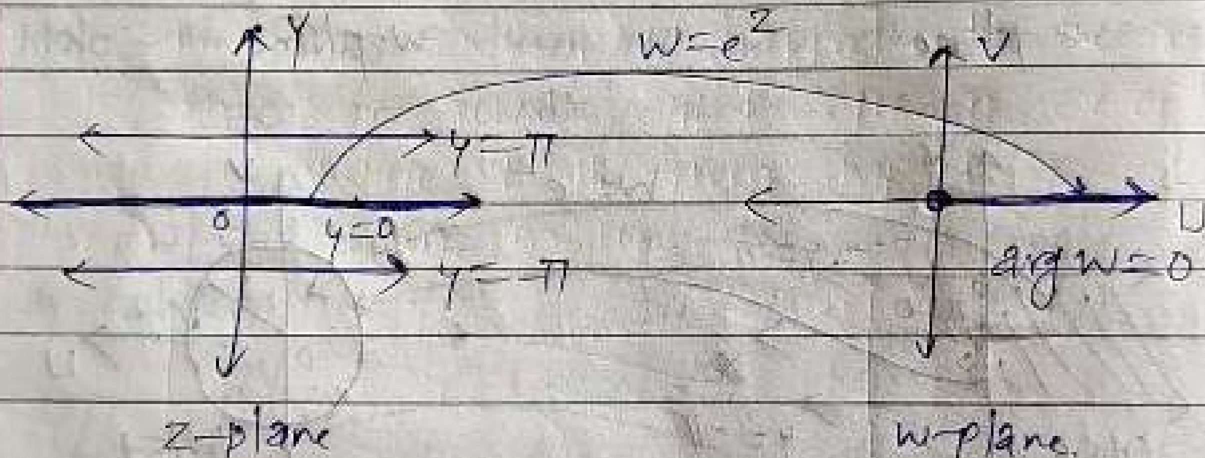
We have, $\arg w = \arg e^z = y$

Hence, $\arg w = b$

This eqn represents a ray inclined at an angle b with +ve U-axis as shown in fig.



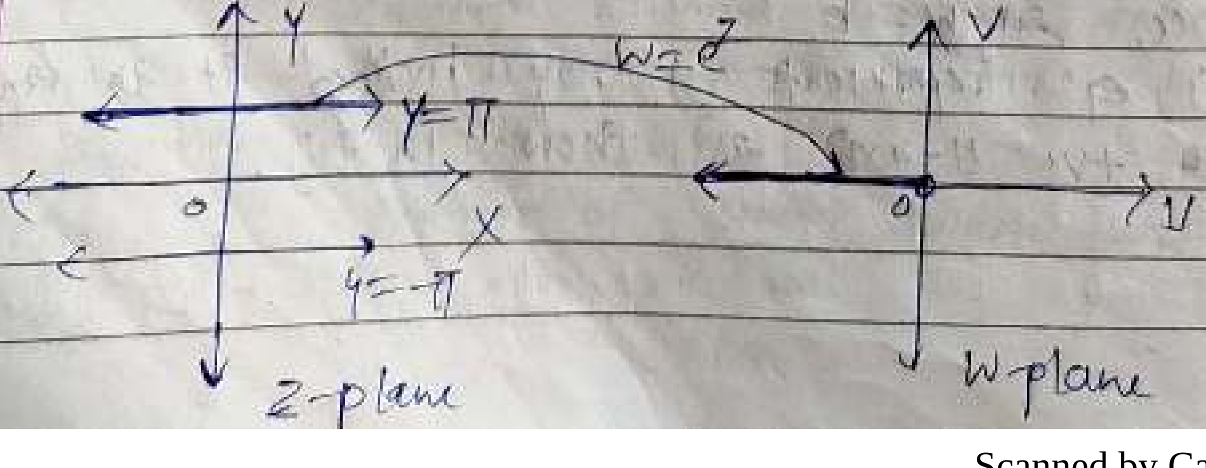
Note:- ① If $y=0$ (x-axis) then $\arg w=\arg e^z=y=0 \Rightarrow w$ is purely real.
Thus, image of x-axis, $y=0$ is $\arg w=0$
i.e., the positive u-axis



(ii) If $y=\pi$, then

$\arg w=\arg e^z=y=\pi \Rightarrow w$ is purely real -ve number

∴ Image of the line $y=\pi$ is -ve u-axis as shown in fig.



(iii) The line of $y = \pi/2$ $\arg w = \arg e^z = y = \pi/2$

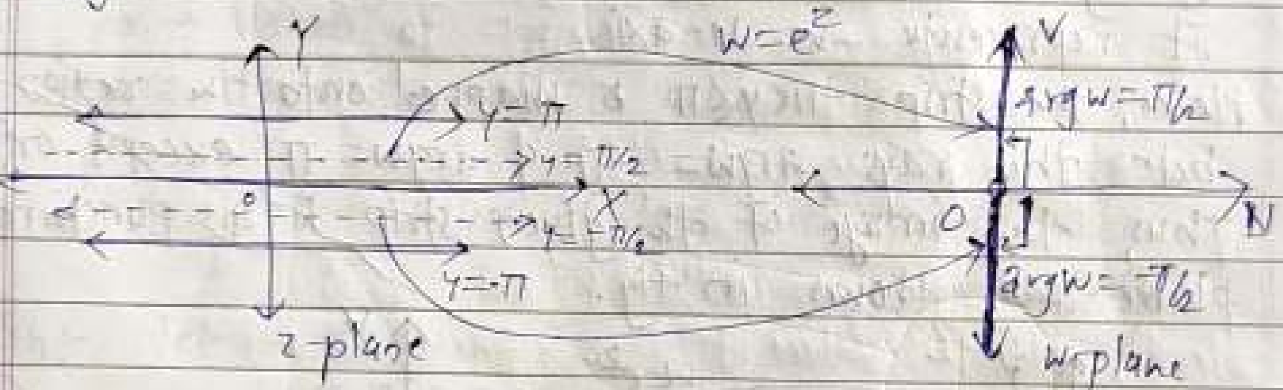
i.e. $\arg w = \pi/2$

$\Rightarrow$ w is purely positive imaginary axis as shown in fig.

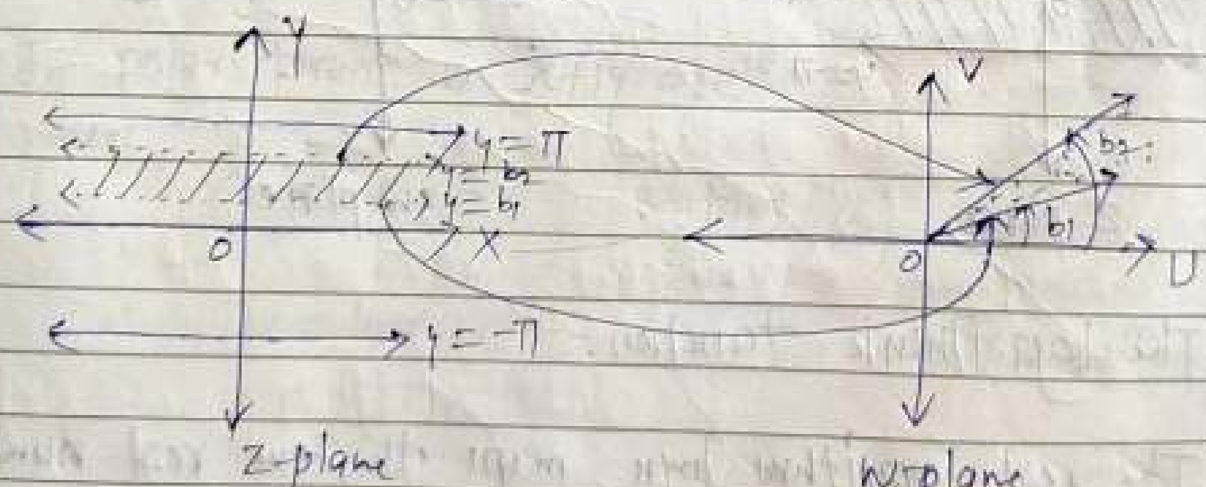
(iv) If $y = -\pi/2$

$\Rightarrow \arg w = \arg e^z = y = -\pi/2$

It represents negative imaginary axis as shown in fig.



5] Consider the strip $-\pi < b_1 \le y \le b_2 < \pi$



$y = b_1 \Rightarrow \arg w = b_1$ i.e, a ray inclined at an angle b_1

$y = b_2 \Rightarrow \arg w = b_2$ i.e, a ray inclined at an angle b_2

(iii) The line of $y=\pi/2$ $\arg w = \arg e^z = y = \pi/2$

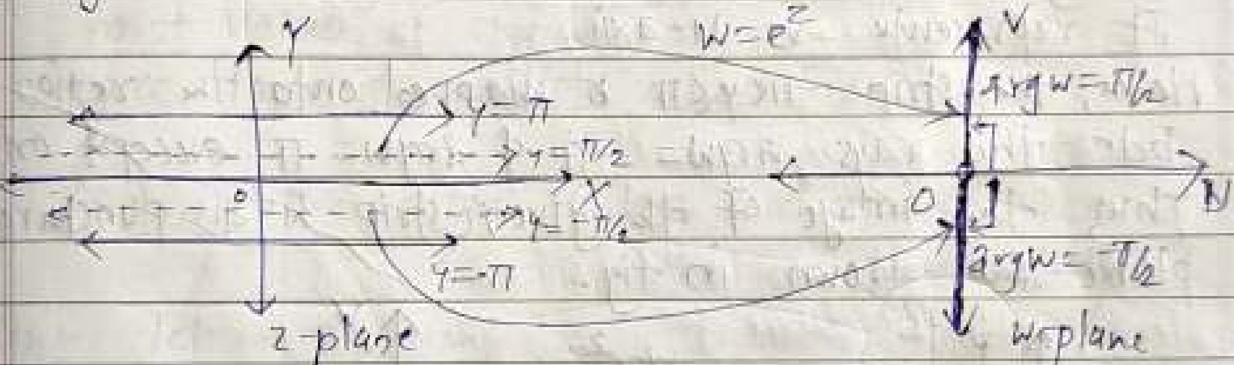
i.e. $\arg w = \pi/2$

$\Rightarrow$ w is purely positive imaginary axis as shown in fig.

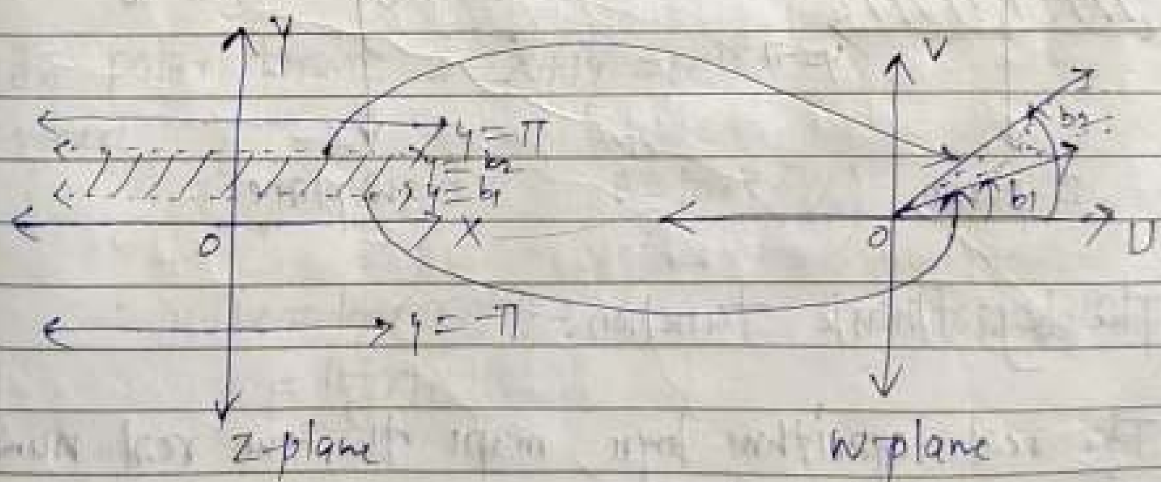
(iv) If $y = -\pi/2$

$\Rightarrow \arg w = \arg e^z = y = -\pi/2$

It represents negative imaginary axis as shown in fig.



5] Consider the strip $-\pi < b_1 < y \le b_2 < \pi$



$y=b_1 \Rightarrow \arg w = b_1$ i.e, a ray inclined at an angle b_1

$y=b_2 \Rightarrow \arg w = b_2$ i.e, a ray inclined at an angle b_2

Date 09 09 18

Hence the image of strip $b_1 \le y \le b_2$ is the closed region between the two rays $\arg w = b_1$ & $\arg w = b_2$ as shown in fig.

6] The strip $-\pi < y \le \pi$

$\Rightarrow$ if $y = \pi$

$\Rightarrow$ $\arg w = \arg e^z = y = \pi$

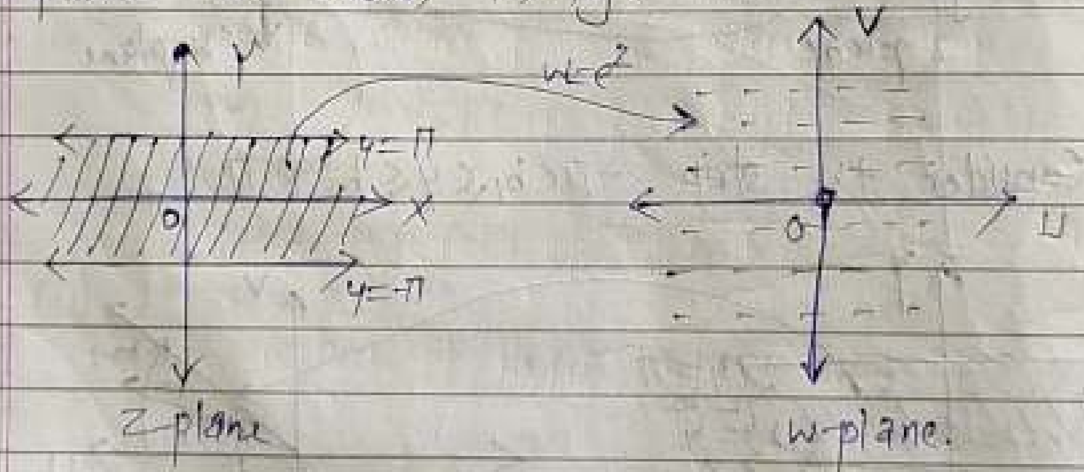
It represents $-ve$ u-axis

& if $y = -\pi$

$\arg w = -\pi$

It represents $-ve$ u-axis.

Hence, the strip $-\pi < y \le \pi$ is mapped onto the region betn the rays $\arg w = -\pi$ & $\arg w = \pi$ except origin. Thus, the image of the given strip is a punctured plane as shown in fig.



z-plane   w-plane.

* The logarithmic function :-

The real logarithm $\log x$ maps the +ve real numbers onto the real numbers.

i.e, $\log x : R^+ \to R$

Moreover $(\log x)^{-1} = e^x$

Here, the inverse exist because of e^x exist because $e^x : R \to R^+$ is one-one and onto function.
But in complex analytic $e^z : C \to C$ is not one-one and not onto function.
i.e, if $e^z = 1$ then $e^{z+2\pi k i} = 1$ $k = 0, \pm 1, \pm 2, \ldots$
Thus, e^z assumes infinitely many times the same value
i.e, if $e^z = w$ then $e^{z+2k\pi i} = w$ $k = 0, \pm 1, \pm 2, \ldots$

$$\therefore \log w = z + 2k\pi i \quad k = 0, \pm 1, \pm 2, \ldots$$

Thus, for single value of w there exist an infinite no. of values of log w.
Hence log w is multiple valued function.

Q. find all the values of log z such that where $z = x+iy$ ~~$\log z = a+ib$~~.

Solution:- let

$$w = \log z$$
$$u+iv = \log(x+iy)$$

In polar form $z = x+iy = re^{i\theta}$
where,

$$r = \sqrt{x^2+y^2} \neq 0 \qquad \theta = \tan^{-1}(y/x)$$

$$\therefore u+iv = \log(re^{i\theta}) \qquad \because x+iy = re^{i\theta}$$
$$= \log r + i\theta$$

$$\therefore u = \log r \ \& \ v = \theta$$
$$\therefore u = \log\sqrt{(x^2+y^2)} \qquad v = \tan^{-1}(y/x)$$

$\because \log z = w = u + iv$

$\therefore \log z = \log\sqrt{(x^2+y^2)} + i\tan^{-1}(y/x)$

$\therefore \log z = \log|z| + i[\arg z + 2k\pi] \quad k = 0, \pm 1, \ldots$

Q. Find all the values of $\log z$ where $z = 1-i$

Solution:- We have $z = 1-i$

comparing with $z = x+iy$

we get $x = 1 \quad y = -1$

$$|z| = \sqrt{x^2+y^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$$

$$\arg z = \tan^{-1}(y/x) = \tan^{-1}(-1/1) = -\pi/4$$

$\because 1-i$ lie in 4th quadrant.

Hence, all the values of $\log z$, where $z = 1-i$

$$\log z = \log|z| + i[\arg z + 2k\pi] \quad k = 0, \pm 1, \ldots$$

$$\log(1-i) = \log(\sqrt{2}) + i[-\pi/4 + 2k\pi] \quad k = 0, \pm 1, \ldots$$

H.W.

Q. Find all values of $\log z$ where $z = 2i$

Solution:- Comparing with $z = x+iy$ $\quad w = \log z$.

we get, $x = 0 \quad y = 2$

$$|z| = \sqrt{x^2+y^2} = \sqrt{0^2+(2)^2} = \sqrt{4} = 2$$

$$\arg z = \tan^{-1}(y/x) = \tan^{-1}(2/0) = \tan^{-1}\infty = \pi/2$$

$\because z = 2i$ lie on the y axis.

Hence all the values of $\log z$ where $z = 2i$

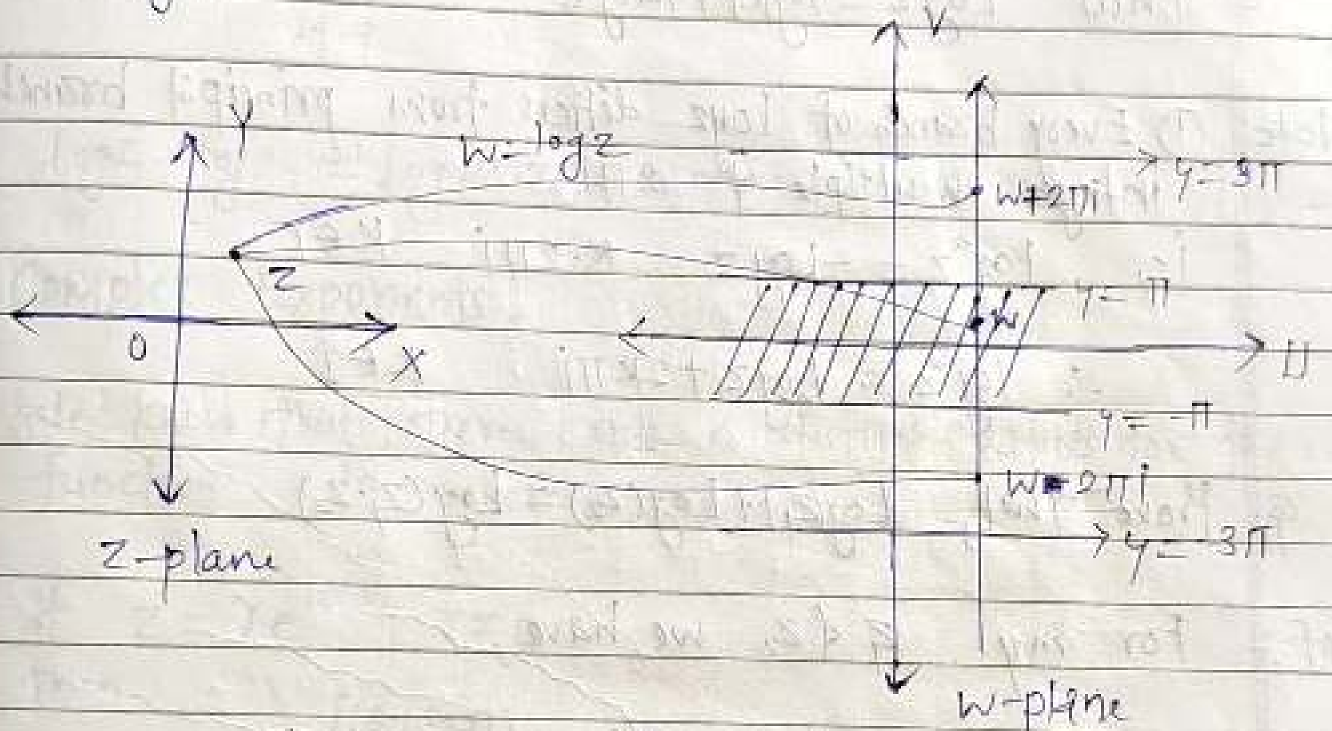
$\log 2i = \log 2 + i[\pi/2 + 2k\pi] \quad k = 0, \pm 1, \ldots$

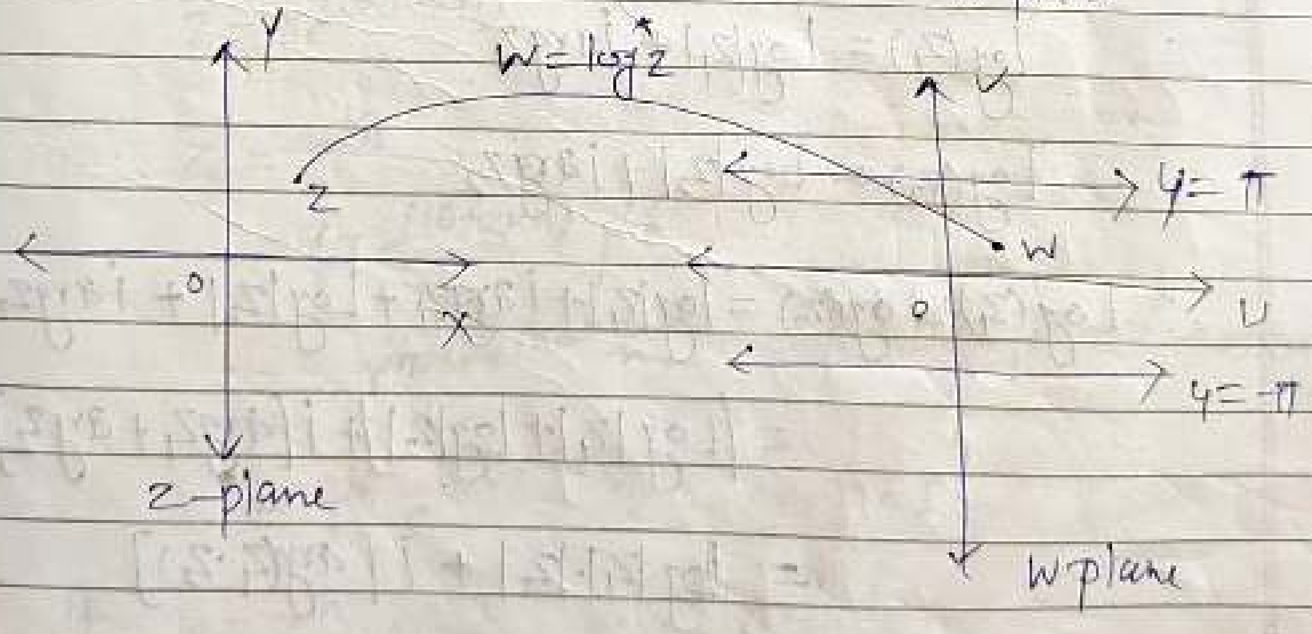
* Branch of Logarithm (log z) :

Branch of log z is a single valued function $\log^* z$ which satisfies the condition :

$$e^w = e^{\log^* z} = z$$

Note :- There are infinitely many branches of log z as shown in figure.





$\arg z_1 + \arg z_2 = \arg(z_1 \cdot z_2)$

Page No.

Date

2016 2M, 2015 8M Define & prove $\log|z| \le |\log z| \le \log|z| + |\arg z|$

* Principal Branch of $\log z$:

- Amongst all the branches of $\log z$, the branch of $\log z$ satisfying $-\pi < \arg z \le \pi$ is called as principal branch of logarithm.

- The principal branch of logarithm is denoted by $\mathrm{Log}\, z$
- Thus $\mathrm{Log}\, z = \log|z| + i \arg z$

Note: ① Every branch of $\log z$ differs from principal branch by integer multiple of $2\pi i$

i.e. $\log^* z - \mathrm{Log}\, z = k \cdot 2\pi i \quad k \in I$

$\therefore \log^* z = \mathrm{Log}\, z + 2k\pi i \quad k \in I$

Q. Prove that, $\mathrm{Log}(z_1) + \mathrm{Log}(z_2) = \mathrm{Log}(z_1 \cdot z_2)$

Proof:- For any z_1 & z_2 we have,

$$\mathrm{Log}(z_1) = \mathrm{Log}|z_1| + i \arg z_1$$

$$\mathrm{Log}(z_2) = \mathrm{Log}|z_2| + i \arg z_2$$

$$\therefore \mathrm{Log}(z_1) + \mathrm{Log}(z_2) = \mathrm{Log}|z_1| + i \arg z_1 + \mathrm{Log}|z_2| + i \arg z_2$$

$$= [\mathrm{Log}|z_1| + \mathrm{Log}|z_2|] + i[\arg z_1 + \arg z_2]$$

$$= \mathrm{Log}|z_1| \cdot |z_2| + i[\arg(z_1 \cdot z_2)]$$

$\because |z_1|$ & $|z_2|$ are real numbers, we have $\log|z_1| + \log|z_2| = \mathrm{Log}(|z_1| \cdot |z_2|)$

Scanned by CamScanner

and $\arg z_1 + \arg z_2 = \arg(z_1 \cdot z_2)$

Hence, we have,

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(|z_1 \cdot z_2|) + i\arg(z_1 \cdot z_2)$$

$$\therefore \operatorname{Log} z_1 + \operatorname{Log} z_2 = \operatorname{Log}(z_1 \cdot z_2)$$

H.P.

Note:- $\log z_1 + \log z_2 = \operatorname{Log}(z_1 \cdot z_2) + 2k\pi i \quad k \in I$

* Complex Exponents:

We know that, there exist n distinct values for the function $z^{1/n}$: where $z \neq 0$

If $z = re^{i\theta}$, $r \neq 0$

then

$$z = re^{i(\theta + 2k\pi)}$$

$$z_k = z^{1/n}$$

$$= \left[re^{i(\theta+2k\pi)}\right]^{1/n}$$

$$= r^{1/n} e^{i\left[\theta/n + 2k\pi/n\right]} \quad (\because \text{by De-Moivre's theorem})$$

$$= r^{1/n} e^{i\left(\theta/n + 2k\pi/n\right)} \quad k \in I.$$

If $w = z^{1/n}$ then $w = e^{\log z^{1/n}}$

$$= e^{\frac{1}{n}\log z}$$

$$= e^{\frac{1}{n}[\log|z| + i\arg z + 2k\pi i]}$$

Problem: Find all values of

2016 4M i) i^i

ii) $5^{1/2}$

2019 8M iii) $i^{i/2}$

Solution:- i] let $w = i^i$

$$= e^{\log i^i}$$

$$= e^{i\log i}$$

$$= e^{i[\log|i| + i\arg i]}$$

$$= e^{i[\log 1 + \frac{\pi}{2}i + 2k\pi i]} \quad \left\{ \text{Log} = \log \right.$$

$k = 0, \pm 1, \pm 2, \ldots$

$$= e^{i[0 + i(\frac{\pi}{2} + 2k\pi)]}$$

$$= e^{i^2(\frac{\pi}{2} + 2k\pi)} \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\therefore W = e^{-(\frac{\pi}{2} + 2k\pi)} \quad k \in I$$

ii] let $w = 5^{1/2}$

$$= e^{\log 5^{1/2}}$$

$$= e^{\frac{1}{2}\text{Log} 5}$$

$$= e^{\frac{1}{2}[\log|5| + i\arg 5 + 2k\pi i]} \quad k \in I$$

$$= e^{\frac{1}{2}[\log 5 + 0 + 2k\pi i]}$$

$$= e^{\frac{1}{2}\log 5 + k\pi i} \quad k \in I$$

$$= e^{\log 5^{1/2}}\, e^{k\pi i}$$

$$= e^{\log\sqrt{5}}[\cos k\pi + i\sin k\pi] \quad k \in I$$

$W = \sqrt{5}\cdot(\pm 1 + i\,0)$

$\therefore \cos K\pi = \pm 1$ & $\sin K\pi = 0 \quad \forall\, K \in I$

$\therefore \; W = \pm\sqrt{5}$

iii] $i^{1/2}$ $\qquad$ (Ans : $\pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$)

let $w = i^{1/2}$

$= e^{\log i^{1/2}}$

$= e^{\frac{1}{2}\log i}$

$= e^{\frac{1}{2}[\log|i| + i\arg i + 2K\pi i]} \qquad K \in I$

$= e^{\frac{1}{2}[\log 1 + i\pi/2 + 2K\pi i]}$

$= e^{\frac{1}{2}[0 + \frac{\pi}{2}i + 2K\pi i]}$

$= e^{\frac{1}{2}\, i[2K\pi + \pi/2]}$

$= e \; e^{i[K\pi + \pi/4]}$

$= e^{K\pi i} \cdot e^{\pi/4\, i}$

$= e^{K\pi i}\left(\cos\pi/4 + i\sin\pi/4\right)$

$= (\cos K\pi + i\sin K\pi)(\cos\pi/4 + i\sin\pi/4)$

$= (\pm 1 + 0)(\cos\pi/4 + i\sin\pi/4)$

$= \pm 1\left(1/\sqrt{2} + i\,1/\sqrt{2}\right)$

$i^{1/2} = \left(\pm 1/\sqrt{2} \pm \frac{i}{\sqrt{2}}\right)$

* Inverse Trigonometric Function.

Theorem: P.T $\sin^{-1}(z) = -i \log[iz + \sqrt{1-z^2}]$

Proof:- Suppose, $w = \sin^{-1} z$

$\therefore z = \sin w$

$\therefore z = \dfrac{e^{iw} - e^{-iw}}{2i}$

$\therefore e^{iw} - e^{-iw} = 2iz$

Multiply e^{iw} on both sides,

$\therefore e^{2iw} - e^{-iw}e^{iw} = 2ize^{iw}$

$e^{2iw} - e^{0} - 2ize^{iw} = 0$

$(e^{iw})^2 - 2ize^{iw} - 1 = 0$

put $e^{iw} = y$

we have, $y^2 - 2izy - 1 = 0$

This is quadratic eq^n in y

$\therefore$ roots of this eq^n are,

$y = \dfrac{2iz \pm \sqrt{4i^2z^2 + 4}}{2}$

$= \dfrac{2iz \pm 2\sqrt{-z^2+1}}{2}$

$= \dfrac{2iz \pm 2\sqrt{1-z^2}}{2}$

$y = \frac{iz \pm \sqrt{1-z^2}}{}$

$\therefore y = iz \pm \sqrt{1-z^2}$

$\therefore e^{iw} = iz \pm \sqrt{1-z^2}$

$\therefore iw = \log[iz \pm \sqrt{1-z^2}]$

$\therefore w = \frac{1}{i} \log[iz \pm \sqrt{1-z^2}]$ $\quad \because \frac{1}{i} = -i$

$\therefore \sin^{-1} z = -i \log[iz \pm \sqrt{1-z^2}]$

Hence proved.

Q. i] $\cos^{-1} z = -i \log[z + \sqrt{z^2-1}]$

ii] $\tan^{-1} z = \frac{1}{2i} \log\left[\frac{1+zi}{1-zi}\right]$

iii] $\cot^{-1} z = \frac{1}{2i} \log\left[\frac{z+i}{z-i}\right]$

iv] $\sec^{-1} z = \frac{1}{i} \log\left[\frac{1+(1-z^2)^{1/2}}{z}\right]$

v] $\operatorname{cosec}^{-1} z = \frac{1}{i} \log\left[\frac{i+(z^2-1)^{1/2}}{z}\right]$

Solution:- i] $\cos^{-1} z = i \log[z + \sqrt{z^2-1}]$

Suppose,

$w = \cos^{-1} z$

$\therefore z = \cos w$

$z = \frac{e^{iw} + e^{-iw}}{2}$

$e^{iw} + e^{-iw} = 2z$

$e^{2iw} + e^{0} - 2z e^{iw} = 0$

Scanned by CamScanner

$$(e^{iw})^2 - 2ze^{iw} + 1 = 0$$

put $e^{iw} = y$

We have, $y^2 - 2zy + 1 = 0$

This is quadratic eqn in y

$\therefore$ roots of this eqn are

$$y = \frac{2z \pm \sqrt{4z^2 - 4}}{2}$$

$$y = \frac{2z \pm 2\sqrt{z^2-1}}{2}$$

$$y = z + \sqrt{z^2-1}$$

$\tan^{-1} z = \frac{1}{2i} \log$ $\quad e^{iw} = z + \sqrt{z^2-1}$

$$iw = \log\left(z + \sqrt{z^2-1}\right)$$

$$w = \frac{1}{i} \log\left(z + \sqrt{z^2-1}\right)$$

$$w = -i \log\left(z + \sqrt{z^2-1}\right)$$

$$\cos^{-1} z = -i \log\left(z + \sqrt{z^2-1}\right)$$

Example: find all the values of

i] $\sin^{-1}(0)$, ii] $\sin^{-1}(i)$, iii] $\sin^{-1}(1/2)$

iv] $\tan^{-1}(1+i)$, v] $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$, vi] $\sec^{-1}(i)$

Solution: We know that,

iii] $\sin^{-1} z = -i \log\left[iz + \sqrt{1-z^2}\right]$

put $z = 1/2$

$\therefore \sin^{-1}(1/2) = -i \log\left[i/2 + \sqrt{1-(1/2)^2}\right]$

$\therefore \sin^{-1}(1/2) = -i \log\left[i/2 + \sqrt{1-1/4}\right]$

$\therefore \sin^{-1}(1/2) = -i \log\left[i/2 + \sqrt{3/4}\right]$

$= -i \log\left[i/2 + \sqrt{3}/2\right]$

$= -i \log\left[\frac{i+\sqrt{3}}{2}\right]$

$= -i\left[\log\left|\frac{\sqrt{3}}{2} + \frac{i}{2}\right| + i \arg\left[\frac{\sqrt{3}}{2} + \frac{i}{2}\right] + 2k\pi i\right]$ $k \in I$

$\because \log z = \log|z| + i \arg z + 2k\pi i$

$= -i\left[\log\sqrt{\frac{3}{4}+\frac{1}{4}} + i\tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) + 2k\pi i\right]$

$\because |x+iy| = \sqrt{x^2+y^2}$

$= -i\left[\log\sqrt{\frac{4}{4}} + i\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + 2k\pi i\right]$

$$\sin^{-1}(1/2) = -i\left[\log 1 + i\,\pi/6 + 2k\pi i\right]$$

$$= -i^2\left[\pi/6 + 2k\pi\right] \quad \because \log 1 = 0$$

$$\sin^{-1}(1/2) = \pi/6 + 2k\pi \qquad k \in I$$

iv] We know that,

$$\tan^{-1}(z) = \frac{1}{2i}\log\left[\frac{1+iz}{1-iz}\right]$$

put $z = 1+i$

$\therefore$ We have, $\tan^{-1}(1+i) = \dfrac{1}{2i}\log\left[\dfrac{1+i(1+i)}{1-i(1+i)}\right]$

$$= \frac{1}{2i}\log\left[\frac{1+i+i^2}{1-i-i^2}\right]$$

$$= \frac{1}{2i}\log\left[\frac{i}{2-i}\right] \qquad \because i^2 = -1$$

$$\tan^{-1}(1+i) = \frac{1}{2i}\log\left[\frac{i}{2-i}\right]$$

$$= \frac{1}{2i}\log\left[\frac{i}{2-i} \times \frac{2+i}{2+i}\right]$$

$$= \frac{1}{2i}\log\left[\frac{i(2+i)}{2^2+1^2}\right] \qquad \left\{ \begin{array}{l}(a+ib)(a-ib) \\ = (a^2 - i^2b^2) = a^2+b^2\end{array}\right.$$

$$= \frac{1}{2i}\log\left[\frac{2i+i^2}{5}\right]$$

$$= \frac{1}{2i}\left[\log(2i-1) - \log 5\right] \quad \because$$

$$\because \log\frac{z_1}{z_2} = \log z_1 - \log z_2$$

Page No.
Date

$\tan^{-1}(1+i) = \frac{1}{2i}\left[\log(2i-1) - \log 5\right]$ ....①

Consider,

$\log(2i-1) = \log|2i-1| + i\arg(2i-1) + 2k_1\pi i \quad k_1 \in I$

$= \log\sqrt{4+1} + i\tan^{-1}\left(\frac{2}{-1}\right) + 2k_1\pi i \quad k_1 \in I$

$= \log\sqrt{5} + i\tan^{-1}\left(\frac{2}{-1}\right) + 2k_1\pi i \quad k_1 \in I$

Now, $\log 5 = \log|5| + i\arg 5 + 2k_2\pi i \quad k_2 \in I$

$= \log 5 + i\,0 + 2k_2\pi i \quad k_2 \in I \quad \because \arg 5 = 0$

$= \log 5 + 2k_2\pi i \quad k_2 \in I$

put all values of $\log(2i-1)$ & $\log 5$ in ① we get,

$\tan^{-1}(1+i) = \frac{1}{2i}\left[\log\sqrt{5} + i\tan^{-1}\left(\frac{2}{-1}\right) + 2k_1\pi i - \log 5 - 2k_2\pi i\right]$

$\tan^{-1}(1+i) = \frac{1}{2i}\left[\log\sqrt{5} - \log 5 + i\tan^{-1}\left(\frac{2}{-1}\right) + 2(k_1-k_2)\pi i\right]$

$k_1 - k_2 \in I$

$\tan^{-1}(1+i) = \frac{1}{2i}\left[\log\frac{\sqrt{5}}{5} + i\tan^{-1}\left(\frac{2}{-1}\right) + 2n\pi i\right]$

$n = k_1 - k_2 \in I$

$= \frac{1}{2i}\left[\log\frac{1}{\sqrt{5}} + i\tan^{-1}\left(\frac{2}{-1}\right) + 2n\pi i\right]$

$= \frac{1}{2i}\left[\log 1 - \log\sqrt{5}\right] + \frac{1}{2i}\,i\tan^{-1}\left(\frac{2}{-1}\right) + \frac{1}{2i}\,2n\pi i$

$$\tan^{-1}(1+i) = -\frac{i}{2}\left(-\log 5^{1/2}\right) + \frac{1}{2}\tan^{-1}\left(\frac{2}{-1}\right) + n\pi \quad , n \in I$$

$$= \frac{i}{2}\log 5^{1/2} + \frac{1}{2}\tan^{-1}\left(\frac{2}{-1}\right) + n\pi \quad , n \in I$$

$$\tan^{-1}(1+i) = i\log 5^{1/4} + \frac{1}{2}\tan^{-1}\left(\frac{2}{-1}\right) + n\pi \quad , n \in I$$

$$\tan^{-1}(1+i) = \frac{i}{4}\log 5 + \frac{1}{2}\tan^{-1}\left(\frac{2}{-1}\right) + n\pi \quad , n \in I$$

V] We know that

$$\cos^{-1}(z) = -i\log\left[z + \sqrt{z^2-1}\right]$$

put $z = \sqrt{2}/2 = 1/\sqrt{2}$

$$\therefore \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = -i\log\left[\frac{1}{\sqrt{2}} + \sqrt{\frac{1}{2}-1}\right]$$

$$= -i\log\left[\frac{1}{\sqrt{2}} + \sqrt{\frac{-1}{2}}\right]$$

$$= -i\log\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] \quad \because \sqrt{-1} = i \quad \ldots (1)$$

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = -i\log\left[\frac{1+i}{\sqrt{2}}\right]$$

Now consider,

$$\log\frac{1}{\sqrt{2}} = \log 1 - \log\sqrt{2}$$
$$= 0 - \log 2^{1/2}$$
$$= -\frac{1}{2}\log 2$$

$$\log\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \log\left|\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right| + i\arg\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + 2k\pi i \quad k \in I$$

Scanned by CamScanner

$$\log\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \log\sqrt{\frac{1}{2}+\frac{1}{2}} + i\tan^{-1}\left(\frac{1/\sqrt{2}}{1/\sqrt{2}}\right) + 2K\pi i$$

$$= \log\sqrt{\frac{2}{2}} + i\tan^{-1}(1) + 2K\pi i \quad K\in I$$

$$= \log 1 + i\tan^{-1}(1) + 2K\pi i$$

$$\log\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = i\pi/4 + 2K\pi i, K\in I \; (\because \log 1 = 0)$$

put this value in (1) then we get,

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = -i\times\left[i\,\pi/4 + 2K\pi i\right], \quad K\in I$$

$$= -i^2\left[\pi/4 + 2\pi K\right] \quad K\in I$$

$$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi/4 + 2\pi K.$$

iv] $\sec^{-1}(i)$

We know that,

$$\sec^{-1}(z) = \frac{1}{i}\log\left[\frac{1\pm(1-z^2)^{1/2}}{z}\right]$$

$$= -i\log\left[\frac{1\pm(1-z^2)^{1/2}}{z}\right]$$

put $z=i$ then we get,

$$\sec^{-1}(i) = -i\log\left[\frac{1\pm(1-i^2)^{1/2}}{i}\right]$$

$$= -i\log\left[\frac{1\pm\sqrt{1-i^2}}{i}\right]$$

$$\sec^{-1}(i) = -i\log\left[\frac{1+\sqrt{1+1}}{i}\right]$$

$$= -i\log\left[\frac{1+\sqrt{2}}{i}\times\frac{(-i)}{(-i)}\right]$$

$$= -i\log\left[\frac{-i(1+\sqrt{2})}{-i^2}\right]$$

$$= -i\log\left[-i(1+\sqrt{2})\right] \quad \because i^2=-1$$

$$= -i\left[\log\left|-i(1+\sqrt{2})\right| + i\arg\left(-i(1+\sqrt{2})\right) + 2k\pi i\right]$$

$k \in I$

$$= -i\left[\log(1+\sqrt{2}) + i\left(-\pi/2\right) + 2k\pi i\right] \quad k \in I$$

$$\sec^{-1}(i) = -i\left[\log(1+\sqrt{2}) + i\left(2k\pi - \pi/2\right)\right], \; k \in I.$$

2016 2M

i) $\sin^{-1}(0)$

We know that,

$$\sin^{-1}(z) = -i\log\left[iz+\sqrt{1-z^2}\right]$$

put $z=0$

$$\sin^{-1}(0) = -i\log\left[i\times 0+\sqrt{1-0^2}\right]$$

$$= -i\log(1)$$

$$= -i\times 0 \quad \because \log(1)=0$$

$$= 0$$

$$\log|-1+\sqrt{2}| = \sqrt{x^2+y^2}$$
$$= \sqrt{(-1+\sqrt{2})^2+0^2}$$
$$= -1+\sqrt{2}$$

iii) $\sin^{-1}(i)$

We know that,

$$\sin^{-1}z = -i\log\left[iz+\sqrt{1-z^2}\right]$$

put $z = i$

$$\sin^{-1}(i) = -i\log\left[i\times i+\sqrt{1-i^2}\right]$$
$$= -i\log\left[i^2+\sqrt{1+1}\right]$$
$$\sin^{-1}(i) = -i\log\left[-1+\sqrt{2}\right]$$

Now consider

$$\log(-1+\sqrt{2}) = \log|-1+\sqrt{2}| + i\arg(-1+\sqrt{2}) + 2k\pi i \quad k\in I,$$
$$= \log(-1+\sqrt{2}) + i\tan^{-1}\left(\frac{0}{-1+\sqrt{2}}\right) + 2k\pi i$$
$$= \log(-1+\sqrt{2}) + i\tan^{-1}(0) + 2k\pi i$$
$$= \log(-1+\sqrt{2}) + 2k\pi i$$

$$\sin^{-1}(i) = -i\left[\log(-1+\sqrt{2})+2k\pi i\right]$$

$=$.

Scanned by CamScanner