UNIT-III

Differentiation

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* **Differentiable funⁿ :->**

Let, $A \subseteq \mathbb{R}$. ~~&~~ let $\phi : A \to \mathbb{R}$

Suppose, A contains a nbd of the point a. We define, the derivetive of ϕ at a

$$\phi'(a) = \lim_{t \to 0} \frac{\phi(a+t) - \phi(a)}{t}$$

provided the Limit exist.
In this case, We say that ϕ is differenti-ble at 'a'.

IMP. A.M 2016

* **Directional derivetive :->**

Let, $A \subseteq \mathbb{R}^m$. let $f : A \to \mathbb{R}^n$

Suppose, A contains a ~~neighbhour~~ nbd of $\bar{a}$. given $\bar{u} \in \mathbb{R}^m$ with $\bar{u} \neq 0$,
We define,

$$f'(\bar{a}, \bar{u}) = \lim_{t \to 0} \frac{f(\bar{a} + t\bar{u}) - f(\bar{a})}{t}$$

provided the limit exist.
This limit depends both on $\bar{a}$ and $\bar{u}$, it is called the directional derivetive of f at $\bar{a}$ w.r.t. vector $\bar{u}$.

Ex:- Let, $f : R^2 \rightarrow R$, be given by the eqⁿ,

$f(x_1, x_2) = x_1 x_2$, find the directional derivetive of f at $\bar{a} = (a_1, a_2)$ w.r.t. the vector $\bar{u} = (1,0)$ and $\bar{v} = (1,2)$.

I] The directional derivetive of f at $\bar{a} = (a_1, a_2)$ w.r.t. the vector $\bar{u} = (1,0)$

Here, $f(x_1, x_2) = x_1 x_2$

$\therefore \quad f(\bar{a}) = f(a_1, a_2) = a_1 a_2$

Now,

$$f(\bar{a} + t\bar{u}) = f[(a_1, a_2) + t(1,0)]$$
$$= f[(a_1, a_2) + (t, 0)]$$
$$= f[(a_1 + t), a_2]$$
$$= (a_1 + t) \cdot a_2$$

We have,

$$f'(\bar{a}, \bar{u}) = \lim_{t \to 0} \frac{f(\bar{a} + t\bar{u}) + f(\bar{a})}{t}$$

$$\therefore \quad f'(\bar{a}, \bar{u}) = \lim_{t \to 0} \frac{(a_1 + t)\, a_2 - a_1 a_2}{t}$$

$$= \lim_{t \to 0} \frac{a_1 a_2 + t a_2 - a_1 a_2}{t}$$

$$= \lim_{t \to 0} \frac{t a_2}{t}$$

$$f'(\bar{a}, \bar{u}) = a_2$$

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II] The directional derivetive of f at $\bar{a} = (a_1, a_2)$ w.r.t. the vector $\bar{v} = (1,2)$

Here, $f(x_1, x_2) = x_1 \cdot x_2$

$f(\bar{a}) = f(a_1 a_2) = a_1 a_2$.

$f(\bar{a} + t\bar{v}) = f[(a_1, a_2) + t(1,2)]$.

$= f[(a_1, a_2) + (t, 2t)]$

$= f[a_1 + t, a_2 + 2t]$

$= (a_1 + t) \cdot (a_2 + 2t)$.

$$f'(\bar{a}, \bar{v}) = \lim_{t \to 0} \frac{f(\bar{a} + t\bar{v}) - f(\bar{a})}{t}$$

$= \cancel{\lim_{t \to 0} f[(a_1, a_2)}$

$$= \lim_{t \to 0} \frac{(a_1 + t) \cdot (a_2 + 2t) - a_1 a_2}{t}$$

$$= \lim_{t \to 0} \frac{a_1 a_2 + 2a_1 t + t a_2 + 2t^2 - a_1 a_2}{t}$$

$$= \lim_{t \to 0} \frac{2a_1 t + t a_2 + 2t^2}{t} = \lim_{t \to 0} \frac{\cancel{t}(2a_1 + a_2 + 2t)}{\cancel{t}}$$

$f'(\bar{a}, \bar{v}) = 2a_1 + a_2$

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Remark :-

① Let, $A \subseteq \mathbb{R}$ let, $\phi : A \to \mathbb{R}$ suppose A contains a nbd of a.
We say that ϕ is differentiable at 'a', if there is number λ such that

$$\frac{\phi(a+t) - \phi(a) - \lambda t}{t} \to 0 \quad \text{as } t \to 0$$

The number λ which is unique, is called the derivetive of ϕ at a and is denoted by, $\phi'(a)$.

② Let, $A \subseteq \mathbb{R}^m$ let, $f : A \to \mathbb{R}^n$.
Suppose, A contains a nbd of $\bar{a}$.
We say that, f is differentiable at $\bar{a}$, if there is an $n \times m$ matrix B such that.

$$\frac{f(\bar{a}+\bar{h}) - f(\bar{a}) - B\bar{h}}{|h|} \to \bar{0} \quad \text{as } \bar{h} \to \bar{0}$$

the matrix B which is unique, is called the derivetive of f at $\bar{a}$ and is denoted by, $Df(\bar{a})$.

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* Thm :→ Let, $A \subseteq \mathbb{R}^m$, Let $f: A \to \mathbb{R}^n$, if f is differentiable at $\bar{a}$ then prove that all the directional derivetive of f at $\bar{a}$ exist and,

$$f'(\bar{a}, \bar{u}) = Df(\bar{a})\bar{u}.$$

→ Proof :→ Given that, f is differentiable at $\bar{a}$, We have

$$\frac{f(\bar{a}+\bar{h}) - f(\bar{a}) - B\bar{h}}{|\bar{h}|} \to 0 \quad \text{as } \bar{h} \to 0 \quad \text{---} (1)$$

Let, $B = Df(\bar{a})$, set $\bar{h} = t\bar{u}$

$\therefore$ eq^n (1) becomes

$$\frac{f(\bar{a}+t\bar{u}) - f(\bar{a}) - Bt\bar{u}}{|t\bar{u}|} \to \bar{0} \quad \text{as } t \to 0 \quad \text{---} (2)$$

if, t approaches $\bar{0}$ through +ve values, We multiply eq^n (2) by $|\bar{u}|$

eq^n (2) becomes,

$$\frac{|\bar{u}|.f(\bar{a}+t\bar{u}) - f(\bar{a})}{|t\bar{u}|} - \frac{Bt\bar{u}\,|\bar{u}|}{|t\bar{u}|} \to \bar{0} \quad \text{as } t \to 0$$

$$\therefore \frac{f(\bar{a}+t\bar{u}) - f(\bar{a})}{t} - B\bar{u} \to 0 \quad \text{as } t \to 0.$$

$$\lim_{t \to 0} \frac{f(\bar{a}+t\bar{u}) - f(\bar{a})}{t} = B\bar{u}$$

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i.e. $\boxed{f'(\bar{a},\bar{u}) = Df(\bar{a})(\bar{u})}$

if t approache $\bar{o}$ through $-$ve values, We multiply ② by $-|\bar{u}|$, to reach same conclusion.

Thus,

$$f'(\bar{a},\bar{u}) = Df(\bar{a}).\bar{u}$$

* **Remark**:→ The converse of the above theorem does not hold.

Ex:→ Define, $f : R^2 \to R$ by setting $f(\bar{o}) = 0$ and $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$ if $f(x,y) \neq \bar{o}$.

→ To show that, all directional derivetive exist at $\bar{o}$ but, that f is not differentiable at $\bar{o}$.

Let, $\bar{u} = (h,k) \neq 0$

consider, $\dfrac{f(\bar{o}+t\bar{u}) - f(\bar{o})}{t} = \dfrac{f(t\bar{u}) - f(\bar{o})}{t}$

$$= \frac{f(t(h,k))}{t} = \frac{f(th,tk)}{h}$$

$$= \frac{t^2h^2tk}{(t^4h^4+t^2k^2)t}$$

$$= \frac{t^3h^2k}{t^2(t^2h^4+\cancel{k^2})t} = \frac{h^2k}{t^2h^4+k^2}$$

$$= \frac{kh^2}{k^2\left(\frac{t^2h^4}{k^2}+1\right)} = \frac{h^2}{k\left(1+\frac{t^2h^4}{k^2}\right)}$$

We have,

$$f'(\bar{o}, \bar{u}) = \lim_{t \to 0} \frac{f(o + t\bar{u}) - f(\bar{o})}{t}$$

$$= \lim_{t \to 0} \frac{h^2}{k\left(1 + \frac{t^2 h^4}{k^2}\right)}$$

$$= \frac{h^2}{k}$$

So that,

$$f'(\bar{o}, \bar{u}) = \begin{cases} \frac{h^2}{k} & , \text{if } k \neq 0 \\ = 0 & , \text{if } k = 0 \end{cases}$$

Thus,

$f'(\bar{o}, \bar{u})$ exist for all $\bar{u} = (h, k) \neq 0$

Now,

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^4 + y^2}$$

- The limiting value of f along the line $y = x^2$ is given by,

$$\lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \neq 0$$

Also, $f(\bar{o}) = f(0,0) = 0$.

$\therefore$ The funⁿ $f(x,y)$ is discontinous and $\bar{o}$

$\therefore$ The given funⁿ is not differentiable at origin.

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Thm :→ let, $A \subseteq \mathbb{R}^m$, let $f : A \to \mathbb{R}^n$. if f is differentiable at $\bar{a}$, then P.T. f is continous at $\bar{a}$.

→ Given that, the funⁿ f is differentiable at $\bar{a}$.

$$\therefore \frac{f(\bar{a}+\bar{h}) - f(\bar{a}) - B\bar{h}}{|\bar{h}|} \to 0 \text{ as } \bar{h} \to 0.$$

let, $B = Df(\bar{a})$.

for $\bar{h}$ near $\bar{0}$ but diffrent from $\bar{0}$. We Write,

$$f(\bar{a}+\bar{h}) - f(\bar{a}) = |\bar{h}| \left[\frac{f(a+\bar{h}) - f(\bar{a}) - B\bar{h}}{|\bar{h}|}\right] + B\bar{h}$$

Taking Limit as $\bar{h} \to \bar{0}$

$$\therefore \lim_{\bar{h} \to \bar{0}} f(\bar{a}+\bar{h}) - f(\bar{a}) = \lim_{\bar{h} \to 0} \left[|\bar{h}| \frac{f(a+\bar{h}) - f(\bar{a}) - B\bar{h}}{|\bar{h}|} + B\bar{h}\right]$$

$$\therefore \lim_{\bar{h} \to \bar{0}} f(\bar{a}+\bar{h}) - f(\bar{a}) = 0$$

$$\lim_{\bar{h} \to 0} f(\bar{a}+\bar{h}) = f(\bar{a})$$

i.e. f is continous at $\bar{a}$.

* j^{th} partial Derivetive :→

Let, $A \subset \mathbb{R}^m$ and let $f : A \to \mathbb{R}$. We define the j^{th} partial derivetive of f at $\bar{a}$ to be the directional derivetive of f at $\bar{a}$ w.r.t. the vector $\bar{e}_j$ provided this derivetive exist and we denote it by,

$$D_j f(\bar{a}) = \lim_{t \to 0} \frac{f(\bar{a} + t\bar{e}_j) - f(\bar{a})}{t}$$

$$= f'(\bar{a}, \bar{e}_j).$$

* **Remark** :→ ① If $f : \mathbb{R}^1 \to \mathbb{R}^3$ is differentiable funⁿ it's derivetive is the column matrix

1×3

$$Df(t) = \begin{bmatrix} f_1'(t) \\ f_2'(t) \\ f_3'(t) \end{bmatrix}$$

② If, g is a funⁿ i.e. $g : \mathbb{R}^3 \to \mathbb{R}^1$, it's derivetive is, row matrix.

$$Dg(a) = \begin{bmatrix} D_1 g(a) & D_2 g(a) & D_3 g(a) \end{bmatrix}$$

2M Imp Ex

Let, $f : R^2 \to R^3$; $f(x,y) = (x^2, x^3y, x^4y^2)$
compute Df.

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 0 \\ 3x^2y & x^3 \\ 4x^3y^2 & 2x^4y \end{bmatrix}$$

Where, $f_1(x,y) = x^2$, $f_2(x,y) = x^3y$, $f_3(x,y) = x^4y^2$

5M

Thm :→ Let $A \subseteq R^m$, Let $f : A \to R$. if f is differentiable at $\bar{a}$ then prove that,

$$Df(\bar{a}) = [D_1f(\bar{a}) \;\; D_2f(\bar{a}) \; \ldots \; D_mf(\bar{a})].$$

→ Given that, the funⁿ f is differentiable at $\bar{a}$.

∴ $Df(\bar{a})$ exist and is a matrix of order $1 \times m$.

Let, $Df(\bar{a}) = [\lambda_1 \;\; \lambda_2 \; \ldots \; \lambda_m] = \lambda_j$

We know that,

"Let, $A \subseteq R^m$, let $f : A \to R^n$, if f is differentiable at $\bar{a}$ then all ~~then~~ the directional derivetive of f at $\bar{a}$ exist and

$$f'(\bar{a}, \bar{u}) = Df(\bar{a})\bar{u}."$$

it follows that,

$$D_jf(\bar{a}) = f'(\bar{a}, \bar{e}_j) = Df(\bar{a})\bar{e}_j = \lambda_j \cdot \bar{e}_j = \lambda_j$$

Ex:→ Let, $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(\bar{0})=0$ and $f(x,y)= \frac{xy}{x^2+y^2}$, if $(x,y) \neq \bar{0}$

(a) for which vectors $\bar{u} \neq 0$ does $f'(\bar{0},\bar{u})$ exist? Evaluate if when it exists?

(b) Do $D_1 f$ and $D_2 f$ exist at $\bar{0}$?

(c) Is f is differentiable at $\bar{0}$?

(d) Is f is continous at $\bar{0}$?

→ (I) Here, $f(x,y) = \frac{xy}{x^2+y^2}$, if $(x,y) \neq 0$

(a)

and $f(\bar{0}) = 0$

and Let, $\bar{u} = (1,0) \neq 0$, we have

$$f'(\bar{0},\bar{u}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{u}) - f(\bar{0})}{t}$$

$$= \lim_{t\to 0} \frac{f(t(1,0))}{t} = \lim_{t\to 0} \frac{f(t,0)}{t} = 0$$

$f'(\bar{0},\bar{u}) = 0$

Let, $\bar{u} = (0,1) \neq 0$, we have

$$f'(\bar{0},\bar{u}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{u}) + f(\bar{0})}{t}$$

$$= \lim_{t\to 0} \frac{f(t(0,1))}{t} = \lim_{t\to 0} \frac{f(0,t)}{t}$$

$$= 0.$$

$f'(\bar{0},\bar{u}) = 0$

This shows that the directional derivetive of f at $\bar{a}$ exist for only directional vector $\bar{u}$ in perticular by setting $\bar{u} = (1,0) \neq 0$ and $\bar{u} = (0,1) \neq 0$

(b) (II) We have,

$$D_1 f(\bar{0}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{u}) - f(\bar{0})}{t} = f'(\bar{0},\bar{u})$$

$$= 0 \quad \text{if } \bar{u} = (1,0) \neq 0$$

|||y, $$D_2 f(\bar{0}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{v}) - f(\bar{0})}{t} = f'(\bar{0},\bar{v})$$

$$= 0 \quad \text{if } \bar{v} = (0,1) \neq 0$$

$\therefore$ $D_1 f$ and $D_2 f$ exist at $\bar{0}$.

(d) (iv) Now, $f(x,y) = \dfrac{xy}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x,y\to(0,0)} \frac{xy}{x^2+y^2}.$$

Consider, the line $y = mx$

$$\therefore \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x.mx}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^2 m}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{m}{(1+m^2)}$$

Which depends on m.

$\therefore$ $\lim_{x\to 0} f(x,mx)$ is not unique

Also, $f(\bar{0}) = 0$

Hence, $\lim_{x,y\to(0,0)} f(x,y)$ does not exist.

$\therefore$ The given funⁿ is not continous at $\bar{0}$.

Ⓒ :- from Ⓓ the funⁿ f is not differentiable at $\bar{0}$.

* **Remark :->**

H.W: ① Is $\bar{u} = (1,2)$ and $\bar{v} = (2,1)$, above example is true.

② $f(x,y) = \dfrac{x^2y^2}{x^2y^2 + (y-x)^2}$ if $(x,y) \neq 0$

$f(\bar{0}) = 0$

③ $f(x,y) = \dfrac{x^3}{x^2+y^2}$ if $(x,y) \neq 0$

$f(\bar{0}) = 0$.

2m * **Defⁿ :->** continuously differentiable funⁿ or ~~continuous~~ of class c^1

Let, A be open in $\mathbb{R}^m$ and suppose that the partial derivetive $D_j f_i(\bar{x})$ of the component ~~of A and are continous~~ ~~of~~ ~~then~~ funⁿ of f exist at each point of $\bar{x}$ of A and are continuous on A. Then f is differentiable at each point of A. In this case We say that funⁿ continously differentiable of class c^1 on A

2M

Ex:- 1) let $f : R^2 \to R^2$, We define by the eqⁿ, $f(r\cos\theta, r\sin\theta)$ calculate, Df and determinant of Df

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→ Here, $f(r,\theta) = (f_1, f_2) = (r\cos\theta, r\sin\theta)$.
We have,

$$Df = \begin{vmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial \theta} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\cancel{= r\cos^2\theta}$$

$$\det Df = |Df| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$
$$= r$$

Ex: 12) 2016 Let, $f: \mathbb{R}^2 \to \mathbb{R}^2$, We define by the eqn, $f(x,y) = (e^x\cos y, e^x\sin y)$ calculate Df and $|Df|$.

→ Here, $f(x,y) = (e^x\cos y, e^x\sin y) = (f_1, f_2)$.

$$Df = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \cancel{\cos y}$$

$$\det Df = |Df| = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^x\cos^2 y + e^x\sin^2 y$$

$$= e^x = e^{2x}$$

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H.W

1) Is $\bar{u} = (1,2)$ and $\bar{v} = (2,1)$. For $f(x,y) = \frac{xy}{x^2+y^2}$

→ @Here, $f(x,y) = \frac{xy}{x^2+y^2}$, if $(x,y) \neq (0,0)$

and $f(\bar{0}) = 0$

and let, $\bar{u} = (1,2) \neq 0$ We have

$$f'(0,\bar{u}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{u}) - f(\bar{0})}{t}$$

$$= \lim_{t\to 0} \frac{f(t(1,2)) - f(\bar{0})}{t}$$

$\frac{t.2t}{t^2+4t^2}$

$$= \lim_{t\to 0} \frac{f(t,2t)}{t}$$

$\frac{2\cancel{t^2}}{\cancel{t^2}(1+4)}$

$$= \lim_{t\to 0} \frac{\frac{2}{5}}{t}$$

$\frac{2}{5}$

$$f'(0,\bar{u}) = 0$$

let, $\bar{v} = (2,1)$ ~~$\neq 0$ tim~~ We have,

$$f'(0,\bar{v}) = \lim_{t\to 0} \frac{f(\bar{0}+t\bar{v}) - f(\bar{0})}{t}$$

$\frac{2t.t}{4t^2+t^2}$

$$= \lim_{t\to 0} \frac{f(t(2,1)) - f(\bar{0})}{t}$$

$\frac{2\cancel{t^2}}{(4+1)\cancel{t^2}}$

$$= \lim_{t\to 0} \frac{f(2t,t)}{t}$$

$\frac{2}{5}$

$$= \lim_{t\to 0} \frac{\frac{2}{5}}{t} = 0.$$

$$f'(0,\bar{v}) = 0$$

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This shows that, the directional derivetive of f at $\bar{a}$ exist for only directional vector $\bar{u}$ and $\bar{v}$ inperticular by setting $\bar{u} = (1,2) \neq 0$ and $\bar{v} = (2,1) \neq 0$.

(b) We have,

$$D_1 f(\bar{0}) = \lim_{t \to 0} \frac{f(\bar{0}+t\bar{u}) - f(\bar{0})}{t} = f'(\bar{0}, \bar{u}).$$

$$= 0 \cdot \text{ if } \bar{u} = (1,2) \neq 0$$

Illy, $$D_2 f(\bar{0}) = \lim_{t \to 0} \frac{f(\bar{0}+t\bar{v}) - f(\bar{0})}{t} = f'(\bar{0}, \bar{v}).$$

$$= 0, \text{ if } \bar{v} = (2,1) \neq 0.$$

$\therefore$ $D_1 f$ and $D_2 f$ exist at $\bar{0}$.

(d) Now, $f(x,y) = \frac{xy}{x^2+y^2}$

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}.$$

consider the line $y = mx$.

$$\therefore \lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 x^2} = \lim_{x \to 0} \frac{x^2 m}{x^2(1+m^2)}$$

$$\lim_{x \to 0} \frac{m}{1+m^2}$$

Which depends on m.

$\therefore$ $\lim_{x \to 0} f(x, mx)$ is not unique.

Also, $f(\bar{0}) = 0$.

Hence, $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

$\therefore$ The given fun^n is not continous at $\bar{0}$.

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ⓒ :→ From ⓓ the funⁿ f is not differentiable at $\bar{o}$.

→ ② Here, $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(y-x)^2}$, if $(x,y) \neq (0,0)$

and $f(\bar{o}) = 0$

and let, $\bar{u} = (0,1) \neq 0$ we have,

$$f'(0,\bar{u}) = \lim_{t\to 0} \frac{(\bar{o}+t\bar{u}) + f(\bar{o})}{t}$$

$$= \lim_{t\to 0} \frac{(t(0,1))}{t}$$

$$= \lim_{t\to 0} \frac{(0,t)}{t}$$

$$= 0.$$

$$\boxed{f'(0,\bar{u}) = 0}$$

Let, $\bar{v} = (1,0) \neq 0$ we have,

$$f'(\bar{o},\bar{v}) = \lim_{t\to 0} \frac{f(0+t\bar{v}) - f(\bar{o})}{t}$$

$$= \lim_{t\to 0} \frac{f(t(1,0))}{t}$$

$$= \lim_{t\to 0} \frac{f(t,0)}{t}$$

$$\boxed{f'(\bar{o},\bar{v}) = 0}$$

This shows that, the directional derivetive of f at $\bar{o}$ exist for only directional derivetive vector $\bar{u}$ and $\bar{v}$ inperticular by setting $\bar{u} = (0,1) \neq 0$ and $\bar{v} = (1,0) \neq 0$

(b) We have,

$$D_1 f(\bar{o}) = \lim_{t\to 0} \frac{f(\bar{o}+t\bar{u})-f(\bar{o})}{t} = f'(\bar{o},\bar{u})$$

$$= 0 \quad \text{if } \bar{u} = (0,1) \neq 0.$$

$$D_2 f(\bar{o}) = \lim_{t\to 0} \frac{f(\bar{o}+t\bar{v})-f(\bar{o})}{t} = f'(\bar{o},\bar{v}).$$

$$= 0 \quad \text{if } \bar{v} = (1,0) \neq 0$$

$\because$ $D_1 f$ and $D_2 f$ exist at $\bar{o}$.

(d) Now, $f(x,y) = \dfrac{x^2y^2}{x^2y^2+(y-x)^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(y-x)^2}$$

consider the line $y = mx$.

$$\lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x^2m^2x^2}{x^2m^2x^2+(mx-x)^2}$$

$$= \lim_{x\to 0} \frac{x^4m^2}{x^4m^2+(m^2x^2-2mx^2+x^2)}$$

$$= \cancel{\lim_{x\to 0}} \frac{x^4m^2}{mx^2(x^2m+m-2.}$$

$$= 1 + \frac{x^4m^2}{(mx-x)^2}$$

$$\lim_{x\to 0} f(x,mx) = 1$$

$\therefore$ The given funⁿ is not continous at $\bar{o}$.

(c) :→ from (d) the funⁿ f is not differentiable at $\bar{o}$.

→ ③ Here, $f(x,y) = \dfrac{x^3}{x^2+y^2}$, if $(x,y) \to (0,0)$

and $f(\bar{0}) = 0$

and let $\bar{u} = (0,1) \neq 0$ We have,

$$f'(0,\bar{u}) = \lim_{t \to 0} \frac{f(0+t\bar{u}) - f(\bar{0})}{t}$$

$$= \lim_{t \to 0} \frac{f(t(0,1))}{t}$$

$$= \lim_{t \to 0} \frac{f(0,t)}{t}$$

$$\boxed{f'(0,\bar{u}) = 0}$$

$\bar{v} = (1,0) \neq 0$ We have,

$$f'(0,\bar{v}) = \lim_{t \to 0} \frac{f(0+t\bar{v}) - f(\bar{0})}{t}$$

$$= \lim_{t \to 0} \frac{f(t(1,0))}{t}$$

$$\frac{\dfrac{t^3}{t^2+0}}{t} \qquad = \lim_{t \to 0} \frac{f(t,0)}{t}$$

$$= \lim_{t \to 0} t$$

$$= 0$$

$f'(0,\bar{v}) = 0$

This shoows that, the directional derivetive of f at $\bar{a}$ exist for only directional derivetive vector $\bar{u}$ and $\bar{v}$ inperticular by setting $\bar{u} = (0,1) \neq 0$ and $\bar{v} = (1,0) \neq 0$

(b) We have,

$$D_1 f(\bar{o}) = \lim_{t \to 0} \frac{f(\bar{o} + t\bar{u}) - f(\bar{o})}{t} = f'(\bar{o}, \bar{u})$$

$$= 0 \quad \text{if } \bar{u} = (0,1) \neq 0$$

$$D_2 f(\bar{o}) = \lim_{t \to 0} \frac{f(\bar{o} + t\bar{v}) - f(\bar{o})}{t} = f'(\bar{o}, \bar{v})$$

$$= 0 \quad \text{if } \bar{v} = (1,0) \neq 0$$

$\therefore$ $D_1 f$ and $D_2 f$ exist at $\bar{o}$.

(d) Now, $f(x,y) = \dfrac{x^3}{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}.$$

consider the line $y = mx$.

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^3}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^2(1+m^2)}$$

$$= \lim_{x\to 0} \frac{x}{(1+m^2)}$$

$$= 0$$

The given fun^n is continouse at $f(\bar{o}) = \bar{o}$

(c) from (d) the fun^n f is differentiable at $\bar{o}$.

EX :- Define $f : \mathbb{R} \to \mathbb{R}$ by setting $f(0) = 0$ and $f(t) = t^2 \sin(1/t)$ if $t \neq 0$

i) Show that, f is differentiable at 0, calculate $f'(0)$.

ii) calculate $f'(t)$ if $t \neq 0$

iii) Show that f' is not continous at 0.

iv) conclude that f is differentiable on $\mathbb{R}$ but not of class C^1.

→ Here, $f(t) = t^2 . \sin\left(\frac{1}{t}\right)$, if $t \neq 0$

diff. w.r.t. 't', we get.

$$f'(t) = 2t . \sin\left(\frac{1}{t}\right) + t^2 \cos\left(\frac{1}{t}\right) . \left(\frac{-1}{t^2}\right)$$

$$f'(t) = 2t \sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)$$

$\therefore$ f is differentiable at $t \neq 0$.

I] $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h)}{h}$$

$$= \lim_{h \to 0} \frac{h^{\cancel{2}} . \sin(1/h)}{\cancel{h}}$$

$$= \lim_{h \to 0} h . \sin(1/h)$$

$$= \lim_{h \to 0} h . \sin(1/h)$$

$$\underline{f'(0) = 0}$$

$\therefore$ f is differentiable at 0 with $f'(0)=0$

II] $f'(t) = 2t\sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)$, if $t \neq 0$

III] Here, $f'(t) = 2t\sin\left(\frac{1}{t}\right) - \cos\left(\frac{1}{t}\right)$, if $t \neq 0$

But, $\lim_{t\to 0} f'(t)$ does not exist.

$\therefore$ f' is not continous at 0.

iv] so, f is differentiable on $\mathbb{R}$ and it's derivetive $f'(t)$ is not continous at 0.
$\therefore$ f is differentiable on $\mathbb{R}$ but not of class C^1

* <u>Chain Rule</u> :-

Let, A be open in $\mathbb{R}^m$, Let B be open in $\mathbb{R}^n$, let $f: A \to \mathbb{R}^n$ and $g: B \to \mathbb{R}^m$ with $f(A) \subseteq B$. if f and g are of class C^r so,

$$D(g\circ f)(x) = Dg(f(x)) \cdot Df(x).$$

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* <u>Thm</u> :- Let, A be open in $\mathbb{R}^n$ and Let $f: A \to \mathbb{R}$ be differentiable on A if A contains the line segment with n points $\bar{a}$ and $\bar{a}+\bar{h}$ then prove that there is a point $\bar{c} = \bar{a} + t_0 h$ with $0<t_0<1$ of this line segment such that

$$f(\bar{a}+\bar{h}) - f(\bar{a}) = Df(\bar{c}) \cdot \bar{h}.$$

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→ proof :→ set $\phi(t) = f(\bar{a}+t\bar{h})$
then, ϕ is define for t in an open interval about $[0,1]$
being the composite of differentiable fun^n, ϕ is differentiable and it's derivetive is given by the formula.

$$\phi(t) = f(\bar{a}+t\bar{h})$$
$$\phi'(t) = Df(\bar{a}+t\bar{h})\cdot\bar{h}$$

The ordinary mean value thm implies that,

$\phi(1)-\phi(0) = \phi'(t_0)(1-0)$, for some t_0
with $0<t_0<1$

$$\therefore \quad f(\bar{a}+\bar{h}) - f(\bar{a}) = Df(\bar{a}+t_0\bar{h})\cdot\bar{h}$$

$$\therefore \quad f(\bar{a}+\bar{h}) - f(\bar{a}) = Df(\bar{c})\cdot\bar{h}$$

$(\because \bar{c} = \bar{a}+t_0\bar{h})$

Hence the proof.

5m
* Thm :→ Let, A be open in $\mathbb{R}^m$, Let $f : A \to \mathbb{R}^n$, let, $f(\bar{a}) = \bar{b}$ suppose that, g maps a neibhourhood of $\bar{b}$ into $\mathbb{R}^n$ that $g(\bar{b}) = \bar{a}$ and $g(f(\bar{x})) = \bar{x}$, $\forall x$ in a nbd of $\bar{a}$, if f is differentiable at $\bar{x}$ and g is differentiable at $\bar{b}$ then P.T. $Dg(\bar{b}) = [Df(\bar{a})]^{-1}$.

→ proof :→ Let, $i: \mathbb{R}^m \to \mathbb{R}^n$ be the identity funⁿ. it's derivetive is the identity matrix I_n.

We are given that,

$$g(f(\bar{x})) = i(\bar{x}) \quad , \text{ for all } \bar{x} \text{ in a nbd of } \bar{a}$$

We know the chain rule,

$$D(gof) = Dg(f(x)) \cdot Df(x)$$

$$Dg(f(\bar{a})) \cdot Df(\bar{a}) = Dg(\bar{b}) \cdot Df(\bar{a}) = I.$$

$$\Rightarrow \quad Dg(\bar{b}) = [Df(\bar{a})]^{-1}$$

Thus, $Dg(\bar{b})$ in the inverse matrix to $Df(\bar{a})$.

Imp.

Ex-1) Let, $f: \mathbb{R}^3 \to \mathbb{R}^2$ satisfy the condition $f(\bar{0}) = (1,2)$ and $Df(\bar{0}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Let, $g: \mathbb{R}^2 \to \mathbb{R}^2$, we define by the eqⁿ

$$g(x,y) = (x+2y+1, 3xy)$$

find, $D(gof)(\bar{0})$

→ We know that,

$$D(gof)(\bar{x}) = Dg(f(\bar{x}))\, Df(\bar{x}).$$

$$D(gof)(\bar{0}) = Dg(f(\bar{0})) \cdot Df(\bar{0}). \quad \text{—— (1)}$$

Here, $f(\bar{0}) = (1,2)$ and $Df(\bar{0}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

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We have, $g(x,y) = (x+2y+1, 3xy) = (g_1, g_2)$

Now, $Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 \\ 3y & 3x \end{bmatrix}$

$Dg(f(\bar{o})) = Dg(1,2) = \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$

$\therefore$ eq^n (1) becomes,

$D(gof)(\bar{o}) = \begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 5 \\ 6 & 12 & 21 \end{bmatrix}$

EX:-2) Let, $f: \mathbb{R}^2 \to \mathbb{R}^3$ and $g: \mathbb{R}^3 \to \mathbb{R}^2$ We given by the eq^n ~~$f(\bar{x}) = (e^{2x_1+x_2}$~~

$f(\bar{x}) = (e^{2x_1+x_2}, 3x_2 - \cos x_1, x_1^2 + x_2 + 2)$.

$g(\bar{y}) = (3y_1 + 2y_2 + y_3^2, y_1^2 - y_3 + 1)$.

i) if $F(\bar{x}) = g(f(\bar{x}))$ find $DF(\bar{o})$

ii) if $G(\bar{y}) = f(g(\bar{y}))$ find $DG(\bar{o})$

Here,

$f(\bar{x}) = (e^{2x_1+x_2},\ 3x_2 - \cos x_1,\ x_1^2 + x_2 + 2) = (f_1, f_2, f_3)$ —(*)

$g(\bar{y}) = (3y_1 + 2y_2 + y_3^2,\ y_1^2 - y_3 + 1) = (g_1, g_2)$.

i) $F(\bar{x}) = g(f(\bar{x}))$

$DF(\bar{x}) = Dg(f(\bar{x})) \cdot Df(\bar{x})$.

$\Rightarrow DF(\bar{0}) = Dg(f(\bar{0})) \cdot Df(\bar{0})$ —(1)

We have,

$f(\bar{0}) = (1, -1, 2)$. put $x=0$ in (*)

So,

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{bmatrix}$$

$$\Rightarrow Df(\bar{0}) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \quad \text{—(2)}$$

Now,

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_3}{\partial y_3} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{bmatrix}$$

$$Dg(f(\bar{0})) = Dg(1,-1,2) = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \quad \text{—(3)}$$

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using eq^n ② and ③ in eq^n ①, We ge

$$Df(\bar{o}) = Dgf(\bar{o}) \cdot Df(\bar{o})$$

$$= \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3+6+4 \\ 4 & 2-1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 13 \\ 4 & 1 \end{bmatrix}$$

2) Here, $G(\bar{y}) = f(g(\bar{y}))$

$$DG(\bar{y}) = Df(g(\bar{y})) \cdot Dg(\bar{y})$$

$$\Rightarrow DG(\bar{o}) = Df\, g(\bar{o}) \cdot Dg(\bar{o}) \quad \text{—①}$$

We have, $g(\bar{o}) = (0, 1)$

~~so,~~ $Dg =$

We have, $Dg = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} \end{bmatrix}$

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$$= \begin{bmatrix} 3 & 2 & 2y_3 \\ 2y_1 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow Dg(\bar{0}) = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad —(2)$$

Now, $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2e^{2x_1+x_2} & e^{2x_1+x_2} \\ \sin x_1 & 3 \\ 2x_1 & 1 \end{bmatrix}$

$$\Rightarrow Df(g(\bar{0})) = Df((0,1)) = \begin{bmatrix} 2e^1 & e^1 \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \quad —(3)$$

using eqn (2) and (3) in eqn (1); we get.

$$DG(\bar{0}) = Df\, g(\bar{0}) \cdot Dg(\bar{0})$$

$$= \begin{bmatrix} 2e & e \\ 0 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6e & 4e & -e \\ 0 & 0 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

V. Imp. 2M

* The inverse function theorem :-

Let A be open in $\mathbb{R}^n$, let $f : A \to \mathbb{R}^n$ be of class C^r. if $Df(\bar{x})$ is non-singular at the point $\bar{a}$ of A there is a nbd U of the point $\bar{a}$ such that, f carries U in a one-to-one fashion on to an open set V of $\mathbb{R}^n$ and the inverse funⁿ is of class C^r

2M Imp. ex:- Let, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the eqⁿ,

~~$f(x,y) = e^x \cos y$.~~

$f(x,y) = (e^x \cos y, e^x \sin y)$

a) Show that f is one to one on the set A consisting of all (x,y) with $0 < y < \pi$

b) What is the set $B = f(A)$?

c) if g is inverse find $Dg(0,1)$?

proof:- a) We know that,

if $f(x,y) = f(a,b)$ then $\|f(x,y)\| = \|f(a,b)\|$

Here, $f(x,y) = (e^x \cos y, e^x \sin y) = (f_1, f_2)$

$\therefore (e^x \cos y, e^x \sin y) = (e^a \cos b, e^a \sin b)$

$\Rightarrow \|(e^x \cos y, e^x \sin y)\| = \|(e^a \cos b, e^a \sin b)\|$

$\Rightarrow$

$\sqrt{e^{2x}(\cos^2 y + \sin^2 y)} = \sqrt{e^{2a}(\cos^2 b + \sin^2 b}$

$\Rightarrow e^x = e^a$ (taking log on both side).

$\Rightarrow x = a \quad , y = b.$

Thus, f is one-to-one

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b) We have,

$$B = f(A)$$

$$= \{ f(x,y) \in \mathbb{R}^2 / 0 < y < 2\pi \}.$$

$$= \{ (e^x \cos y,\ e^x \sin y) / 0 < y < 2\pi \}.$$

$$= \mathbb{R}^2 - \{0\}.$$

c) We know that,

$$Dg(y) = [Df(x)]^{-1}$$

$$Dg(y_1, y_2) = [Df(x_1, x_2)]^{-1}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} e^x \cos y & -\sin y\, e^x \\ e^x \sin y & e^x \cos y \end{bmatrix}^{-1}$$

$$= (e^x \cos y,\ e^x \sin y) = (0,1)$$

$$e^x \cos y = 0,\ e^x \sin y = 1$$

$$\Rightarrow x = 0,\quad y = \pi/2$$

$$\therefore (x, y) = (0, \pi/2).$$

$$\therefore Dg(0,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1}$$

2) Let, $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the eqn
$f(x, y) = (x^2 - y^2, 2xy)$

a) Show that f is one-to-one on the set A consisting for all (x, y) with $x > 0$.

b) What is the set $B = f(A)$?

c) if g is inverse funn Find $Dg(0,1)$?

→ a) We know that,
if $f(x, y) = f(a, b)$ then,
$\|f(x,y)\| = \|f(a,b)\|$

Here, $f(x,y) = (x^2 - y^2, 2xy) = (f_1, f_2)$

$(x^2 - y^2, 2xy) = (a^2 - b^2, 2ab)$

$\Rightarrow \|(x^2 - y^2, 2xy)\| = \|(a^2 - b^2, 2ab)\|$

$\Rightarrow \sqrt{(x^2 - y^2)^2 + 4x^2y^2} = \sqrt{(a^2 - b^2)^2 + 4^2x^2y}$

$\cancel{\Rightarrow \sqrt{x^4 + y^4 - 2x^2y^2) + 4x^2y^2}}$

$\Rightarrow \sqrt{((x - y)(x + y))^2 + 4x^2y^2} = \sqrt{((a - b)(a + b))^2 + 4x^2y^2}$

$\Rightarrow (x - y)(x + y) + \cancel{2xy} = (a - b)(a + b) + \cancel{2xy}$

$\Rightarrow (x - y)(x + y) = (a - b)(a + b)$

$\Rightarrow x - y = a - b, \quad x + y = a + b$

$\Rightarrow x = a, \quad y = b$

$\Rightarrow$ Thus,

$\Rightarrow \sqrt{x^4 - 2x^2y^2 + y^4 + 4x^2y^2} = \sqrt{a^4 - 2a^2b^2 + b^4 + 4a^2b^2}$

$\Rightarrow \sqrt{x^4 + 2x^2y^2 + y^4} = \sqrt{a^4 + 2a^2b^2 + b^4}$

$\Rightarrow (x^2 + y^2) = a^2 + b^2$

$\Rightarrow x^2 = a^2 \quad , \quad y^2 = b^2$

$\Rightarrow x = a \quad , \quad y = b.$

$\Rightarrow$ Thus, f is one-to-one.

(2) We have,

$B = f(A).$

$= \{f(x,y) \in R^2 \mid x > 0\}.$

$= \{(x^2 - y^2, 2xy) \mid x > 0\}.$

$= R^2 - \{0\}.$

(3) We know that,

$Dg(y) = [Df(x)]^{-1}$

$Dg(y_1, y_2) = [Df(x_1, x_2)]^{-1}$

$= \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}^{-1}$

$= \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}^{-1}$

$(x^2 - y^2, 2xy) = (0, 1)$

$x^2 - y^2 = 0 \quad , \quad 2xy = 1$

$\Rightarrow x^2 = y^2 \qquad \Rightarrow 2x^2 = 1$

$\Rightarrow x = y \qquad \Rightarrow x^2 = \frac{1}{2}$

$\Rightarrow x = \frac{1}{\sqrt{2}} \quad , \quad y = \frac{1}{\sqrt{2}}$

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$$= \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}^{-1}$$

$$= \sqrt{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \sqrt{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

V.Imp
2M

* The implicit Function theorem :-

Statement :- Let, A be open in $\mathbb{R}^{k+n}$ let $f : A \to \mathbb{R}^n$ be of class C^r. Write f in the form $f(\bar{x}, \bar{y})$ for $\bar{x} \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$ suppose that, $f(\bar{a}, \bar{b})$ is a point of A such that, $f(\bar{a}, \bar{b}) = 0$ and

$$\det \frac{\partial f}{\partial y}(\bar{a}, \bar{b}) \neq 0$$

Then there is a nbd B of $\bar{a}$ in $\mathbb{R}^k$ and a unique continous funⁿ $g : B \to \mathbb{R}^n$ such that $g(\bar{a}) = \bar{b}$ and $f(\bar{x}, g(\bar{x})) = \bar{0}$, $\forall \bar{x} \in B$.

The funⁿ g is in fact of class C^r.

ex :- 1) Given, $f : \mathbb{R}^5 \to \mathbb{R}^2$ of class C'
Let $\bar{a} = (1, 2, -1, 3, 0)$ suppose $f(\bar{a}) = 0$
and

$$Df(\bar{a}) = \begin{bmatrix} 1 & 3 & 1 & -1 & 2 \\ 0 & 0 & 1 & 2 & 4 \end{bmatrix}$$

1) Show that there is funⁿ $g : B \to \mathbb{R}^2$ of class C' defined on an open set B of $\mathbb{R}^3$ such that, $f(x, g_1(x), g_2(x), x_2, x_3) = \bar{0}$ for $\bar{x} = (x_1, x_2, x_3) \in B$ and $g(1, 3, 0) = (2, -1)$

2) find $Dg(1, 3, 0)$

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Let, $\bar{a} = (1,3,0)$, $\bar{b} = (2,-1)$

We have,

$$\det \frac{\partial f}{\partial y}(\bar{a},\bar{b}) = \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} = 3 \neq 0$$

By implicit funⁿ thm there is a nbd B of $\bar{a} = (1,3,0)$ in $\mathbb{R}^3$ and unique continous funⁿ $g : B \to \mathbb{R}^2$ of class C' such that, $g(1,3,0) = (2,-1)$

$$f(x, g_1(x), g_2(x), x_2, x_3) = 0$$

We know that,

$$\frac{\partial f}{\partial x}(\bar{a},\bar{b}) + \frac{\partial f}{\partial y}(\bar{a},\bar{b}) \cdot Dg(\bar{a}) = 0$$

$$\therefore Dg(\bar{a}) = -\frac{\partial f}{\partial x}(\bar{a},\bar{b})\left[\frac{\partial f}{\partial y}(\bar{a},\bar{b})\right]^{-1}$$

$$= -\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj} A$

$= \frac{-1}{3}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$

$$\cancel{= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \frac{-1}{3}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

$$\cancel{= \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}} = \frac{-1}{3}\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

$$Dg(1,3,0) = \cancel{\begin{bmatrix} -3 & 3 & -6 \\ 1 & 1 & -2 \end{bmatrix}} = \frac{-1}{3}\begin{bmatrix} 1 & -3 & -2 \\ 0 & 6 & 12 \end{bmatrix}$$

Ex 2) Let, $f : \mathbb{R}^3 \to \mathbb{R}^2$ be of class c^1, write f in the form $f(x, y_1, y_2)$, assume that,
$f(3,-1,2)=0$ and $Df(3,-1,2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

i) Show there is funⁿ $g : B \to \mathbb{R}^2$ of class c^1 defined on an open set B of $\mathbb{R}^3$ such that, $f(x, g_1(x), g_2(x)) = 0$, $x \in B$ and $g(3) = (-1,2)$

2) Find $Dg(3)$

→ Let, $\bar{a} = 3$, $\bar{b} = (-1,2)$ We have,

$$\det \frac{\partial f}{\partial y}(\bar{a},\bar{b}) = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = 2+1 = 3 \neq 0$$

By implicit funⁿ thm there is a nbd B of $\bar{a} = (3)$ in $\mathbb{R}$ and unique continous funⁿ $g : B \to \mathbb{R}^2$ of class c^1 such that,

$$g(3) = (-1,2)$$

We know that,

$$Dg(\bar{a}) = -\frac{\partial f}{\partial \bar{x}}(\bar{a},\bar{b}) \left[\frac{\partial f}{\partial \bar{y}}(\bar{a},\bar{b})\right]^{-1}$$

$$= \left[\frac{\partial f}{\partial \bar{y}}(\bar{a},\bar{b})\right]^{-1} \frac{\partial f}{\partial \bar{x}}(\bar{a},\bar{b})$$

$$= -1 \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 1-1 \\ 1+2 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{6}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$