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Metric Spaces

Analysis is mainly concerned with processes pertaining to limits. In this chapter we shall study these processes in a general setting. We recall that in the theory of functions of real variables the notion of distance plays a vital role in formulating the definition of convergence, continuity and differentiability. Metric spaces are sets in which there is defined a notion of '*distance between pair of points*' and they provide the general setting in which we study convergence and continuity. The concept of metric spaces was formulated in 1906 by M. Fréchet.

The first section of this chapter is devoted mainly to basic definitions and important examples of metric spaces. We shall study the concepts of open sets, closed sets, convergence, continuity, compactness and connectedness in the later sections. We shall also prove in this chapter a simple result about complete metric spaces, Banach's Fixed point theorem, that has interesting and important applications in classical analysis.

1. DEFINITIONS AND EXAMPLES

Definition. Let X be a non-empty set. A *metric* on X is a real-valued function $d: X \times X \to \mathbf{R}$ which satisfies the following conditions:

(1) $d(x, y) \geq 0, \forall\, x, y \in X$,

(2) $d(x, y) = 0$ if and only if $x = y$, $\forall\, x, y \in X$,

(3) $d(x, y) = d(y, x)$, $\forall\, x, y \in X$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$ (triangle inequality).

A metric d is also called a *distance function*, and the non-negative real number $d(x, y)$ is to be thought of as the distance between x and y.

A *metric space* is a non-empty set X equipped with a metric d on X and is denoted by the pair (X, d) or simply X. Different metrics can be defined on a single non-empty set and this gives rise to distinct metric spaces.

ILLUSTRATIONS

1. The function $d : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by

$$d(x, y) = |x - y|, \ \forall\, x, y \in \mathbf{R}$$

is a metric on the set $\mathbf{R}$ of all real numbers, since for $x, y, z \in \mathbf{R}$, we have

$$d(x, y) = |x - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

The number $d(x, y)$ is, of course, the usual '*distance*' between the points x, y on the real line. Therefore d is sometimes referred to as the *usual metric* on **R**.

2. The function d defined by

$$d(z_1, z_2) = |z_1 - z_2|, \quad \forall\, z_1, z_2 \in \mathbf{C}$$

is a metric on the set C of all complex numbers. To prove the triangle inequality it is sufficient to prove that

$$|z_1 + z_2| \le |z_1| + |z_2|, \quad \forall\, z_1, z_2 \in \mathbf{C}$$

This follows from the following:

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \\ &= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= z_1\bar{z}_1 + 2\,Re\,(z_1\bar{z}_2) + z_2\bar{z}_2 \\ &\le |z_1|^2 + 2|z_1|\,|z_2| + |z_2|^2 = (|z_1| + |z_2|)^2. \end{aligned}$$

3. Let X be an arbitrary non-empty set. The function d defined by

$$d(x, y) = \begin{cases} 1, & \text{if } x \ne y \\ 0, & \text{if } x = y \end{cases} \quad \forall\, x, y \in X$$

is a metric on X and is called the *discrete* (*trivial*) *metric* on X, and (X, d) is called the *discrete metric space* or the *trivial metric space*.

4. The set l_∞ of all bounded sequences $\{x_n\}$ of real numbers with the function d defined by

$$d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in N\}, \forall\, \{x_n\}, \{y_n\} \in l_\infty$$

is a metric on l_∞.

For the triangle inequality, we have $\forall\, \{x_n\}, \{y_n\}, \{z_n\} \in l_\infty$

$$|x_n - y_n| = |x_n - z_n + z_n - y_n| \le |x_n - z_n| + |z_n - y_n|,$$

$$\therefore \quad \sup_n |x_n - y_n| \le \sup_n |x_n - z_n| + \sup_n |z_n - y_n|$$

i.e., $d(\{x_n\}, \{y_n\}) \le d(\{x_n\}, \{z_n\}) + d(\{z_n\}, \{y_n\})$.

5. The set $C[0, 1]$ consisting of all real-valued continuous functions defined on $[0, 1]$ with the function d given by

$$d(f, g) = \int_0^1 |f(x) - g(x)|\,dx, \quad \forall\, f, g \in C[0, 1]$$

is a metric space.

6. The set $C\,[0, 1]$ with another metric d defined by

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|, \quad \forall\, f, g \in C[0, 1]$$

is a metric space. The metric d is called the *Tehebyshev metric* or *sup metric*.

7. The set **C** of complex numbers with the metric d defined by

$$d(x, y) = \begin{cases} |x| + |y|, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

is a metric space.

In the following examples the conditions (1), (2) and (3) of definition of the metric are easy to verify and are left to the reader. We shall verify only the triangle inequality.

Example 1. Show that the set $\mathbf{R}^n$ of all ordered n-tuples with the function d defined by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}, \forall\, x = (x_1, x_2 \ldots, x_n),\ y = (y_1, y_2 \ldots, y_n) \in \mathbf{R}^n$$

is a metric space (d is called the *Euclidean metric* on $\mathbf{R}^n$).

■ To prove the triangle inequality we shall use the following *Cauchy-Schwarz inequality*:

$$\left| \sum_{k=1}^{n} a_k\, b_k \right| \leq \sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}$$

where $a = (a_1, a_2, \ldots, a_n)$, and $b = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n$

If $b_k = 0$, for $1 \leq k \leq n$, then there is nothing to prove.

Assume that $b_k \neq 0$ for some k, $1 \leq k \leq n$, then $\sum_{k=1}^{n} b_k^2 > 0$.

If x is any real number, then we have

$$\sum_{k=1}^{n} (a_k - x b_k)^2 \geq 0$$

i.e.,

$$\sum_{k=1}^{n} a_k^2 - 2x \sum_{k=1}^{n} a_k\, b_k + x^2 \sum_{k=1}^{n} b_k^2 \geq 0$$

This is true for all $x \in R$, and $\sum_{k=1}^{n} b_k^2 > 0$, therefore the discriminant of the quadratic in x is non-positive and hence

$$\left(\sum_{k=1}^{n} a_k\, b_k \right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$$

i.e.,

$$\left| \sum_{k=1}^{n} a_k\, b_k \right| \leq \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \left(\sum_{k=1}^{n} b_k^2 \right)^{1/2}.$$

For the triangle inequality, consider

$$\left[\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2}\right]^2$$

$$= \sum_{i=1}^{n}(x_i - z_i)^2 + \sum_{i=1}^{n}(z_i - y_i)^2 + 2\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2}$$

$$\geq \sum_{i=1}^{n}(x_i - z_i)^2 + \sum_{i=1}^{n}(z_i - y_i)^2 + 2\sum_{i=1}^{n}(x_i - z_i)(z_i - y_i)$$

(Using Cauchy-Schwarz inequality)

$$= \sum_{i=1}^{n}[(x_i - z_i) + (z_i - y_i)]^2 = \sum_{i=1}^{n}(x_i - y_i)^2$$

$$\therefore \quad \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2} \geq \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$$

i.e., $\quad d(x, y) \leq d(x, z) + d(z, y), \quad \forall\, x, y, z \in \mathbf{R}^n.$

We now generalise $\mathbf{R}^n$ to '*infinite-tuples*' which are sequences, and generalise the above Euclidean metric to the function

$$d(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty}(x_n - y_n)^2\right)^{1/2}$$

d is well defined if the series $\sum_{n=1}^{\infty}(x_n - y_n)^2$ converges, and so we must restrict ourselves to those sequences $\{x_n\}$ for which the series $\sum_{n=1}^{\infty} x_n^2$ converges.

Let l_2 be the set of all real sequences $\{x_n\}$, for which the series $\sum_{n=1}^{\infty} x_n^2$ converges, *i.e.*, $\sum_{n=1}^{\infty} x_n^2 < \infty$.

Example 2. Show that the function d defined by

$$d(\{x_n\}, \{y_n\}) = [\sum_{n=1}^{\infty}(x_n - y_n)^2]^{1/2}, \{x_n\}, \{y_n\} \in l_2$$

is a metric on l_2. The metric space (l_2, d) is known as a *Hilbert space*.

- First we have to show that d is well defined. For this, let $\{x_n\}, \{y_n\} \in l_2$, then by the Cauchy-Schwarz inequality

$$\sum_{k=1}^{n}|x_k||y_k| \leq \sqrt{\left(\sum_{k=1}^{n}|x_k|^2\right)\left(\sum_{k=1}^{n}|y_k|^2\right)}, \quad \forall\, n \in \mathbf{N}$$

$$\sum_{k=1}^{n} |x_k y_k| \le \sqrt{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)} \le \sqrt{\left(\sum_{k=1}^{\infty} x_k^2\right)\left(\sum_{k=1}^{\infty} y_k^2\right)} < \infty$$

This implies $\sum_{k=1}^{\infty} x_k y_k$ converges absolutely and hence converges.

Thus, the series $\sum_{k=1}^{\infty} (x_k - y_k)^2$ being the sum of the three convergent series $\sum_{n=1}^{\infty} x_n^2$, $-2\sum_{n=1}^{\infty} x_n y_n$

and $\sum_{k=1}^{\infty} y_k^2$ is also convergent.

Now from the Cauchy-Schwarz inequality we have by taking limits as $n \to \infty$

$$\left|\sum_{k=1}^{\infty} x_k y_k\right| \le \sqrt{\left(\sum_{k=1}^{\infty} x_k^2\right)\left(\sum_{k=1}^{\infty} y_k^2\right)} \quad \ldots(1)$$

To prove triangle inequality, let $(a_n), (b_n), (c_n) \in l_2$

Put $x_n = a_n - b_n$, $y_n = b_n - c_n$, $\forall n \in \mathbf{N}$

Then the triangle inequality in l_2 takes the form:

$$\sqrt{\sum_{n=1}^{\infty} (x_n + y_n)^2} \le \sqrt{\sum_{n=1}^{\infty} x_n^2} + \sqrt{\sum_{n=1}^{\infty} y_n^2}$$

This follows from equation (1), since

$$\sum_{n=1}^{\infty} (x_n + y_n)^2 = \sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} x_n y_n + \sum_{n=1}^{\infty} y_n^2$$

$$\le \sum_{n=1}^{\infty} x_n^2 + 2\sqrt{(\sum x_n^2)(\sum y_n^2)} + \sum_{n=1}^{\infty} y_n^2$$

$$= \left(\left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}}\right)^2$$

...le 3. Show that the set $\mathbf{R}^n$ with d' defined by

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

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$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$$

For the triangle inequality, consider

$$\left[\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2}\right]^2$$

$$= \sum_{i=1}^{n}(x_i - z_i)^2 + \sum_{i=1}^{n}(z_i - y_i)^2 + 2\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2}\left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2}$$

$$\geq \sum_{i=1}^{n}(x_i - z_i)^2 + \sum_{i=1}^{n}(z_i - y_i)^2 + 2\sum_{i=1}^{n}(x_i - z_i)(z_i - y_i)$$

(Using Cauchy-Schwarz inequality)

$$= \sum_{i=1}^{n}[(x_i - z_i) + (z_i - y_i)]^2 = \sum_{i=1}^{n}(x_i - y_i)^2$$

$\therefore$
$$\left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(z_i - y_i)^2\right)^{1/2} \geq \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}$$

i.e.,
$$d(x, y) \leq d(x, z) + d(z, y), \quad \forall\, x, y, z \in \mathbf{R}^n.$$

We now generalise $\mathbf{R}^n$ to '*infinite-tuples*' which are sequences, and generalise the above Euclidean metric to the function

$$d(\{x_n\}, \{y_n\}) = \left(\sum_{n=1}^{\infty}(x_n - y_n)^2\right)^{1/2}$$

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$$\sum_{k=1}^{n}|x_k||y_k| \leq \sqrt{\left(\sum_{k=1}^{n}|x_k|^2\right)\left(\sum_{k=1}^{n}|y_k|^2\right)}, \quad \forall\, n \in \mathbf{N}$$

$$\therefore \quad \sum_{k=1}^{n} |x_k\, y_k| \le \sqrt{\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)} \le \sqrt{\left(\sum_{k=1}^{\infty} x_k^2\right)\left(\sum_{k=1}^{\infty} y_k^2\right)^2} < \infty$$

This implies $\sum_{n=1}^{\infty} x_n y_n$ converges absolutely and hence converges.

Thus, the series $\sum_{n=1}^{\infty} (x_n - y_n)^2$ being the sum of the three convergent series $\sum_{n=1}^{\infty} x_n^2$, $-2\sum_{n=1}^{\infty} x_n\, y_n$, and $\sum_{n=1}^{\infty} y_n^2$ is also convergent.

Now from the Cauchy-Schwarz inequality we have by taking limits as $n \to \infty$

$$\left|\sum_{k=1}^{\infty} x_k\, y_k\right| \le \sqrt{\left(\sum_{k=1}^{\infty} x_k^2\right)\left(\sum_{k=1}^{\infty} y_k^2\right)} \qquad ...(1)$$

To prove triangle inequality, let $\{a_n\}, \{b_n\}, \{c_n\} \in l_2$.

Put $x_n = a_n - b_n,\ y_n = b_n - c_n,\ \forall\, n \in \mathbf{N}$

Then the triangle inequality in l_2 takes the form:

$$\sqrt{\sum_{n=1}^{\infty} (x_n + y_n)^2} \le \sqrt{\sum_{n=1}^{\infty} x_n^2} + \sqrt{\sum_{n=1}^{\infty} y_n^2}$$

This follows from equation (1), since

$$\sum_{n=1}^{\infty} (x_n + y_n)^2 = \sum_{n=1}^{\infty} x_n^2 + 2\sum_{n=1}^{\infty} x_n\, y_n + \sum_{n=1}^{\infty} y_n^2$$

$$\le \sum_{n=1}^{\infty} x_n^2 + 2\sqrt{(\Sigma\, x_n^2)\,(\Sigma\, y_n^2)} + \sum_{n=1}^{\infty} y_n^2$$

$$= \left(\left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty} y_n^2\right)^{\frac{1}{2}}\right)^2.$$

Example 3. Show that the set $\mathbf{R}^n$ with d' defined by

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

is a metric space (d' is called the *rectangular metric* on $\mathbf{R}^n$).

■ For the triangle inequality, consider

$$d'(x, y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$$

$$\leq \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i|$$

$$= d'(x, z) + d'(z, y), \quad \forall\, x, y, z \in \mathbf{R}^n$$

Since the metrics d and d' of examples (1) and (3) are different functions, therefore we get different metric spaces $(\mathbf{R}^n, d)$, and $(\mathbf{R}^n, d')$ with the same set $X = \mathbf{R}^n$.

Note that these metrics satisfy the inequality:

$$d(x, y) \leq d'(x, y) \leq \sqrt{n}\, d(x, y).$$

Example 4. Prove that the set $C[a, b]$ of all real-valued functions continuous on the interval $[a, b]$ with the function d defined by

$$d(f, g) = \left(\int_a^b (f(x) - g(x))^2 \, dx \right)^{1/2}$$

is a metric space.

To establish triangle inequality we need the following:

- Consider the function, for $t \in [a, b]$

$$\phi(t) = \int_a^b (t\, f(x) + g(x))^2 \, dx$$

$$= t^2 \int_a^b f^2(x)\, dx + 2t \int_a^b f(x)\, g(x)\, dx + \int_a^b g^2(x)\, dx$$

Since $\phi(t) \geq 0$, $\forall\, t \in [a, b]$, therefore the discriminant of the quadratic in t should be non-positive, and so

$$\left(\int_a^b f(x)\, g(x)\, dx \right)^2 \leq \int_a^b f^2(x)\, dx \int_a^b g^2(x)\, dx$$

i.e.,

$$\int_a^b f(x)\, g(x)\, dx \leq \left(\int_a^b f^2(x)\, dx \right)^{1/2} \left(\int_a^b g^2(x)\, dx \right)^{1/2} \qquad \ldots(1)$$

Now consider

$$\left[\left(\int_a^b (f(x) - h(x))^2 \, dx \right)^{1/2} + \left(\int_a^b (h(x) - g(x))^2 \right)^{1/2} \right]^2$$

$$= \int_a^b (f(x) - h(x))^2 \, dx + \int_a^b (h(x) - g(x))^2 \, dx$$

$$+ 2 \left(\int_a^b (f(x) - h(x))^2 \, dx \right)^{1/2} \left(\int_a^b (h(x) - g(x))^2 \, dx \right)^{1/2}$$

$$\geq \int_a^b (f(x) - h(x))^2 \, dx + \int_a^b (h(x) - g(x))^2 \, dx$$

$$+ 2 \int_a^b (f(x) - h(x))\, (h(x) - g(x))\, dx \qquad \text{[using (1)]}$$

$$= \int_a^b (f(x) - h(x) + h(x) - g(x))^2 \, dx = \int_a^b (f(x) - g(x))^2 \, dx$$

Hence

$$\left(\int_a^b (f(x) - g(x))^2 \, dx\right)^{1/2} \leq \left(\int_a^b (f(x) - h(x))^2 \, dx\right)^{1/2} + \left(\int_a^b (h(x) - g(x))^2 \, dx\right)^{1/2}.$$

Example 5. Let (X, d) be any metric space. Show that the function d_1 defined by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \ \forall \, x, y \in X$$

is a metric on X.

- For the triangle inequality we proceed as follows:

  Using the triangle inequality for the metric d, we have for all $x, y, z \in X$

$$d(x, y) \leq d(x, z) + d(z, y)$$

or

$$1 + d(x, y) \leq 1 + d(x, z) + d(z, y)$$

or

$$1 - \frac{1}{1 + d(x, y)} \leq 1 - \frac{1}{1 + d(x, z) + d(z, y)}$$

or

$$\frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)}$$

$$\leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)}$$

i.e.,

$$d_1(x, y) \leq d_1(x, z) + d_1(z, y).$$

Example 6. ***Fréchet Space.*** Let X be the set of all sequences of complex numbers. We define the function d by

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{(1 + |x_n - y_n|)}, \ \forall \, x = \{x_n\}, \ y = \{y_n\} \in X$$

The function d is well defined, since the nth term of the above series is less than $\frac{1}{2^n}$, therefore it is convergent. To prove triangle inequality we first establish the following inequality.

Let $0 \leq \alpha \leq \beta$, then

$$\alpha + \alpha\beta \leq \beta + \alpha\beta$$

Dividing both sides by $(1 + \alpha)(1 + \beta)$, we obtain

$$\frac{\alpha}{1+\alpha} \le \frac{\beta}{1+\beta} \qquad ...(1)$$

Now for any $x = \{x_n\}$, $y = \{y_n\}$ and $z = \{z_n\}$ in X, we have

$$0 \le |x_n - y_n| \le |x_n - z_n| + |z_n - y_n|$$

So from (1) it follows that

$$\frac{|x_n - y_n|}{1+|x_n - y_n|} \le \frac{|x_n - z_n| + |z_n - y_n|}{1 + |x_n - z_n| + |z_n - y_n|}$$

$$\le \frac{|x_n - z_n|}{1+|x_n - z_n|} + \frac{|z_n - y_n|}{1+|z_n - y_n|}$$

Multiplying both sides by 2^{-n} and summing w.r.t. n, we get

$$d(x, y) \le d(x, z) + d(z, y)$$

Hence, (X, d) is a metric space.

The very definition of a metric presents the concept of the distance from one point to another. We now define the distance from a point to a set and the distance between two non-empty subsets of a metric space.

For any two non-empty subsets A and B of a metric space (X, d), the *distance* between them denoted by $d(A, B)$ is defined as:

$$d(A, B) = \inf \{d(a, b): a \in A, b \in B\}.$$

If x is a point of X, then the *distance from x to A* denoted by $d(x, A)$ is defined as:

$$d(x, A) = \inf \{d(x, a): a \notin A\}.$$

If $A = \{x \in \mathbf{R}: 0 < x \le 1\}$ and d is the usual metric, then

$$d(0, A) = 0, \text{ although } 0 \notin A.$$

Similarly $d(A, B) = 0$ does not imply that A and B have common elements, as can be seen by the following example.

Example 7. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and d is the usual metric defined on X.

Let $A = \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}, \ldots\right\}$, and

$$B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2n}, \ldots\right\}$$

Then $d(A, B) = 0$, although $A \cap B = \phi$.

Diameter of a Non-Empty Set

Definition. The *diameter* of any non-empty subset $A \subseteq X$ denoted by $d(A)$ is defined as:

$$d(A) = \sup \{d(a, b): a, b \in A\}.$$

If $d(A) < \infty$, then the diameter of A is said to be finite otherwise infinite.

By convention $d(\phi) = -\infty$.

Definition. A metric d on a non-empty set X is said to be *bounded* if there exists a real number $k > 0$ such that

$$d(x, y) \leq k, \quad \forall\, x, y \in X$$

i.e.,

$$d(X) \leq k,$$

(X, d) is then called a *bounded metric space,* otherwise unbounded.

Example 8. Let $\mathbf{R}_\infty$ be the *extended set* of real numbers (*i.e.*, the set of real numbers including $-\infty$ and $+\infty$).

The function d defined by

$$d(x, y) = |f(x) - f(y)|, \quad \forall\, x, y \in \mathbf{R}_\infty$$

where $f(x)$ is given by

$$f(x) = \begin{cases} \dfrac{x}{1+|x|}, & \text{when } -\infty < x < \infty \\ 1, & \text{when } x = \infty \\ -1, & \text{when } x = -\infty \end{cases}$$

Show that $(\mathbf{R}_\infty, d)$ is a bounded metric space.

- For the triangle inequality

$$d(x, y) = \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right|$$

$$= \left| \frac{x}{1+|x|} - \frac{z}{1+|z|} + \frac{z}{1+|z|} - \frac{y}{1+|y|} \right|$$

$$\leq \left| \frac{x}{1+|x|} - \frac{z}{1+|z|} \right| + \left| \frac{z}{1+|z|} - \frac{y}{1+|y|} \right|$$

$$= d(x, z) + d(z, y), \quad \forall\, x, y, z \in \mathbf{R}$$

If $x = \infty$, $y = -\infty$, then

$$d(x, y) = |1 - (-1)| \le \left|1 - \frac{z}{1+|z|}\right| + \left|\frac{z}{1+|z|} - (-1)\right|$$

$$= d(x, z) + d(z, y)$$

Similarly when $x = -\infty$, $y = +\infty$, the triangle inequality holds.

Hence $(\mathbf{R}_\infty, d)$ is a metric space.

Moverover, if x and y are two elements of $\mathbf{R}_\infty$, then

$$-1 \le f(x) \le 1, \text{ and } -1 \le f(y) \le 1$$

$\therefore$
$$d(x, y) = |f(x) - f(y)| \le 2, \quad \forall\, x, y \in \mathbf{R}_\infty$$

Hence $(R_\infty\ d)$ is a bounded metric space.

Remark: It is to be noted that even when a metric space is unbounded, we can define another metric in many ways, so that the resulting metric space is bounded. As from example 5 if (X, d) is any metric space, then (X, d_1) is a bounded metric space, where

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \; x, y \in X$$

Since
$$0 \le d_1(x, y) < 1, \quad \forall\, x, y \in X.$$

EXERCISE

1. Let X be a non-empty set and a function d from $X \times X$ into $\mathbf{R}$ satisfies :
   (*i*) $d(x, y) = 0$, if and only if $x = y$, and
   (*ii*) $d(x, y) \le d(x, z) + d(y, z)$. $\forall\, x, y, z \in X$.

   Prove that (X, d) is a metric space.

   [*Hint*: Take $y = x$, in (*ii*), $d(x, z) \ge 0$. Take $x = z$ in (*ii*), $d(z, y) \le d(y, z)$ and interchange the role of y and z.]

2. Show that the conditions
   (*i*) $d(x, y) = 0$, if and only if $x = y$, and
   (*ii*) $d(x, y) \le d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$

   are not sufficient to ensure that the function $d : X \times X \to \mathbf{R}$ is a metric on a non-empty set X.

3. Give an example of a function $d : X \times X \to R$ defined on a non-empty set X satisfying the following three conditions but not a metric on X.
   (*i*) $d(x, y) \ge 0$, and $x = y \Rightarrow d(x, y) = 0$
   (*ii*) $d(x, y) = d(y, x)$, $\forall\, x, y \in X$
   (*iii*) $d(x, y) \le d(x, z) + d(z, y)$, $\forall\, x, y, z \in X$

   [*Hint*: Take $X = \mathbf{R}^2$, $d(x, y) = |x_1 - y_1|$, where $x = (x_1, x_2)$, $y = (y_1, y_2)$. Such a metric is called a *Pseudometric*.]

4. Prove that if (X, d) is a metric space, then

$$|d(x, z) - d(y, z)| \le d(x, y), \quad \forall\, x, y, z \in X.$$

[*Hint*: Apply triangle inequality to $d(x, z)$ and $d(y, z)$ separately.]

5. Prove that, if (X, d) is a metric space, and $x_1, x_2, x_3, \ldots x_n \in X$, then

$$d(x_1, x_n) \le d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + \ldots + d(x_{n-1}, x_n).$$

6. Functions d_1 and d_2 from $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are defined by

$$d_1(x, y) = \exp\{|x - y|\}, \text{ and } d_2(x, y) = \max\{x - y, 0\}.$$

Is either d_1 or d_2 a metric on $\mathbf{R}$?

7. Prove that the function $d^*: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ defined by

$$d^*(x, y) = \max_{1 \le i \le n} |x_i - y_i|, \quad \forall\, x = (x_1, x_2 \ldots x_n), y = (y_1, y_2 \ldots y_n) \in \mathbf{R}^n$$

is a metric on $\mathbf{R}^n$. Also prove that

$$d^*(x, y) \le d\,(x, y) \le \sqrt{n}\, d^*(x, v), \quad \forall\, x. y \in \mathbf{R}^n$$

where d is the *Euclidean metric* on $\mathbf{R}^n$.

8. Consider the set l_p of all sequences $\{x_n\}$ of complex numbers satisfying the convergence condition $\sum_{n=1}^{\infty} |x_n|^p < \infty$, for some fixed $p \ge 1$, where the distance between points is defined by

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}, \quad \forall\, x = \{x_n\}, y = \{y_n\} \in l_p$$

Show that (l_p, d) is a metric space.

[*Hint*: $d(x, y)$ is well defined, can be seen by using, *Minkowski's inequality*:

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{1/p} \le \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{1/p}, \quad \forall\, \{x_n\}, \{y_n\} \in l_p]$$

9. Let H_∞ denote the set of all real sequences $\{x_n\}$ such that $|x_n| \le 1, \forall\, n \in N$, then prove that the function d defined by

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}, \{x_n\}, \{y_n\} \in H_\infty$$

is a metric on H_∞ [(H_∞, d) is called the *Hilbert-cube*].

10. Let $(X, d_1), (X, d_2)$ be two metric spaces and k is a positive real number. Which of the following are metric spaces:

$$(X, kd_1), (X, \sqrt{d_1}), (X, d_1^2), (X, \min(1, d_1)), (X, d_1 d_2),$$
$$(X, d_1 + d_2), (X, \max(d_1, d_2)), (X, \min(d_1, d_2))?$$

11. Prove that the function $d: \mathbf{C} \times \mathbf{C} \to \mathbf{R}$ defined by

$$d(x, y) = \frac{2|x - y|}{\sqrt{1 + |x^2|}\sqrt{1 + |y|^2}}$$

is a metric on the set of all complex numbers.

12. Let $R[0, 1]$ be the set of all Riemann integrable functions defined on $[0, 1]$ and let d be defined by

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx, \ \forall\, f, g \in R[0, 1].$$

Show that d is not a metric on $R[0, 1]$.

13. If d is a metric on X, then show that min $\{d(x, y), 1\}$ is a bounded metric on X.

14. If $d : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$ is defined by, for $m, n \in \mathbf{N}$,

$$d(m, n) = 0, \text{ if } m = n, \text{ and for } m \neq n, d(m, n) = 1/5^k,$$

where $m - n = 5^k r$, are r is not a multiple of 3. Prove that (N, d) is a bounded metric space.

15. Let $C'[a, b]$ denote the set of all real-valued functions defined on the closed interval $[a, b]$ having continuous first order derivatives on $[a, b]$. Prove that the function d defined on $C'[a, b]$ by

$$d(f, g) = \sup_{a \le x \le b} |f(x) - g(x)| + \sup_{a \le x \le b} |f'(x) - g'(x)|, \ \forall\, f, g \in C'[a, b]$$

is a metric on $C'[a, b]$.

16. Let A and B be any two non-empty subsets of a metric space X.
Prove that
    (i) If $A \subseteq B$, then $d(A) \le d(B)$.
    (ii) $d(A \cup B) \le d(A) + d(B) + d(A, B)$
    (iii) If $A \cap B \neq \phi$, then $d(A \cup B) \le d(A) + d(B)$.

2. OPEN AND CLOSED SETS

In this section we shall study the concept of neighbourhoods, open sets and closed sets in a metric space (X, d) and develop some of the important results relating to these concepts. We begin by defining open spheres and closed spheres.

2.1 Open and Closed Spheres

Let (X, d) be any metric space, and $a \in X$. Then for any $r > 0$, the set

$$S_r(a) = \{x \in X : d(x, a) < r\}$$

is called an *open sphere* (or *open ball*) of radius r centered at a.

The set

$$S_r[a] = \{x \in X : d(x, a) \le r\}$$

is called a *closed sphere* of radius r centered at a. It is clear that

$$S_r(a) \subset S_r[a]$$

for every $a \in X$, and for every $r > 0$.

ILLUSTRATIONS

1. In the metric space $(\mathbf{R}, d)$ of real numbers with the usual metric d, the open sphere $S_r(a)$ is the open interval $]a - r, a + r[$, and the closed sphere $S_r[a]$ is the closed interval $[a - r, a + r]$, where $a \in \mathbf{R}$, and $r > 0$.

2. In the discrete space (X, d), the open sphere $S_r(a)$ for $a \in X$ is given by

$$S_r(a) = \begin{cases} \{a\}, & \text{if } 0 < r \leq 1 \\ X, & \text{if } r > 1 \end{cases}$$

and the closed sphere $S_r[a]$ is given by

$$S_r[a] = \begin{cases} \{a\}, & \text{if } 0 < r < 1 \\ X, & \text{if } r \geq 1. \end{cases}$$

3. The open sphere in the complex plane is the inside of a circle with center at a and radius r.
4. The open sphere $S_r(f_0)$ in the metric space $C[a, b]$ of all real-valued continuous functions defined on $[a, b]$ is a strip of width $2r$ centered on the graph of f_0.

Example 9. Let (X, d) be a metric space and $S_r(x)$ the open sphere with centre x and radius r. Let A be a subset of X with diameter less than r, which intersects $S_r(x)$, then $A \subseteq S_{2r}(x)$.

■ Since $A \cap S_r(x) \neq \phi$, let $a \in A \cap S_r(x)$

then $$d(a, x) < r, \; a \in A, \; r > 0.$$

Let y be an arbitrary element of A then by triangle inequality

$$d(y, x) \leq d(y, a) + d(a, x) \qquad [\because \; y, a \in A \text{ and } d(A) < r]$$
$$< r + r = 2r$$

This implies $y \in S_{2r}(x)$.

2.2 Neighbourhood of a Point

Let (X, d) be a metric space and $a \in X$. A subset N_a of X is called a *neighbourhood* of a point $a \in X$, if there exists an open shpere $S_r(a)$ centered at a and contained in N_a; *i.e.*, $S_r(a) \subseteq N_a$, for some $r > 0$.

Example 10. Every open sphere is a neighbourhood of each of its points.

■ Let $S_r(a)$ be an open sphere, and $x \in S_r(a)$. If $x = a$, then $a \in N_a \subset S_r(a)$. Therefore suppose that $x \neq a$. In order to show that $S_r(a)$ is a neighbourhood of x, we must show that there exists $r_1 > 0$ such that

$$S_{r_1}(x) \subseteq S_r(a)$$

Now $x \in S_r(a)$ implies $d(x, a) < r$. Take $r_1 = r - d(x, a)$.

Then $y \in S_{r_1}(x)$ implies by using triangle inequality

$$d(y, a) \leq d(y, x) + d(x, a) < r_1 + d(x, a) = r$$

i.e., $$y \in S_r(a)$$

Hence $$S_{r_1}(x) \subseteq S_r(a).$$

2.3 Open Set

Definition. A subset G of a metric space (X, d) is said to be **open** in X with respect to the metric d, if G is a neighbourhood of each of its points. In other words, if for each $a \in G$, there is an $r > 0$ such that

$$S_r(a) \subseteq G.$$

ILLUSTRATIONS

1. The empty set ϕ and the entire space X with any metric are open sets.
2. Every open sphere is an open set.
3. Let A be the annulus consisting of the complex numbers z such that $1 < |z| < 2$ with the usual metric d, then A is open.
4. The subset $S = \{(x, y): x^2 + y^2 < 1,\ x, y \in \mathbf{R}\}$ of $\mathbf{R}^2$ with the Euclidean metric is an open set.
5. The subset $A = \{(x, y): y^2 < x;\ x, y \in \mathbf{R}\}$ of $\mathbf{R}^2$ with the Euclidean metric is an open set.
6. Let $A = \{f \in C[a, b] : \inf_{x \in [a, b]} f(x) > 0\}$. Then A is open with respect to the sup metric in $C[a, b]$.

Example 11. Show that every set in a discrete space (X, d) is open.

- Let G be any non-empty subset of the discrete space (X, d) and x be any point of G. Then the open sphere $S_r(x)$ with $r \leq 1$ is the singleton set $\{x\}$ which is contained in G *i.e.*, each point of G is the centre of some open sphere contained in G. In particular, each singleton set is open.

Example 12. Show that on the real line with the usual metric the singleton set $\{x\}$ is not open.

- For the metric space $(\mathbf{R}, d)$ each open sphere $S_r(x)$ is the bounded open interval $]x - r, x + r[$ and for no value of r (how so-ever small it may be) this sphere is contained in $\{x\}$. Hence $\{x\}$ is not open in $(\mathbf{R}, d)$.

Remark: It is to be observed that a given subset of a metric space may be open with respect to one metric but may not be open with respect to another metric as can be seen by the following example.

Example 13. Let $\mathbf{R}$ be the set of reals, d the usual metric and d' the discrete metric on the same set $\mathbf{R}$. Then show that every singleton set $\{x\},\ x \in \mathbf{R}$ is open in $(\mathbf{R}, d')$ but not so in $(\mathbf{R}, d)$.

- Every singleton set $\{x\},\ x \in \mathbf{R}$ is open in $(\mathbf{R}, d')$ being an open sphere $S_r(x), r < 1$, *i.e.*, bounded open interval $]x - r, x + r[$ for $r < 1$ contains only one point of the set, x of the space. But $\{x\}$ is not open in $(\mathbf{R}, d)$. Because every open sphere $S_r(x), r < 1$, is the bounded open interval $]x - r, x + r[\not\subset \{x\}$.

Example 14. Show that the subset $A = [0, 1[$ of the metric space (X, d), where $X = [0, 2[$, and d is the usual metric, is an open set.

- Let $x \in A = [0, 1[$.

  If $x = 0$, then $S_{1/2}(0) = [0, \frac{1}{2}[\subseteq A$. If $x \neq 0$, choose $r = \min\{x, 1 - x\}$. Clearly $r > 0$ and

$$S_r(x) =]x - r, x + r[\subseteq [0, 1[= A.$$

Hence, A is open in X.

Definition. Two metrics d and d' on the same set X are said to be *equivalent*, if every set open in (X, d) is open in (X, d'), and vice versa.

Example 15. Let (X, d) be any metric space and let

$$d(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad \forall\, x, y \in X$$

Show that d and d' are equivalent.

- We have already shown that d' is a metric on X (Example 5).

Let G be any open subset of (X, d). Then for each $x \in G$, $\exists$ an open sphere,

$$S_r(x) = \{y \in X : d(y, x) < r\} \subseteq G$$

Let $r_1 = \dfrac{r}{1+r}$, then $r_1 < r$.

Now,

$$\{y \in X : d'(y, x) < r_1\} \subseteq \{y \in X : d(y, x) < r\}$$

$$d'(y, x) < r_1 \Rightarrow \frac{d(y, x)}{1 + d(y, x)} < \frac{r}{1+r}.$$

Thus every point of G is the centre of some open sphere in (X, d') contained in G.

Consequently, every set open in (X, d) is open in (X, d'). Again, let G be any open set in (X, d') so, $\exists$ an open sphere

$$S_r(x) = \{y \in X : d'(y, x) < r\} \subseteq G$$

Since $d'(x, y) < 1$, we may assume $r < 1$.

Let $r' = \dfrac{r}{1-r}$.

Now each point of G is the centre of an open sphere contained in G implying that G is open in (X, d).

Ex. Let (X, d) be a metric space, and let $d'(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Then show that d and d' are equivalent.

Theorem 1. *In any metric space (X, d),*

(*i*) *the union of an arbitrary family of open sets is open,*

(*ii*) *the intersection of a finite number of open sets is open.*

(*i*) Let $\{G_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of open sets in X, where Λ is any non-empty index set.

Let $G = \bigcup_{\alpha \in \Lambda} G_\alpha$.

If $G = \phi$ then G is open. Suppose $G \neq \phi$. Let x be any element of G. Since $G = \bigcup_{\alpha \in \Lambda} G_\alpha$, therefore there is an $\alpha_0 \in \Lambda$ such that $x \in G_{\alpha_0}$.

The set G_{α_0} being open implies that there exists $r > 0$, such that

$$S_r(x) \subseteq G_{\alpha_0},$$

and so

$$S_r(x) \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha \qquad (\because G_{\alpha_0} \subseteq \bigcup_{\alpha \in \Lambda} G_\alpha)$$

Hence, G is an open set.

(ii) Let $G_1, G_2, \ldots, G_n$ be any finite number of open sets in X, and

$$G = \bigcap_{i=1}^{n} G_i.$$

If $G = \phi$, then G is open. Suppose $G \neq \phi$.

Let $x \in G = \bigcap_{i=1}^{n} G_i$, then $x \in G_i$, $\forall\, i = 1, 2, \ldots n$

Since each G_i, $i = 1, 2 \ldots n$, is open $\exists\, r_i > 0$, such that

$$S_{r_i}(x) \subseteq G_i, \quad \forall\, i = 1, 2, \ldots n$$

Let $r = \min\{r_1, r_2, \ldots, r_n\}$.

Then $r > 0$ and

$$S_r(x) \subseteq S_{r_i}(x) \subseteq G_i \quad \forall\, i = 1, 2, \ldots n$$

This implies

$$S_r(x) \subseteq \bigcap_{i=1}^{n} G_i = G$$

Hence, G is open in X.

Remarks:

1. The part (ii) of the above theorem need not be true for the intersection of an arbitrary family of open sets. For example, consider a family of open sets $G_n =]-1/n, 1/n[$, $n \in \mathbf{N}$ in $(\mathbf{R}, d)$, where d is the usual metric. The intersection

   $\bigcap_{n=1}^{\infty} G_n = \{0\}$, which is not open in $\mathbf{R}$.

   Definition. If X is any set, and $\mathbf{F}$ is a collection of subsets of X satisfying,

   (i) $\phi, X \in \mathbf{F}$

   (ii) The union of an arbitrary family of sets in $\mathbf{F}$ is a member of $\mathbf{F}$.

   (iii) The intersection of a finite number of sets in $\mathbf{F}$ is a number of $\mathbf{F}$, then $\mathbf{F}$ is called a *topology* for X.

   For example, the collection of all open subsets of a metric space X is a topology for X.

2. We note that, just as every open set on the real line is a union of open intervals, similarly every open set in the metric space (X, d) can be written as a union of open spheres. If $G \subseteq X$ is open, then for each $x \in G, \exists\, r_x > 0$ such that

$$S_{r_x}(x) \subseteq G.$$

Hence $G = \bigcup_{x \in G} S_{r_x}(x)$.

Conversely, the union of open spheres is always an open set in (X, d) (the first part of the above theorem). However, in the case of real line an explicit description of open sets can be given.

Lemma. *Let* $\mathbf{F}$ *be the family of open intervals in* $\mathbf{R}$, *no two of which are disjoint, then* $\bigcup_{\alpha \in \Lambda} I$ *is an open interval.*

Let $I_0 = \bigcup_{I \in F} I$, and let $a, b \in I_0$, and $a < c < b$.

Then $a \in I_1$ and $b \in I_2$, for some $I_1, I_2 \in \mathbf{F}$.

Let $I_1 =]a_1, b_1[$ and $I_2 =]a_2, b_2[$, then

$$a_1 < a < c < b < b_2 \quad \text{...(1)}$$

If $b_1 \le a_2$, then $I_1 \cap I_2 = \phi$, which is impossible, so $b_1 > a_2$. Then either $c < b_1$ or $c \ge b_1$. In the former case $c \in I_1$, and in the later $c \in I_2$ ($\because a_2 < b_1$) and consequently in either case $c \in I_0$. Thus I_0 is an interval. The interval I_0 must be an open interval because it is an open set (being union of open intervals).

Theorem 2. *Every non-empty open set on the real line is the union of a countable collection of pairwise disjoint open intervals.*

Let G be a non-empty open subset of **R**. For each $x \in G$, let I_x be the union of all the open intervals which contain x, and are contained in G (such intervals exist because G is open). By the above Lemma, each I_x is an open interval. Obviously I_x contains every open interval which contains x and is contained in G. Moreover

$$G = \bigcup_{x \in G} I_x$$

We shall show that any two members in the above union are either disjoint or identical. For this let $x, y \in G$, and suppose that

$$I_x \cap I_y \neq \phi$$

Then, by the above lemma, the set $I_x \cup I_y$ is an open interval which contains both x and y. Therefore by definition of I_x and I_y it follows that

$$I_x \cup I_y \subseteq I_x, \text{ and } I_x \cup I_y \subseteq I_y.$$

Consequently $I_x = I_y$.

Let **F** be the collection of all distinct sets of the form I_x with $x \in G$. This being a disjoint collection of open intervals whose union is in G.

Now it remains to show that **F** is countable. The set $Q \cap G$ of all rational numbers in G is countable.

We define a function f from $Q \cap G$ into **F** as follows:

Given $r \in Q \cap G$, let $f(r)$ be the unique set I_r in **F** that contains r (the set I_r is unique because the sets in **F** are pairwise disjoint).

f is obviously onto, since each open interval in **R** contains a rational number. Hence **F** is countable.

2.4 Limit Points

Definition. Let A be any subset of a metric space (X, d). A point 'a' of X is called an *adherent point* of A, if every open sphere centered at 'a' contains a point of A.

Adherent points are of two types:

(*i*) isolated points,

(*ii*) limit points.

An adherent point 'a' of a subset A of X is called an *isolated point* if every open sphere centered at 'a' contains no point of A other than a itself.

An adherent point 'a' of a subset A of X is said to be a *limit point* of A if every open sphere centered at 'a' contains at least one member of A other than a

i.e., $$S_r(a) \cap (A - \{a\}) \neq \phi, \quad \forall\, r > 0$$

The essential idea here is that points of A different from 'a' get arbitrarily close to a or 'pile up' at a.

The limit point is also known as a *cluster point*, a *condensation point* or an *accumulation point*. If 'a' is a limit point of A then every open sphere centered at a contains infinitely many elements of A, and conversely. The limit point of A may or may not be a member of A.

Derived Set. The set of all limit points of A is called the *derived set* of A and is denoted by A'.

ILLUSTRATIONS

1. Let **R** be the set of reals with the usual metric d. Let $A = [0, 1[$. Every point of A is a limit point of A. Further '1' is also a limit point of A which is not a member of A. Here $A' = [0, 1]$.
2. Let $A = \{1, \frac{1}{2}, \frac{1}{3} \ldots\}$ with the usual metric in **R**. '0' is the only limit point of A which is not a member of A, so that $A' = \{0\}$.
3. The derived set of every subset of a discrete space is empty.
4. Every real number is a limit point of the set of rationals.
5. The set of integers has no limit point.
6. A finite set has no limit point.
7. Let A be the annulus consisting of complex numbers z such that $1 < |z| < 2$, with the usual metric, then $A' = \{z : 1 \le |z| \le 2\}$.

2.5 Closed Sets

Definition. A subset F of a metric space (X, d) is said to be closed if F contains all its limit points.

ILLUSTRATIONS

1. The empty set ϕ and the whole space X are closed sets in every metric space (X, d).
2. On the real line with the usual metric the set **N** of natural numbers is closed.
3. The set **Q** of rational numbers is not closed.
4. Every closed interval on the real line is a closed set.
5. Every finite subset of a metric space is closed.
6. Let $A = \{f \in C[a, b] : f(a) = 1\}$. Then A is closed with respect to the sup metric in $C[a, b]$.
7. The set $\left\{\dfrac{1+i}{n} : n \in N\right\}$ is neither open nor closed with respect to the usual metric in the complex plane.

Theorem 3. *Let (X, d) be any metric space. A subset F of X is closed if and only if its complement in X is open.*

Suppose F is closed. If $F^c = X - F = \phi$, then it is open.

Assume that $X - F \neq \phi$. Let $x \in X - F$; then $x \notin F$. F being closed implies x is not a limit point of F and so there exists $r > 0$, such that

$$S_r(x) \cap F = \phi$$

i.e,

$$x \in S_r(x) \subseteq X - F$$

Hence F^c is open.

Conversely, suppose that $X - F$ is open. If $x \in X - F$, then there exists $r > 0$, such that

$$S_r(x) \subseteq X - F$$

i.e., $S_r(x) \cap F = \phi \Rightarrow x$ cannot be a limit point of F.

Since $x \in X - F$ is arbitrary, therefore F does not have any limit point outside it. Consequently F is closed.

From the above theorem we can easily see that every subset of a discrete space is closed. Since in a discrete space (X, d) every subset of it is open so if F is an arbitrary subset of X, then $X - F$ being a subset of X is open and therefore F is closed.

Example 16. Every closed sphere is a closed set.

■ Let $S_r[x]$ be any closed sphere in a metric space (X, d). If $X - S_r[x] = \phi$, then ϕ is open.

Assume $X - S_r[x] \neq \phi$. Let $y \in X - S_r[x]$. Then $y \notin S_r[x]$.

This implies $d(y, x) > r$. Let $r_1 = d(y, x) - r$.

The open sphere $S_{r_1}(y) \subseteq X - S_r[x]$, for if $z \in S_{r_1}(y)$, then $d(z, y) < r_1$, and so

$$d(z, y) < d(y, x) - r$$

i.e., $$r < d(y, x) - d(z, y) = d(x, z) \quad \text{(by triangle inequality)}$$

Thus $z \in S_{r_1}(y) \subseteq X - S_r[x]$

This implies $X - S_r[x]$ is open. Hence $S_r[x]$ is closed.

Theorem 4. *In any metric space* (X, d),

(*i*) *the intersection of an arbitrary family of closed sets is closed,*

(*ii*) *the union of a finite number of closed sets is closed.*

(Proofs follow from Theorems 6 and 7 of Chapter 2.)

[For alternative proof. *Hint* : (*i*) Let $\{F_\alpha : \alpha \in \Lambda\}$ be any family of closed sets, then $\bigcap_{\alpha \in \Lambda} F_\alpha$ is closed, $\because X - \bigcap_{\alpha \in \Lambda} F_\alpha = \bigcup_{\alpha \in \Lambda} (X - F_\alpha)$, Λ is any index set.]

Remark: The arbitrary union of closed sets in a metric space is not necessarily a closed set.
For example, consider the family $\{F_n : n \in \mathbb{N}\}$, where $F_n = [1/n, 1]$ is closed for each $n = 1, 2, \ldots,$

$$\bigcup_{n=1}^{\infty} F_n =]0, 1],$$

which is not closed.

2.6 Subspaces

Definition. Let (X, d) be a metric space. Let Y be a non-empty subset of X. Then the restriction map d_Y of the metric d to $Y \times Y$ is a metric for Y called the *induced metric* and the metric space (Y, d_Y) is called a *subspace* of (X, d).

The closed unit interval $[0, 1]$ and the set of all rational numbers are subspaces of $\mathbf{R}$; and the unit circle, the closed unit disc and the open unit disc are subspaces of the space $(\mathbf{C}, d)$ of complex numbers. In fact the real line itself is a subspace of the space complex numbers.

If $Y \subseteq X$, (X, d) is a metric space, and $y \in Y$, then we shall denote the open sphere centered at y with radius r in (Y, d_Y) by $S_r^Y(y)$.

i.e.,
$$S_r^Y(y) = \{x \in Y : d_Y(x, y) < r\}$$

It is easy to verify that
$$S_r^Y(y) = S_r(y) \cap Y.$$

From this it follows that a subset of Y which is open in X is also open in Y. However, the converse may not be true as can be seen by the following examples.

1. Take $Y = [0, 1]$, $X = \mathbf{R}$, d the usual metric, then $S_{\frac{1}{2}}^Y(0) = [0, \frac{1}{2}[$ is open in Y, but not in $\mathbf{R}$. Note that Y is not open in $\mathbf{R}$.
2. An open interval of the real line is not an open subset of the complex plane.

The following theorem gives a criterion for a subset to be open in a subspace.

Theorem 5. *Let (X, d) be a metric space and $Y \subseteq X$, then a subset A of Y is open in (Y, d_Y) if and only if there exists a set G open in (X, d) such that*
$$A = G \cap Y.$$

Assume that A is open in (Y, d_Y). Then for each $a \in A$ there exists $r_a > 0$ such that
$$S_{r_a}^Y(a) \subseteq A$$
so that
$$A = \bigcup_{a \in A} S_{r_a}^Y(a)$$

But since
$$S_{r_a}^Y(a) = S_{r_a}(a) \cap Y$$
$$\therefore \quad A = \bigcup_{a \in A} (S_{r_a}(a) \cap Y) = G \cap Y,$$
where
$$G = \bigcup_{a \in A} S_{r_a}(a) \text{ is open in } (X, d).$$

Conversely, suppose that there is a set G which is open in (X, d) with $G \cap Y = A$. Let $a \in A$, then $a \in G$, and so there exists $r > 0$, such that
$$S_r(a) \subseteq G$$

This implies
$$S_r^Y(a) = S_r(a) \cap Y \subseteq G \cap Y = A$$

i.e.,

$$S_r^Y(a) \subseteq A$$

Hence, A is open in (Y, d_Y).

From the above theorem, it follows that an open subset of Y is open in X if and only if Y itself is open in X.

A similar criterion for closed sets is the following:

Theorem 6. *Let (X, d) be a metric space and $Y \subseteq X$, then a subset A of Y is closed in (Y, d_Y) if and only if there exists a set F closed in (X, d) such that*

$$A = F \cap Y.$$

(Proof follows from the above theorem by taking complements.)

2.7 Closure of a Set

Definition. Let A be any subset of a metric space (X, d). The *closure of A* denoted by $\overline{A}$ is the set of all adherent points of A.

i.e., $\overline{A} = A \cup A'$

Symbolically $\overline{A} = \{x \in X : S_r(x) \cap A \neq \phi, \text{ for all } r > 0\}$.

Properties:

Let A and B be any two subsets of a metric space (X, d). Then

(1) $\overline{A}$ is a closed set.

(2) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

(3) $\overline{A}$ is the smallest closed superset of A.

(4) $A = \overline{A}$ if and only if A is closed.

(5) $\overline{A}$ is the intersection of all closed sets containing A.

(6) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(7) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

(1) In order to show that $\overline{A}$ is a closed set we shall show that its complement $(\overline{A})^c$ is open.

If $(\overline{A})^c = \phi$ then ϕ is open. Assume that $(\overline{A})^c \neq \phi$.

Let $x \in (\overline{A})^c$, then $x \notin \overline{A} \Rightarrow$ there exists at least one $r > 0$ such that

$$S_r(x) \cap A = \phi.$$

Now to show $S_r(x) \cap \overline{A} = \phi$, we take a $y \in S_r(x)$, then $d(y, x) < r$.

Let $r_1 = r - d(y, x)$.

Clearly $r_1 > 0$ and $S_{r_1}(y) \subseteq S_r(x)$

$\Rightarrow \quad S_{r_1}(y) \cap A = \phi$, for at least one r_1 $[\because S_{r_1}(y) \cap A \subseteq S_r(x) \cap A]$

$\Rightarrow \quad y \notin \overline{A}$.

Since y is an arbitrary member of $S_r(x)$, therefore

$S_r(x) \subseteq (\overline{A})^c$. This implies $(\overline{A})^c$ is open.

Hence $\overline{A}$ is open.

(2) Let $x \in \overline{A}$ then

This implies $\quad S_r(x) \cap A \neq \phi$, for all $r > 0$

i.e., $\quad S_r(x) \cap B \neq \phi, \quad (\because A \subseteq B)$

$x \in \overline{B}$.

Hence, $\overline{A} \subseteq \overline{B}$.

(3) We know that $\overline{A}$ is a closed set, and $A \subseteq \overline{A}$. To show that $\overline{A}$ is the smallest closed set containing A, we assume that if F is any other closed set containing A, then $A \subseteq F \Rightarrow \overline{A} \subseteq \overline{F} = F$ [$\because F$ is closed]. Since F is arbitrary, so $\overline{A}$ is the smallest closed set containing A.

(4) If $A = \overline{A}$, then by (1) $\overline{A}$ is closed, and so A is closed.

Conversely, let A be any closed set.

Since $A \subseteq \overline{A}$, so we need to show that $\overline{A} \subseteq A$.

Let x be any element of $\overline{A}$, then either $x \in A$ or $x \notin A$.

If $x \in A$, then the result is proved.

If $x \notin A$, and $x \in \overline{A}$, then for every $r > 0$, the open sphere $S_r(x)$ contains a point of A other than x.

$\Rightarrow \quad x$ is a limit point of A.

But A being closed, therefore x must belong to A. Hence $\overline{A} \subseteq A$.

(5) Let F be the intersection of all closed sets containing A. Then F is closed.

$$A \subseteq F \Rightarrow \overline{A} \subseteq \overline{F} = F$$

i.e., $\quad \overline{A} \subseteq F$,

Thus every closed set which contains A, contains $\overline{A}$. But $\overline{A}$ is a closed set containing A. F, being the intersection of all closed sets containing A, is contained in $\overline{A}$. Therefore $\overline{A} = F$.

(6) We know that

$$A \subseteq A \cup B, \text{ and } B \subseteq A \cup B$$

$\therefore \quad \overline{A} \subseteq \overline{A \cup B}$, and $\overline{B} \subseteq \overline{A \cup B}$

And so $\quad \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Now to show that

$$\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$$

We proceed as follows:

Let, if possible $x \in \overline{A \cup B}$, but $x \notin \overline{A} \cup \overline{B}$. The x is neither an adherent point of A nor that of B. Consequently, there exist open spheres $S_{r_1}(x)$, and $S_{r_2}(x)$ containing no point of A and B respectively.

Take $r = \min\{r_1, r_2\}$, then

$S_r(x)$ contains no point of A as well as no point of B, and therefore of $A \cup B$.

$\therefore$ x is not an adherent point of $A \cup B$.

i.e., $x \notin \overline{A \cup B}$.

Thus we arrive at a contradiction.

Hence $x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cup \overline{B}$.

(7) Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$

$\therefore$ $\overline{A \cap B} \subseteq \overline{A}$, and $\overline{A \cap B} \subseteq \overline{B}$. Hence $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

The result can be extended to the intersection of an arbitrary family $\{A_\alpha\}$ of subsets of X, *i.e.*,

$$\overline{\bigcap_{\alpha \in \Lambda} A_\alpha} \subseteq \bigcap_{\alpha \in \Lambda} \overline{A_\alpha}$$

Remark: Let (X, d) be a metric space and $A \subseteq Y \subseteq X$. Then the closure of A in (Y, d_Y) is denoted by $\overline{A}^Y$. It is easy to verify that $\overline{A}^Y = \overline{A} \cap Y$.

2.8 Interior, Exterior, Frontier and Boundary Points

Definition. Let A be any subset of a metric space (X, d). A point 'a' in A is an *interior point* of A if there exists $r > 0$, such that

$$a \in S_r(a) \subseteq A.$$

Interior of a Set. The set of all interior points of A is called the *interior* of A, and we write

$$\text{int } A = \{x \in A : S_r(x) \subseteq A, \text{ for some } r > 0\}$$

clearly int A is an open subset of A.

A point $x \in X$ is said to be an *exterior point* of A, if it is an interior point of the complement of A. *i.e.*, if there exists an open sphere $S_r(x)$ such that

$$S_r(x) \subseteq A^c, \text{ or } S_r(x) \cap A = \phi.$$

Exterior of a Set. The set of all exterior points of A, denoted by ext A, is called the *exterior* of A.

By definition

$$\text{ext } A = \text{int } A^c$$

$\therefore$ ext A is open and is the largest open set contained in A^c.

Also a point is an *exterior point* of A if and only if it is not an adherent point of A.

$\therefore$ $$\text{ext } A = (\overline{A})^c$$

Moreover, int A = ext $A^c = (\overline{A^c})^c$, and int $A^c = (\overline{A})^c$.

For example, if $X = \mathbf{R}$, and $A = [0, 1[$, then

$$\text{int } A =]0, 1[$$

$$\text{ext } A =]-\infty, 0[\cup]1, \infty[.$$

Note that the points '0' and '1' are neither interior points nor exterior points of A. Such points of A are called *Frontier points* of A.

In another example, let A be the set of complex numbers $x + iy$, such that $y = -1$, and $0 \leq x \leq 1$. Then $A' = A$ and there are no interior points.

A point $x \in X$ is said to be a *frontier point* of $A \subseteq X$ if it is neither an interior nor an exterior point of A. If the frontier point belongs to A it is then called a *boundary point* of A.

Frontier and Boundary of a Set. The set of all frontier points and boundary points are denoted by $F_r(A)$, and $bd(A)$, respectively. Clearly

$$bd(A) \subseteq F_r(A).$$

In the above example $F_r(A) = \{0, 1\}$, and $bd(A) = \{0\}$. Note that the interior points, exterior points and frontier points of any subset A of X fill up the whole space X.

i.e.,
$$X = \text{int } A \cup \text{ext } A \cup F_r(A)$$

Now since int A and ext A both are open in X, therefore $F_r(A)$ is a closed set.

ILLUSTRATIONS

1. $X = \mathbf{R}$, and d the usual metric, $A = \mathbf{Q}$, then int $A = \phi$, ext $A = \phi$, $F_r(A) = \mathbf{R}$, and $bd(A) = \phi$.
2. $X = \mathbf{R}$, and d is the usual metric, $A =]1, 2] \cup]3, 4[$, then
$$\text{int } A = A, \text{ ext } A =]-\infty, 1[\cup]2, 3[\cup]4, \infty[$$
$$F_r(A) = \{1, 2, 3, 4\}, \text{ and } bd(A) = \{2\}.$$
3. If (X, d) is a discrete metric space, and $A \subseteq X$, then
$$\text{int } A = A, \text{ ext } A = A^c, F_r(A) = bd(A) = \phi.$$
4. If $X = \mathbf{N}$, and A be a finite subset of it, say $A = \{1, 2, 3\}$, then
$$\text{int } A = \phi, \text{ ext } A = \phi$$
$$F_r(A) = \mathbf{N}, \text{ and } bd(A) = A.$$

Properties:

Let A and B be any two subsets of a metric spaces (X, d), then

(*i*) int A is the largest open set contained in A.
i.e., $A = \cup \{G : G \text{ is open, and } G \subseteq A\}$

(*ii*) A is open if and only if $A = \text{int } A$.

(*iii*) $A \subseteq B$ implies int $A \subseteq$ int B

(*iv*) $\text{int }(A \cap B) = (\text{int } A) \cap \text{int } B$.

(*v*) $\text{int }(A \cup B) \supseteq \text{int } A \cup \text{int } B$.

Proof of (*i*) and (*ii*) follow from theorems 1 and 2, chapter 2 (change open interval to open sphere in the proof).

(*iii*) Let $x \in \text{int } A$. Then there exists $r > 0$ such that

Note that the points '0' and '1' are neither interior points nor exterior points of A. Such points of A are called *Frontier points* of A.

In another example, let A be the set of complex numbers $x + iy$, such that $y = -1$, and $0 \le x \le 1$. Then $A' = A$ and there are no interior points.

A point $x \in X$ is said to be a *frontier point* of $A \subseteq X$ if it is neither an interior nor an exterior point of A. If the frontier point belongs to A it is then called a *boundary point* of A.

Frontier and Boundary of a Set. The set of all frontier points and boundary points are denoted by $F_r(A)$, and $bd\,(A)$, respectively. Clearly

$$bd(A) \subseteq F_r(A).$$

In the above example $F_r(A) = \{0, 1\}$, and $bd(A) = \{0\}$. Note that the interior points, exterior points and frontier points of any subset A of X fill up the whole space X.

i.e.,
$$X = \text{int } A \cup \text{ext } A \cup F_r(A)$$

Now since int A and ext A both are open in X, therefore $F_r(A)$ is a closed set.

ILLUSTRATIONS

1. $X = \mathbf{R}$, and d the usual metric, $A = \mathbf{Q}$, then int $A = \phi$, ext $A = \phi$, $F_r(A) = \mathbf{R}$, and $bd(A) = \phi$.
2. $X = \mathbf{R}$, and d is the usual metric, $A =]1, 2] \cup]3, 4[$, then
$$\text{int } A = A, \text{ ext } A =]-\infty, 1[\cup]2, 3[\cup]4, \infty[$$
$$F_r(A) = \{1, 2, 3, 4\}, \text{ and } bd\,(A) = \{2\}.$$
3. If (X, d) is a discrete metric space, and $A \subseteq X$, then
$$\text{int } A = A, \text{ ext } A = A^c, F_r(A) = bd(A) = \phi.$$
4. If $X = \mathbf{N}$, and A be a finite subset of it, say $A = \{1, 2, 3\}$, then
$$\text{int } A = \phi, \text{ ext } A = \phi$$
$$F_r(A) = \mathbf{N}, \text{ and } bd(A) = A.$$

Properties:

Let A and B be any two subsets of a metric spaces (X, d), then

(*i*) int A is the largest open set contained in A.
i.e., $A = \cup\,\{G: G$ is open, and $G \subseteq A\}$

(*ii*) A is open if and only if $A = \text{int } A$.

(*iii*) $A \subseteq B$ implies int $A \subseteq$ int B

(*iv*) int $(A \cap B) = (\text{int } A) \cap \text{int } B$.

(*v*) int $(A \cup B) \supseteq \text{int } A \cup \text{int } B$.

Proof of (*i*) and (*ii*) follow from theorems 1 and 2, chapter 2 (change open interval to open sphere in the proof).

(*iii*) Let $x \in \text{int } A$. Then there exists $r > 0$ such that

$$S_r(x) \subseteq A$$

Therefore

$$S_r(x) \subseteq B \qquad (\because\ A \subseteq B)$$

This implies $x \in \text{int } B$.

Hence $\text{int } A \subseteq \text{int } B$.

(*iv*) By definition

$\text{int } A \subseteq A$, and $\text{int } B \subseteq B$

$\therefore$ $\text{int } A \cap \text{int } B$, being the intersection of two open sets, is open. Therefore

$\text{int } A \cap \text{int } B \subseteq \text{int } (A \cap B)$ (by (*ii*))

Also $A \cap B \subseteq A \Rightarrow \text{int } (A \cap B) \subseteq \text{int } A$, and $\text{int } (A \cap B) \subseteq \text{int } B$

$\therefore$ $\text{int } (A \cap B) \subseteq \text{int } A \cap \text{int } B$.

(*v*) $A \subseteq A \cup B \Rightarrow \text{int } A \subseteq \text{int } (A \cup B)$

and $B \subseteq A \cup B \Rightarrow \text{int } B \subseteq \text{int } (A \cup B)$

$\therefore$ $\text{int } A \cup \text{int } B \subseteq \text{int } (A \cup B)$.

Note that equality may not hold in (*v*) as can be seen by the following example:

Let $A =]2, 5[$, and $B = [5, 7[$ be the subsets of the metric space $(\mathbf{R}, d)$ with the usual metric d, then $\text{int } A =]2, 5[$, $\text{int } B =]5, 7[$

$\text{int } A \cup \text{int } B =]2, 7[- \{5\}$

But $\text{int } (A \cup B) =]2, 7[$

$\therefore$ $\text{int } (A \cup B) \neq \text{int } A \cup \text{int } B$

Theorem 7. *Let (X, d) be a metric space, and A, B be any two subsets of X, then*

(*i*) *ext A is the largest open set contained in A^c*

(*ii*) *A^c is open if and only if $A^c = \text{ext } A$*

(*iii*) *$A \subseteq B$ implies ext $B \subseteq$ ext A*

(*iv*) *$\text{ext } (A \cap B) \supseteq \text{ext } A \cup \text{ext } B$*

(*v*) *$\text{ext } (A \cup B) = \text{ext } A \cap \text{ext } B$*

Proof follows from the above theorem by taking complements.

Theorem 8. Let *(X, d) be a metric space, and A, B are subset of X, then*

(*i*) *$Fr(A) = \overline{A} \cap (\overline{A})^c = \overline{A} - \text{int } A$*

(*ii*) *$Fr(A) = \phi$ if and only if A is both open and closed*

(*iii*) *A is closed if and only if $A \supseteq Fr(A)$*

(*iv*) *A is open if and only if $A^c \supseteq Fr(A)$*

(*v*) *$Fr(A \cap B) \subseteq Fr(A) \cup Fr(B)$. The equality holds if $\overline{A} \cap \overline{B} = \phi$*

(*vi*) *$Fr(\text{int } A) \subseteq Fr(A)$.*

(*i*) $Fr(A) = (\text{int } A \cup \text{ext } A)^c = (\text{int } A^c) \cap (\text{ext } A)^c = \overline{A}^c \cap \overline{A}$

i.e., $Fr(A) = \overline{A} \cap \overline{A}^c = \overline{A} - (\overline{A}^c)^c = \overline{A} - \text{int } A$

(*ii*) Let $Fr(A) = \phi$, then by (*i*)

$$\overline{A} - \text{int } A = \phi \quad i.e., \ \overline{A} \subseteq \text{int } A$$

$$\Rightarrow \qquad A \subseteq \overline{A} \subseteq \text{int } A \subseteq A$$

Hence, A is both open and closed.

Conversely, let A be both open and closed then

$$Fr(A) = \overline{A} - \text{int } A = A - A = \phi.$$

(*iii*) Let A be closed, then [by (*i*)]

$$Fr(A) = \overline{A} \cap (\overline{A^c})$$
$$= A \cap (\overline{A^c}) \subseteq A$$

Conversely, let $Fr\ A \subseteq A$.

If possible, let A be not closed, then there exists an element x belonging to $\overline{A}$, but not belonging to A, *i.e.*, $x \in \overline{A} - A$.

But [by (*i*)]

$$\overline{A} - A = \overline{A} \cap A^c \subseteq \overline{A} \cap (\overline{A^c}) = Fr(A)$$

$$\therefore \qquad x \in Fr(A)$$

So that $x \in A$, which is a contradiction.

Hence, A must be closed.

(*iv*) A is open if and only if A^c is closed, and A^c is closed if and only if

$$A^c \supseteq Fr(A^c)$$

But since [by (*i*)]

$$Fr(A^c) = (\overline{A^c}) \cap (\overline{A^{c^c}})$$
$$= \overline{A}^c \cap \overline{A} = Fr(A)$$

Hence, A is open if and only if $A^c \supseteq Fr(A)$.

(*v*) [by (*i*)]

$$Fr(A \cap B) = (\overline{A \cap B}) \cap (\overline{A \cap B})^c$$
$$\subseteq \overline{A} \cap \overline{B} \cap (\overline{A^c \cup B^c})$$
$$= \overline{A} \cap \overline{B} \cap (\overline{A}^c \cup \overline{B}^c)$$
$$= (\overline{A} \cap \overline{B} \cap \overline{A}^c) \cup (\overline{A} \cap \overline{B} \cap \overline{B}^c)$$
$$= (Fr(A) \cap \overline{B}) \cup (Fr(B) \cap \overline{A})$$
$$\subseteq (Fr(A) \cup Fr(B)$$

(*vi*) $Fr(A \cup B) = Fr(A \cup B)^c = Fr(A^c \cap B^c) \subseteq Fr(A^c) \cup Fr(B^c)$

$$= Fr(A) \cup Fr(B)$$

2.9 Dense Sets

Definition. A subset A of a metric space (X, d) is said to be *dense* (or *everywhere dense*) in X, if the closure of A is X, *i.e.*, $\overline{A} = X$.

For example the set of rationals is dense in **R** with the usual metric. Every interval is dense-in-itself.

A set A is said to be *nowhere dense* in X, if the complement of the closure of A is dense in X.

i.e.,
$$(\overline{A})^c = X, \text{ or } \overline{\text{ext } A} = X$$

Equivalently, A is nowhere dense in X iff int $(\overline{A}) = \phi$, since
$$\text{int } \overline{A} = \text{ext } (\overline{A})^c, \text{ and } \text{ext } X = \phi$$

Clearly every finite subset of X is nowhere dense.

A set is said to be *somewhere* dense in X, if it is not nowhere dense in X.

It is clear that A is somewhere dense in X if and only if $\overline{A}$ contains a non-empty open sphere.

Also since $\overline{A}$ is closed, A is nowhere dense in X if and only if $\overline{A}$ is nowhere dense in X. Clearly a subset of a nowhere dense set, is nowhere dense.

A set 'A' is said to be *dense-in-itself* if every point of A is a limit point of A.

i.e., $A \subseteq A'$

A set 'A' is said to be perfect if it is closed and dense-in-itself,

i.e., $A = A'$

Every closed interval, the empty set, ϕ, and the whole space X are perfect sets.

Example 17. The *cantor set* is a perfect set.

Recall that the cantor set is the set obtained from the closed interval $[0, 1]$ by removing the sequence of open intervals $\left]\frac{1}{3}, \frac{2}{3}\right[$; $\left]\frac{1}{9}, \frac{2}{9}\right[\cup \left]\frac{7}{9}, \frac{8}{9}\right[$; which are middle thirds of $[0, 1]$; $\left[0, \frac{1}{3}\right]$; $\left[\frac{2}{3}, 1\right]$;... respectively. Thus the *cantor set* is the intersection of the family of sets $\{F_n : n \in \mathrm{N}\}$, where

$$F_1 = [0, 1]$$

$$F_2 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

$$F_3 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}$$

are closed sets.

($\because$ Each F_n is the complement of the union of removed open intervals and the intervals $]-\infty, 0[$ and $]1, \infty[$), and so the cantor set

$$F = \bigcap_{n=1}^{\infty} F_n$$

is closed.

All that remains is to show that it is dense-in-itself.

For this let $x \in F$, then $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where each a_k is either 0 or 2 be the ternary expansion of x (expansion in the scale of 3). We shall show that x is a limit point of F.

Choose the sequence $\{x_n\}$ in F, such that

$$x_1 = \cdot a_1' a_2 a_3 \ldots a_n \ldots$$

$$x_2 = \cdot a_1 a_2' a_3 \ldots a_n \ldots$$

$$\vdots$$

$$x_n = \cdot a_1 a_2 \ldots a_n' \; a_{n+1} \ldots$$

$$\vdots$$

where $a_n' = 0$, if $a_n = 2$, and $a_n' = 2$, if $a_n = 0$.

The sequence $\{x_n\}$ of distinct points of F differ from x at the nth place in the ternary expansion.

Therefore $$\lim_{n \to \infty} x_n = x$$

Thus every point of the cantor set is a limit point of the set and so it is dense-in-itself.

Ex. Prove that the cantor set is nowhere dense.

Definition. A metric space X is said to be *separable* if there is a countable subset of X, which is dense in X.

Since the set of all rational numbers is countable, and dense in **R**, therefore the metric space **R** is separable. Let (X, d) be a discrete metric space where X is any uncountable set, then (X, d) is not separable ($\because$ the only dense subset of (X, d) is X itself). Hence the discrete metric space is separable if and only if it is countable.

Ex. 1. A subset A of (X, d) is dense in X if and only if A has non-empty intersection with each non-empty open sphere in X, or equivalently if and only if A has non-empty intersection with each non-empty open subset of X.

[*Hint*: Given $S_r(x) \cap A \neq \phi$, for all $r > 0$, and for each $x \in X$, and by definition

$$\bar{A} = \{x \in X : S_r(x) \cap A \neq \phi, \text{ for all } r > 0\}.$$

hence $\bar{A} = X$.]

Ex. 2. Show that the Euclidean space $\mathbf{R}^n$ is separable.

Ex. 3. Prove that the metric space l_∞ of all bounded sequences with sup metric is not separable.

[*Hint*: Show that every dense subset is uncountable.]

EXERCISE

1. Let x_1 and x_2 be distinct points in the metric space (X, d). Show that there exist two disjoint open spheres centered at x_1 and x_2, respectively.
2. Show by example that a set which fails to be closed need not be open.

3. Show by example that a non-empty proper subset A of a metric space (X, d) can be both open and closed.
4. Give an example to show that the closure of the open sphere of radius r centered at x_0 is not necessarily equal to the closed sphere or radius r centered at x_0. Prove that we always have

$$\bar{S}_r(x_0) \subseteq S_r[x_0]$$

5. Let $\{[a_n, b_n]\}$ be a sequence of closed intervals such that $|a_n| \leq 1$, $|b_n| \leq 1$, $\forall\, n$. Then prove that

$$\{\{x_n\} : x_n \in [a_n, b_n]\}$$

is a closed subset of H_∞.

6. If A and B are disjoint closed subsets of a metric space X, show that

$$G = \{x \in X : d(x, A) < d(x, B)\}, \text{ and } H = \{x \in X : d(x, B) < d(x, A)\}$$

are disjoint open sets containing A and B respectively.

7. Determine whether the following subsets of the metric spaces indicated are open, closed, both open and closed, or neither open nor closed
   (*i*) $\{(x_1, x_2 \ldots, x_n) \in \mathrm{R}^n : x_i > 0 \text{ for } i = 1, 2 \ldots n\}$ and d is the Euclidean metric.
   (*ii*) $\{(x_1, x_2 \ldots, x_n) \in \mathrm{R}^n : x_i \text{ is rational for } i = 1, 2 \ldots n\}$
   (*iii*) $\{\{x_n\} \in l : x_n < 1/n, \text{ for } n = 1, 2 \ldots\}$, where l is the set of all sequences $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|$ is convergent with the metric defined by

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} |x_n - y_n|.$$

   (*iv*) $\{f \in C'[a, b] : f(a) + f'(a) = 0\}$, where $C'[a, b]$ is the set of all functions defined on $[a, b]$ having continuous first order derivative on $[a, b]$, with the metric defined by

$$d(f, g) = \sup \{| f(x) - g(x) | : x \in [a, b]\}$$
$$+ \sup \{| f'(x) - g'(x) | : x \in [a, b]\}.$$

8. Give an example of a countable family of closed subsets of **R** whose union is not closed.
9. Prove that an open subset of $\mathbf{R}^n$ can be expressed as the union of a countable family of open spheres in $\mathbf{R}^n$.
10. Show that if $n \geq 2$, then there are open subsets of $\mathbf{R}^n$ which cannot be expressed as the union of a countable family of pairwise disjoint open spheres in $\mathbf{R}^n$.
11. Show that a metric space is discrete if and only if every point of the space is isolated.
12. Find the closures, the interiors, and the frontiers of the following:
    (*i*) a subset A of a discrete metric space X,
    (*ii*) the set of all rational numbers in **R**,
    (*iii*) an open sphere in the Euclidean space $\mathbf{R}^n$.
13. Prove that the cantor set of example 17 has neither isolated points nor interior points.
14. Let A be the subset

$$\left\{\left(\frac{m}{n}, \frac{1}{n}\right) : n = 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots\right\}$$

of the Euclidean space $\mathbf{R}^2$. Prove that $\overline{A}$ is the union of A and the set $\{(x, 0) : x \in \mathbf{R}\}$.

15. Find the Frontier of the subset $\{x_1, x_2) : x_2 = 0\}$ of the Euclidean space $\mathbf{R}^2$.
16. Prove that a set D is dense in a metric space (X, d) if and only if X is the only closed set containing D.
17. Verify that $(\mathbf{R} - \mathbf{Q}, d)$, the irrationals with the usual metric, is a separable metric space.

18. Prove that every subspace of a separable metric space is separable.
19. Prove that the space $(\mathbf{R}^n, d)$ is separable, with the metric d given by $d(x, y) = \max_{1 \le k \le n} |x_k - y_k|$,

$$x = (x_1, x_2, \ldots, x_n), \text{ and } y = (y_1, \ldots, y_n) \in \mathbf{R}^n.$$

20. Prove that the example of a Hilbert space (Example 2) is a separable metric space.
21. Prove that a closed set in a metric space (X, d) either is nowhere dense in X or else contains some non-empty open set.
22. Prove that $A = \{f_n : n \in \mathbf{N}\}$ is a nowhere dense subset of $\mathbf{C}\,[0, 1]$ w.r.t. sup metric, where $f_n(x) = n - n^3 x$, if $x \le 1/n^2$, and $f_n(x) = 0$, otherwise.

• 3. CONVERGENCE AND COMPLETENESS

Definition. Let (X, d) be any metric space. The sequence $\{a_n\}$ of points of X is said to *converge* to a point 'a' of X, if for each $\varepsilon > 0$ there exists a positive integer m, such that

$$d(a_n, a) < \varepsilon, \quad \forall\, n \ge m$$

i.e.,

$$d(a_n, a) \to 0, \quad \text{as } n \to \infty$$

or equivalently, for each open sphere $S_\varepsilon(a)$ centered at 'a' there exist as positive integer m such that a_n is in $S_\varepsilon(a)$, for all $n \ge m$.

The point 'a' is called the *limit* of the sequence $\{a_n\}$, and we write $a_n \to a$, as $n \to \infty$

i.e.,

$$\lim_{n \to \infty} a_n = a.$$

Cauchy sequence. *A* sequence $\{a_n\}$ of points of (X, d) is said to be a *Cauchy sequence* if for each $\varepsilon > 0$ there exists a positive integer n_0, such that

$$d(x_n, x_m) < \varepsilon, \quad \forall\, n, m \ge n_0$$

i.e.,

$$d(x_n, x_m) \to 0, \quad \text{as } n, m \to \infty.$$

Theorem 9. *Every convergent sequence is a Cauchy sequence.*

Let (X, d) be any metric space. Let the sequence $\{a_n\}$ of points in X converge to a.

For every given $\varepsilon > 0$ there exists a positive integer n_0 such that

$$d(a_n, a) < \varepsilon/2, \quad \forall\, n \ge n_0$$

Then for $m, n \ge n_0$ we have

$$d(a_n, a_m) \le d(a_n, a) + d(a, a_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This implies $\{a_n\}$ is a Cauchy sequence.

Note: The following examples show that the converse of the statement need not be true.

Example 18. Consider the space $X =]0, 1]$ of the real line with the usual metric. The sequence $\{a_n\} = \{1/n\}$ is a Cauchy sequence converges to '0', which is not a point of the space.

Example 19. Let $\mathbf{Q}$ be the set of rational numbers in which the metric d is defined by

$$d(x, y) = |x - y|, \quad \forall\, x, y \in \mathbf{Q}$$

($\mathbf{Q}$, d) is a metric space. The sequence $\{1/3^n\}$ is a Cauchy sequence which converges to the limit 0. But the sequence $\{(1 + 1/n)^n\}$ is also a Cauchy sequence in it, which does not converge to a point of $\mathbf{Q}$.

Complete metric space. A metric space (X, d) is said to be *complete* if every Cauchy sequence converges to a point of X.

The spaces in the examples mentioned above are not complete. But if we adjoin the point '0' to the space]0, 1] in the first example it becomes complete.

Remark: Any metric space which is not already complete can be made so by adjoining additional points to it.

ILLUSTRATIONS

1. The discrete space (X, d) is a complete metric space. For in this space a Cauchy sequence must be a constant sequence (*i.e.*, it must consist of a single point repeated from some place on) and so converges.
2. The space (R, d) is a complete metric space. The convergence in **R** is the ordinary convergence of numerical sequences.
3. The space R^n of all ordered n-tuples with the metric d,

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$$

is a complete metric space. The convergence in this space is coordinate wise. This space $(\mathbf{R}^n, d)$ is called n-dimensional Euclidean space.

Example 20. The space $C\,[0, 1]$ of all bounded continuous real-valued functions defined on the closed interval [0, 1] with the metric d given by

$$d(f, g) = \max_{0 \le x \le 1} |f(x) - g(x)|$$

is a complete metric space.

■ Let $\{f_n\}$ be a Cauchy sequence in $C\,[0, 1]$.

Let $\varepsilon > 0$ be given. Then there exists a positive integer n_0, such that

$$d(f_n, f_m) < \varepsilon, \quad \forall\, n, m \ge n_0$$

i.e.,

$$\max_{0 \le x \le 1} |f_n(x) - f_m(x)| < \varepsilon, \quad \forall\, n, m \ge n_0$$

i.e.,

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall\, n, m \ge n_0 \text{ and } \forall\, x \in [0, 1]$$

By Cauchy criterion of uniform convergence, the sequence of functions $\{f_n\}$ converges uniformly on [0, 1]. Let f be the limit of a uniformly convergent sequence of continuous functions so this itself is continuous on [0, 1]. Hence the Cauchy sequence $\{f_n\}$ converges to a point of $C\,[0,1]$.

Example 21. Let l_∞ be the set of all bounded numerical sequences $\{x_n\}$ in which the metric d is defined by

$$d(x, y) = \sup_n | x_n - v_n |, \forall\ x = \{x_n\}, y = \{y_n\} \in l_\infty$$

- Let $\{x_n\}$ be a Cauchy sequence of elements of l_∞ and let

$$x_n = \{a_i^{(n)}\}. \text{ Since } x_n \in l_\infty, \text{ so } \exists\ M > 0,$$
$$| a_i^{(n)} | \leq M, \text{ for } i = 1, 2, 3...$$

Therefore for $\varepsilon > 0$, there exists an integer n_0 such that

$$d(x_n, x_m) < \varepsilon, \quad \forall\ n, m \geq n_0$$

i.e., $$\sup_i | a_i^{(n)} - a_i^{(m)} | < \varepsilon, \ \forall\ n, m \geq n_0$$

$\Rightarrow$ $$| a_i^{(n)} - a_i^{(m)} | < \varepsilon, \ \forall\ n, m \geq n_0, \text{ and for all } i = 1, 2, 3... \qquad ...(1)$$

Let i be fixed. Then (1) implies that the sequence $\{a_i^{(1)}\ a_i^{(2)},... a_i^{(n)},...\}$ is Cauchy and so converges to a_i, by Cauchy's General Principle of convergence. Taking limit in (1) as $m \to \infty$, we have

$$| a_i^{(n)} - a_i | \leq \varepsilon, \quad \forall\ n \geq n_0$$

and this is true for all $i = 1, 2, 3,...$

Hence, $$| a_i | \leq | a_i^{(n)} - a_i | + | a_i^{(n)} | < \varepsilon + M, \forall\ i$$

This implies $\{a_i\}$ is bounded. Let $x = \{a_i\}$. Then $x \in l_\infty$. Hence (l_∞, d) is a complete space.

Example 22. Let l_p be the set of all real numerical sequences for which

$$\sum_{i=1}^{\infty} | x_i |^p < \infty.$$

- We define the metric d in l_p by

$$d(x, y) = \left(\sum_{i=1}^{\infty} | x_i - y_i |^p \right)^{1/p}, \ \forall\ x = \{x_i\}, y = \{y_i\} \in l_p$$

The space (l_p, d) is a complete metric space, and is known as Hilbert sequence space.

Consider a Cauchy sequence $\{x_n\} = \{\{x_i^{(n)}\}\}$ in l_p.

Therefore for a given $\varepsilon > 0$ there exists an integer n_0, such that

$$d(x_n, y_m) = \left(\sum_{i=1}^{\infty} | x_i^{(n)} - x_i^{(m)} |^p \right)^{1/p} < \varepsilon, \ \forall\ n, m \geq n_0 \qquad ...(1)$$

Hence $| x_i^{(n)} - x_i^{(m)} | < \varepsilon, \ \forall\ n, m \geq n_0$, and for all $i \in \mathrm{N}$ ...(2)

Fixing i, we see that the sequence $\{x_i^{(1)}\ x_i^{(2)},... x_i^{(n)},...\}$ converges to a limit x_i

i.e., $$\lim_{n \to \infty} x_i^{(n)} = x_i$$

Let $x = \{x_i\}$. Then the inequality (1) implies

$$\sum_{i=1}^{k} | x_i^{(n)} - x_i^{(m)} |^p < \varepsilon^p, \text{ for every } k, \text{ and for } n, m \geq n_0$$

Taking limit as $m \to \infty$, we have

$$\sum_{i=1}^{k} | x_i^{(n)} - x_i |^p \leq \varepsilon^p, \quad \text{for } n \geq n_0$$

Letting $k \to \infty$, we get

$$\sum_{i=1}^{\infty} | x_i^{(n)} - x_i |^p \leq \varepsilon^p, \quad \text{for } n \geq n_0$$

This implies $x_n - x \in l_p$, and so $x = x_n - (x_n - x) \in l_p$. Also $d(x_n, x) \to 0$ as $n \to \infty$. Hence l_p is complete.

Example 23. Let X be the set of all continuous real-valued functions defined on [0, 1], and let

$$d(x, y) = \int_0^1 | x(t) - y(t) | \, dt, \quad x, y \in X$$

Show that (X, d) is not complete.

■ Let $\{x_n\}$ be a sequence in X defined by

$$x_n(t) = \begin{cases} n, & \text{if } 0 \leq t \leq \dfrac{1}{n^2} \\ \dfrac{1}{\sqrt{t}}, & \text{if } \dfrac{1}{n^2} \leq t \leq 1 \end{cases}$$

For $n > m$, we have

$$d(x_n, x_m) = \int_0^1 | x_n(t) - x_m(t) | \, dt$$

$$= \int_0^{1/n^2} | n - m | \, dt + \int_{1/n^2}^{1/m^2} \left| \frac{1}{\sqrt{t}} - m \right| dt + \int_{1/m^2}^{1} \left| \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t}} \right| dt$$

$$= \frac{(n - m)}{n^2} + (2t^{1/2} - mt) \Big|_{1/n^2}^{1/m^2}$$

$$= \frac{1}{n} - \frac{m}{n^2} + \left(\frac{2}{m} - \frac{1}{m} \right) - \left(\frac{2}{n} - \frac{m}{n^2} \right)$$

$$= \frac{1}{m} - \frac{1}{n} \to 0, \text{ as } m, n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in X.

Now we shall show that this Cauchy sequence does not converge in X. For every $x \in X$

$$d(x_n, x) = \int_0^1 |x_n(t) - x(t)|\, dt$$

$$= \int_0^{1/n^2} |n - x(t)|\, dt + \int_{1/n^2}^1 \left| \frac{1}{\sqrt{t}} - x(t) \right| dt$$

Since integrals are non-negative, so is each integral on the right, and hence $d(x_n, x) \to 0$, as $n \to \infty$ would imply that each integral approaches zero, and since x is in X, so x is continuous.

But

$$x(t) = \begin{cases} t^{-1/2}, & \text{if } 0 < t \le 1 \\ 0\ , & \text{if } t = 0 \end{cases}$$

which is discontinuous at $t = 0$. Hence $d(x_n, x)$ does not tend to zero for each $x \in X$, *i.e.*, the sequence $\{x_n\}$ does not converge to the point of the space. This implies that (X, d) is not complete.

Lemma. *Let (X, d) be any metric space and A be any non-empty subset of X, then $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x$, as $n \to \infty$.*

Let $x \in \overline{A}$ then every open sphere centered at x intersects A. In particular $S_{1/n}(x) \cap A \neq \phi$, for all n. So we get a sequence $\{x_n\}$ in A such that

$$d(x_n, x) < \frac{1}{n}, \forall\, n$$

$$\Rightarrow \qquad \lim_{n\to\infty} x_n = x.$$

Again, let $\{x_n\}$ be a sequence in A which converges to x. To show that $x \in \overline{A}$ we must show that every open sphere centered at x intersects A.

Let $S_r(x)$ be any open sphere. Then for $r > 0$, $\lim_{n\to\infty} x_n = x$ implies that there exists a positive integer n_0, such that

$$d(x_n, x) < r, \quad \forall\, n \ge n_0$$

In particular

$$d(x_{n_0}, x) < r$$

$$\Rightarrow \qquad x_{n_0} \in S_r(x)$$

$$\Rightarrow \qquad S_r(x) \cap A \neq \phi \qquad [\because x_{n_0} \in A]$$

$$\Rightarrow \qquad x \in \overline{A}$$

Theorem 10. *Let (X, d) be a complete metric space and Y be a subspace of X. Then Y is complete if and only if it is closed in (X, d).*

Let Y be a complete subspace of X. In order to show that Y is closed we need to show that $Y = \overline{Y}$. By definition $Y \subset \overline{Y}$, so we shall show that $\overline{Y} \subseteq Y$.

Let x be an element of Y. If $x \in \overline{Y}$, the result is proved. If $x \notin \overline{Y}$, then x is a limit point of Y. By definition of limit point, every neighbourhood $S_{1/n}(x)$ contains at least one member of Y other than x. Thus for each n we get a sequence $\{y_n\}$ in Y such that

$$d(y_n, x) < 1/n. \text{ Thus } y_n \to y, \text{ as } n \to \infty.$$

Now the sequence $\{y_n\}$ being a convergent sequence must be a Cauchy sequence. Since Y is complete, this Cauchy sequence $\{y_n\}$ must converge in Y, hence $x \in Y$. But x is an arbitrary point of $\overline{Y}$, therefore $\overline{Y} \subseteq Y$.

Conversely, we assume that Y is a closed subspace of X, and establish that Y is complete.

Let $\{y_n\}$ be a Cauchy sequence in Y, and since X is given to be a complete space, therefore $\{y_n\}$ must converge to a point y in X. But then $y_n \in Y$, for all n, and $y_n \to y$, as $n \to \infty$

$$\Rightarrow \qquad y \in \overline{Y} = Y \quad (\because Y \text{ is closed})$$

The following is a generalisation of Nested-Interval Theorem (Example 17, Chapter 3).

Theorem 11. ***Cantor's Intersection Theorem.*** *Let (X, d) be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$, as $n \to \infty$. Then $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.*

Since $F_n \neq \phi$, for each $n \in \mathbf{N}$, we can choose a sequence of points $\{x_n\}$ such that $x_n \in F_n$, for $n = 1, 2, 3...$ We shall show that $\{x_n\}$ is a Cauchy sequence in X.

Now $\{F_n\}$ is a decreasing sequence, *i.e.*, $F_{n+1} \subseteq F_n$, for all n, therefore $x_n, x_{n+1}, \ldots$ all lie in F_n.

Moreover $d(F_n) \to 0$, as $n \to \infty$. Therefore given $\varepsilon > 0$ there exists a positive integer n_0, such that

$$d(F_n) < \varepsilon, \quad \forall\, n \geq n_0$$

$$\Rightarrow \qquad x_{n_0}, x_{n_{0+1}}, x_{n_{0+2}}, \ldots \text{ all lie in } F_{n_0}$$

Thus for positive integers $m, n \geq n_0$, we have

$$d(x_n, x_m) \leq d(F_{n_0}) < \varepsilon$$

$\Rightarrow$ $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, $\exists$ a point $x \in X$ such that $\lim_{n \to \infty} x_n = x$.

We shall now show that $x \in \bigcap_{n=1}^{\infty} F_n$

If possible, $x \notin \bigcap_{n=1}^{\infty} F_n$. Then there exists a positive integer m, such that $x \notin F_m$. Since F_m is closed, and $x \notin F_m$.

$$d(x, F_m) > 0. \text{ Let } d(x, F_m) = r > 0,$$

then $$d(x, y) \geq r, \quad \forall\, y \in F_m.$$

Thus, the open sphere $S_{r/2}(x)$, and F_m are clearly disjoint, and therefore

$$n > m \Rightarrow F_n \subset F_m,$$

and this implies

$$x_n \in F_m \ (\because x_n \in F_n) \Rightarrow x_n \notin S_{r/2}(x).$$

This is impossible, since $\{x_n\}$ converges to x. Hence $x \in \bigcap_{n=1}^{\infty} F_n$.

Now to show that $x \in \bigcap_{n=1}^{\infty} F_n$ is unique.

If possible, let y be another point in $\bigcap_{n=1}^{\infty} F_n$

Then $y \in F_n$, for every n.

$\Rightarrow \quad d(x, y) \leq d(F_n)$, for every n (by the definition of the diameter).

But, since it is given that $d(F_n) \to 0$ as $n \to \infty$.

Therefore on taking limit as $n \to \infty$.

$d(x, y) \leq 0$. But $d(x, y) \geq 0$ is always true. Hence $d(x, y) = 0$, and so $x = y$.

Note: The converse of the above theorem is also true.

Ex. If every decreasing sequence of non-empty closed sets whose diameter tends to zero have a non-empty intersection in a metric space (X, d), then (X, d) is complete.

The following examples show that the condition $\lim_{n\to\infty} d(F_n) = 0$, and that the sets F_n's are closed, are both necessary in the above theorem.

Example 24. Let X be the real line **R** with the usual metric, and let $F_n = [n, \infty[$.

Now $X =$ **R** is complete. The sets F_n are closed and $F_1 \supset F_2 \supset F_3 \ldots \supset F_n \ldots$ But $\bigcap_{n=1}^{\infty} F_n$ is empty. Observe that $\lim_{n\to\infty} d(F_n) \neq 0$.

Example 25. Let X be the real line with the usual metric, and let $F_n =]0, 1/n]$

Now $X =$ **R** is complete

$$F_1 \supset F_2 \supset F_3 \ldots\ldots \supset F_n$$

and $$\lim_{n\to\infty} d(F_n) = 0, \text{ but } \bigcap_{n=1}^{\infty} F_n \text{ is empty.}$$

Observe that $F_n's$ are not closed.

Definition. A subset A of a metric space (X, d), possibly the whole space, is said to be of the *first category*, if it is the union of a countable family of nowhere dense sets.

i.e., $A \subseteq X$ can be written as $A = \bigcup_{n=1}^{\infty} A_n$,

where each A_n is nowhere dense in X, *i.e.*, int $(\overline{A_n}) = \phi$, for each n.

Otherwise it is said to be of the *second category*. It is important to note, that in a discrete space the only nowhere dense set is the empty set, *i.e.*, every non-empty set is of the second category. In particular the set I of integers is of the second category in the space **R** of reals with the discrete metric. On the other hand if (**R**, d) is a metric space with the usual metric, then the set **I** of integers is nowhere dense and hence is of the first category. This shows that the set is not of the first or second category in and on its own, rather its category classification also depends on the metric space to which the set belongs.

Ex. Prove that

(*i*) Q is of first category in **R**, w.r.t. usual metric,

(*ii*) every countable subspace of **R** is of first category in **R**,

(*iii*) if X is of second category, and if $X = A \cup B$, then either A or B must be of second category,

(*iv*) X is of second category in itself if and only if the intersection of every countable family of dense open sets in X is non-empty,

(*v*) if A is a dense subset of a complete metric space X, and if $A = \bigcap_{n=1}^{\infty} G_n$, where $G_n's$ are open in X, then $X - A$ is of first category.

Theorem 12. ***Baire's Category Theorem.*** *If* $\{A_n\}$ *is a sequence of nowhere dense sets in a complete metric space* (X, d), *then*

$$X \neq \bigcup_{n=1}^{\infty} A_n.$$

i.e., Every complete metric space is of second category.

To prove the theorem we need the following lemma.

Lemma. *Let A be a nowhere dense subset of a metric space,* (X, d). *Let G be any non-empty open set in X,* and $r > 0$ *be any real number, then there exists an open sphere of radius less than or equal to r contained in G and disjoint from A.*

Since A is nowhere dense *i.e.*, int $\overline{A} = \phi$, and int $\overline{A}$ is the largest open set containing $\overline{A}$. Therefore, if G is any non-empty open set, then

$$G \not\subseteq \overline{A},$$

G being non-empty and $G \not\subseteq \overline{A}$, therefore $\exists$ an $x \in G$ such that $x \notin \overline{A}$.

Moreover G is open, $\exists$ an open sphere $S_r(x)$, for some $r > 0$, such that

$$S_r(x) \subset G.$$

Since $x \notin \overline{A}$, we can choose a positive number $r_1 < r$, such that

$$S_{r_1}(x) \subset S_r(x), \text{ and } S_{r_1}(x) \cap A = \phi.$$

Thus $$x \in S_{r_1}(x) \subset G, \text{ and } S_{r_1}(x) \cap A = \phi.$$

Proof of the main theorem:

X is open, being a non-empty open subset of itself, and A is nowhere dense in X, then by the above lemma, for given $r_1 > 0, 0 < r_1 < 1$, $\exists$ an open sphere $S_{r_1}(x_1)$ in X, such that

$$S_{r_1}(x_1) \cap A_1 = \phi$$

Let F_1 be the concentric closed sphere of radius $\frac{1}{2}r_1$,

i.e., $$F_1 = S_{\frac{1}{2}r_1}[x_1],$$

and consider its interior, int $F_1 \neq \phi$. Let $x_2 \in$ int F_1. Since A_2 is nowhere dense, int F_1 contains an open sphere $S_{r_2}(x_2)$ of radius $r_2 < \frac{1}{2}$, such that

$$S_{r_2}(x_2) \cap A_2 = \phi$$

Let F_2 be the concentric closed sphere of radius $\frac{1}{2}r_2$, *i.e.*, $F_2 = S_{r2/2}[x_2]$. Since A_3 is nowhere dense, int F_2, being non-empty, contains an open sphere $S_{r_3}(x_3)$ centered at $x_3 \in$ int F_2 and radius $r_3 < \frac{1}{3}$, such that

$$S_{r_3}(x_3) \cap A_3 = \phi$$

Let F_3 be the concentric closed sphere of radius $\frac{1}{2}r_3$.

Continuing in this manner we get a decreasing sequence $\{F_n\}$ of non-empty closed subsets of X, where

$$F_n = S_{rn/2}[x_n],$$

and

$$F_{n+1} = S_{\frac{1}{2}r_{n+1}}[x_{n+1}] \subset S_{r_{n+1}}(x_{n+1}) \subseteq S_{\frac{1}{2}r_n}[x_n] = F_n$$

i.e., $$F_{n+1} \subseteq F_n, \quad \forall\, n.$$

Also

$$d(F_n) = 2\, r_n/2 < 1/n, \quad \forall\, n$$

$$\therefore \quad d(F_n) \to 0, \quad \text{as } n \to \infty$$

Since X is given to be complete, therefore by Cantor's intersection theorem we conclude that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point say $x \in X$.

$$\Rightarrow \quad x \in F_n, \quad \forall\, n$$

$$\Rightarrow \quad x \in S_{\frac{1}{2}r_n}[x_n] \subset S_{r_n}(x_n), \quad \forall\, n$$

and $$S_{r_n}(x_n) \cap A_n = \phi$$

$$\Rightarrow \quad x \notin A_n \quad \forall\, n$$

$$\Rightarrow \quad x \notin \bigcup_{n=1}^{\infty} A_n$$

Hence, $$\bigcup_{n=1}^{\infty} A_n \neq X.$$

Ex. Use Baire's Category theorem to prove the existence of everywhere continuous, nowhere differentiable real-valued functions.

[*Hint*: Take

$$A_n = \{f \in C[0,1] : \exists\, x \in [0, 1 - 1/n], \text{ such that } \left|\frac{f(x+h) - f(x)}{h}\right| \leq n, \text{ for } 0 < h < 1/n\}.$$

If $f \in C[0,1]$ has a derivative at some point, then $f \in A_n$, for some n. Show that A_n is closed, and has empty interior.]

• 4. CONTINUITY AND UNIFORM CONTINUITY

Definition. Let (X, d_1), and (Y, d_2) be any two metric spaces. A function $f : X \to Y$ is said to be *continuous* at a point 'a' of X, if for given $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$d_2(f(x), f(a)) < \varepsilon, \text{ whenever } d_1(x, a) < \delta.$$

Equivalently, for each open sphere $S_\varepsilon(f(a))$ centered at $f(a)$ there is an open sphere $S_\delta(a)$ centered at a such that

$$f(S_\delta(a)) \subseteq S_\varepsilon(f(a))$$

The function $f : X \to Y$ is said to be continuous, if it is continuous at each point of X.

Example 26. If (X, d) is a discrete space then every function $f : X \to Y$ is continuous on X.

■ For any $a \in X$ if we choose $\delta < 1$. Then

$$S_\delta(a) = \{a\}$$

and so $$f(S_\delta(a)) = \{f(a)\} \subseteq S_\varepsilon(f(a)) \text{ holds for each positive } \varepsilon.$$

Example 27. If (X, d_1), and (Y, d_2) are any two metric spaces; then the constant function $f : X \to Y$ is continuous on X.

Theorem 13. *Let (X, d_1), and (Y, d_2) be any two metric spaces and f is a function from X into Y. Then f is continuous at $a \in X$ if and only if, for every sequence $\{a_n\}$ converging to 'a' we have*

$$\lim_{n\to\infty} f(a_n) = f(a)$$

i.e., $$a_n \to a \Rightarrow f(a_n) \to f(a).$$

First, let us suppose that the function f is continuous at a point $a \in X$ and $\{a_n\}$ is a sequence in X such that $\lim_{n\to\infty} a_n = a$.

Since f is continuous at a, therefore for any given $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$d_2(f(x), f(a)) < \varepsilon, \text{ whenever } d_1(x, a) < \delta \qquad ...(1)$$

Again, since $\lim_{n\to\infty} a_n = a$, therefore $\exists$ a positive integer n_0 such that

$$d_1(a_n, a) < \delta, \quad \forall\, n \geq m$$

From (1) putting $x = a_n$, we have

$$d_2(f(a_n), f(a)) < \varepsilon, \text{ whenever } d_1(a_n, a) < \delta, \quad \forall\, n \geq m$$

$$\Rightarrow \quad d_2(f(a_n), f(a)) < \varepsilon, \quad \forall\, n \geq m$$

$$\Rightarrow \quad \{f(a_n)\} \text{ converges to } f(a)$$

i.e., $\lim_{n\to\infty} f(a_n) = f(a)$.

We now assume that f is not continuous at a; and show that though there exists a sequence $\{a_n\}$ converging to 'a' yet the sequence $\{f(a_n)\}$ does not converge to $f(a)$.

Since f is not continuous at a, therefore there exists at least one $\varepsilon > 0$ such that for every $\delta > 0$ $d_2(f(x), f(a)) \geq \varepsilon$, and $d_1(x, a) < \delta$, for some $x \in X$.

Therefore, by taking $\delta = \frac{1}{n}$, we find that for each positive integer n there is $a_n \in X$ such that

$$d_2(f(a_n), f(a)) \geq \varepsilon, \text{ and } d_1(a_n, a) < \frac{1}{n}$$

Thus, the sequence $\{f(a_n)\}$ does not converge to $f(a)$.

This shows that continuous functions of one metric space into another are those functions which send every convergent sequence to a convergent sequence or in other words which preserve convergence.

Theorem 14. *Let (X, d_1), and (Y, d_2) be two metric spaces, then $f : X \to Y$ is continuous if and only if $f^{-1}(G)$ is open in X, whenever G is open in Y.*

We first assume that f is continuous. If G is any open subset in Y, we shall show that $f^{-1}(G)$ is open in X. (Recall that if G is a subset of Y then the set $\{x \in X : f(x) \in G\}$, consists of all points of X whose images lie in G, is denoted by $f^{-1}(G)$). If $f^{-1}(G) = \phi$, it is open; so we assume that $f^{-1}(G) \neq \phi$. Let $x \in f^{-1}(G)$. Then $f(x) \in G$ and since G is open, $\exists$ an open sphere $S_\varepsilon(f(x))$ such that

$$S_\varepsilon(f(x)) \subseteq G, \text{ for some } \varepsilon > 0$$

Now by definition of continuity, there exists an open sphere $S_\delta(x)$ such that

$$f(S_\delta(x)) \subseteq S_\varepsilon(f(x)), \text{ for } \delta > 0$$

But $S_\varepsilon(f(x)) \subseteq G$.

$$\therefore \quad S_\delta(x) \subseteq f^{-1}(G) \Rightarrow f^{-1}(G) \text{ is open}$$

Now we assume that $f^{-1}(G)$ is open in X, whenever G is open in Y, and show that f is continuous. Let x be an arbitrary point in X, and let $\varepsilon > 0$ be given. Let $S_\varepsilon(f(x))$ be an open sphere in Y centered at $f(x)$.

This open sphere in an open set, so its inverse image is an open set which contains x.

i.e., $$f^{-1}(S_\varepsilon(f(x))) \text{ is open in } X$$

Since $x \in f^{-1}(S_\varepsilon(f(x)))$, $\exists$ a $\delta > 0$ such that

$$S_\delta(x) \subseteq f^{-1}(S_\varepsilon(f(x)))$$

This implies $f(S_\delta(x)) \subseteq S_\varepsilon(f(x))$

Hence, f is continuous as x. Since x was taken to be an arbitrary point of X. Hence f is continuous at every point of X.

Note: From the above theorem we observe that continuous functions are precisely those which pull open sets back to open sets. It is to be noted that a continuous function need not take open sets to open sets.

Example 28. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \sin x$$

in the metric space $(\mathbf{R}, d)$. It can be easily seen that f is continuous. The open set $]0, 2\pi[$ in $(\mathbf{R}, d)$ is mapped to the closed set $[-1, 1]$ in $(\mathbf{R}, d)$.

Example 29. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x^2.$$

f is continuous on $\mathbf{R}$. f maps the open set $]-2, 2[$ onto the semi-closed set $[0, 4[$.

Ex. The function $f: X \to Y$ of a metric space (X, d_1) into a metric space (Y, d_2) is continuous iff the inverse image of every closed set contained in Y is closed.

[*Hint*: This follows from the preceding theorem by taking complements.]

* **Example 30.** Let (X, d_1) and (Y, d_2) be metric spaces. Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$, for every $A \subseteq X$.

■ Let f be continuous and A be any subset of X. Then $f^{-1}\overline{(f(A))}$ is closed in X.

Now $$f(A) \subseteq \overline{f(A)}$$

$$\Rightarrow \quad A \subseteq f^{-1}\overline{(f(A))}$$

$$\Rightarrow \quad \overline{A} \subseteq \overline{f^{-1}\overline{(f(A))}} = f^{-1}\overline{(f(A))}$$

i.e., $$f(\overline{A}) \subseteq \overline{f(A)}.$$

Conversely, suppose $$f(\overline{A}) \subseteq \overline{f(A)}, \quad \ldots(1)$$

for every subset A of X.

Let F be any closed subset of Y, then

using (1), we have $$f^{-1}(F) \subseteq X$$

$$f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))} = \overline{F} = F$$

$$\therefore \quad \overline{f^{-1}(F)} \subseteq f^{-1}(F)$$

Thus $f^{-1}(F)$ is closed in X.

Hence f is continuous.

Definition. Let (X, d_1) and (Y, d_2) be two metric spaces. A function $f : X \to Y$ is said to be ***uniformly continuous*** if for each $\varepsilon > 0$ there exists a $\delta > 0$ (depending on ε alone) such that

$$d_2(f(x), f(y)) < \varepsilon, \text{ whenever } d_1(x, y) < \delta, \quad \forall\, x, y \in X$$

Every function $f : X \to Y$ which is uniformly continuous on X is necessarily continuous on X.

However, converse may not be true. We shall see later that these two concepts are equivalent on concept metric spaces.

Example 31. For any non-empty subset A of a metric space (X, d) the function $f : X \to \mathbf{R}$ given by

$$f(x) = d(x, A), \text{ for } x \in X$$

is uniformly continuous. Also show that $f(x) = 0 \Leftrightarrow x \in \overline{A}$.

■ By definition we have for $x \in X$

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

By triangle inequality,

$$d(x, a) \le d(x, y) + d(y, a), \ \forall\, a \in A \subseteq X, \ x, y \in X$$

On taking infimum

$$d(x, A) = \inf_{a \in A} d(x, a) \le d(x, y) + \inf_{a \in A} d(y, a)$$

[$\because$ $d(x, y)$ is independent of a]

$$= d(x, y) + d(y, A)$$

$$\therefore \quad d(x, A) - d(y, A) \le d(x, y) \quad \text{...(1)}$$

Since (1) is true for each $x, y \in X$. Therefore on interchanging x and y,

$$d(y, A) - d(x, A) \le d(y, x) = d(x, y).$$

Thus

$$|\, d(x, A) - d(y, A)\, | \le d(x, y) \quad \text{...(2)}$$

Now for each $\varepsilon > 0$, choose a $\delta \le \varepsilon$

then

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x, y) < \delta \leq \varepsilon \quad \text{[using (2)]}$$

i.e., $|d(x, A) - d(y, A)| < \varepsilon$, whenever $d(x, y) < \delta$

Hence f is uniformly continuous on X.

For the second part, let $f(x) = 0$, *i.e.*, $d(x, A) = 0$.

This implies that there exists a sequence $\{a_n\}$ in A such that

$$d(a_n, x) \to d(x, A)$$

i.e., $$\lim_{n\to\infty} d(a_n, x) = d(x, A) = 0$$

$\Rightarrow$ $$a_n \to x, \text{ as } n \to \infty$$

Therefore for a given $\varepsilon > 0$, there exists a positive integer n_0, such that

$$a_n \in S_\varepsilon(x), \quad \forall\, n \geq n_0$$

In particular $a_{n_0} \in S_\varepsilon(x)$. But $a_{n_0} \in A$.

Therefore for each $\varepsilon > 0$, $S_\varepsilon(x)$ contains a point of A other than x.

Hence $x \in \overline{A}$.

Conversely, let $x \in \overline{A}$, then there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x.

$\therefore$ $$d(x_n, x) \to 0 \text{ as } n \to \infty$$

But, since $d(x, A) \leq d(x, x_n)$, $\forall\; n \in \mathrm{N}$, $x_n \in A$

and $$d(x, x_n) \to 0, \text{ as } n \to \infty$$

$\therefore$ $d(x, A) \leq 0$. Hence $d(x, A) = 0$

⁕ **Example 32.** Let (X, d) be a metric space then show that any disjoint pair of closed sets in X can be separated by disjoint open sets in X.

■ Let A and B be any closed subsets of X such that

$$A \cap B = \phi.$$

Define a mapping $f : X \to \mathbf{R}$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x \in X$$

f is well-defined.

Since $d(x, A) + d(x, B) \neq 0$, $\forall x \in X$. For if $d(x, A) + d(x, B) = 0$ for some $x \in X$, then $d(x, A) = 0$ and $d(x, B) = 0$.

This implies $x \in \overline{A} = A$ and $x \in \overline{B} = B$

i.e., $x \in A \cap B$, which is impossible ($\because A \cap B = \phi$)

f is continuous on X, Since the functions $x \to d(x, A)$, and $x \to d(x, B)$ are continuous on X.

Clearly
$$f(x) = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \in B \end{cases}$$

Let
$$G = \{x \in X : f(x) < \tfrac{1}{2}\}$$

then $G = f^{-1}(]-\infty, \frac{1}{2}[)$, being an inverse image of an open interval $]-\infty, \frac{1}{2}[$ under a continuous mapping f is an open subset of X.

Moreover $x \in A$ implies $f(x) = 0 < \frac{1}{2}$, *i.e.*, $x \in G$

$\therefore$
$$A \subseteq G.$$

Similarly, $H = \{x \in X : f(x) > \frac{1}{2}\}$ is an open set of X containing B. Also $G \cap H = \phi$. Hence the result.

Definition. Let (X, d), and (Y, d') be any two metric spaces. A function $f: X \to Y$ is said to be a **homeomorphism** if

(*i*) f is both one-one and onto,

(*ii*) f and f^{-1} are both continuous.

By theorem 14 it follows that a homeomorphism induces a $1-1$ correspondence between the open sets in X and open sets in Y.

Two metric spaces are said to be *homeomorphic* if there exists a homeomorphism between them. But not all metric properties are shared by homeomorphic spaces as is shown by the following example.

Example 33. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...,\}$ with d the usual metric on the subsets of **R**, and let $Y = \mathbf{N}$, the set of natural numbers with the usual metric $d'(= d)$. Then the function $f: X \to Y$ defined by $f(1/n) = n$ is a homeomorphism from (X, d) to (Y, a'). In fact every subset of X and every subset of Y is open in the respective spaces. But (X, d) is a bounded metric space and (Y, d') is not. Also (Y, d') is a complete metric space but (X, d) is not.

Definition. A function $f: X \to Y$ is called an *isometry* if
$$d(x, y) = d'(f(x), f(y)), \quad \forall\, x, y \in X.$$
Clearly each isometry is always one-to-one and uniformly continuous.

Two metric spaces are said to be *isometric* if there exists an isometry between them which is onto. It is easy to verify that if two metric spaces are isometric, then they are necessarily homeomorphic. But its converse may not be true as can be seen by the above example.

By definition it follows that isometric spaces possess all the same metric properties. Such spaces are metrically identical and differ only in names of their elements.

Theorem 15. *The image of a Cauchy sequence under a uniformly continuous function is again a Cauchy sequence.*

Let (X, d_1), and (Y, d_2) be two metric spaces and $f: X \to Y$ be uniformly continuous. Let $\{x_n\}$ be any Cauchy sequence in X; and let $\varepsilon > 0$ be given. Then, f being uniformly continuous, there exists a $\delta > 0$ (depending on ε) such that

$$d_2(f(x_m), f(x_n)) < \varepsilon, \text{ whenever } d_1(x_m, x_n) < \delta \quad \text{...(1)}$$

Since $\{x_n\}$ is Cauchy, corresponding to this $\delta > 0$ there exists a positive integer n_0 (depending on δ and so on ε) such that

$$d_1(x_m, x_n) < \delta, \text{ for } m, n \geq n_0$$

From (1) and (2), we conclude that

$$d_2(f(x_m), f(x_n)) < \varepsilon, \text{ for } m, n \geq n_0$$

Hence $\{f(x_n)\}$ is a Cauchy sequence in Y.

Remark: Note that a continuous function may not send a Cauchy sequence to a Cauchy sequence; as can be seen by the following example.
Let X be the set of all positive real numbers and d the usual metric on X, and $Y = \mathbf{R}$ be the set of all real numbers with d' the usual metric on $\mathbf{R}$. Then $f : X \to Y$ defined by

$$f(x) = \frac{1}{x}, \forall\, x \in X, \text{ is continuous on } X$$

Now $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence in $X \left[\because \left|\frac{1}{n} - \frac{1}{m}\right| \to 0 \text{ as } m, n \to \infty\right.$. But $\left\{f\left(\frac{1}{n}\right)\right\} = \{n\}$ is not a Cauchy sequence in Y. Hence f cannot be a uniformly continuous function.

(Statement only)

Theorem 16. *Let (X, d_1) be a metric space, and (Y, d_2) be a complete metric space. If f is a uniformly continuous function from a subset A of X into Y, then f can be extended uniquely to a uniformly continuous function g from A into Y.*

We shall prove the theorem in the following steps.

(1) Existence of $g : \overline{A} \to Y$

(2) Uniform continuity of g

(3) Uniqueness of g.

(1) Let $\{a_n\}$ be any convergent sequence in A converging to a point $x \in \overline{A}$. Also $\{a_n\}$ being convergent must be a Cauchy sequence and since f is uniformly continuous, its image $\{f(a_n)\}$ is a Cauchy sequence in Y. Again Y is given to be complete, the sequence $\{f(a_n)\}$ must be a convergent sequence in Y, and so there exists a point y in Y such that

$$f(a_n) \to y, \text{ i.e., } \lim_{n\to\infty} f(a_n) = y$$

Now we shall show that y depends only on x and not on the sequence $\{a_n\}$.

Let $\{b_n\}$ be another sequence in A such that $\{b_n\}$ converges to x then by the triangle inequality in (X, d_1) we have

$$d_1(a_n, b_n) \leq d_1(a_n, x) + d_1(x, b_n)$$

$$\Rightarrow \quad d_1(a_n, b_n) \to 0, \text{ as } n \to \infty \quad (\because a_n \to x, \text{ and } b_n \to x, \text{ as } n \to \infty)$$

And by the uniform continuity of f

$$d_2(f(a_n), f(b_n)) \to 0, \text{ as } n \to \infty$$

Now from the triangle inequality in (X, d_2)

$$d_2(f(b_n), y) \le d_2(f(b_n), f(a_n)) + d_2(f(a_n), y)$$

we have

$$d_2(f(b_n), y) \to 0, \text{ as } n \to \infty$$

i.e.,
$$\lim_{n\to\infty} f(b_n) = y.$$

This shows that y is independent of the sequence $\{a_n\}$ in A.

Thus if we define

$$y = g(x)$$

Then g extends f from A to $\overline{A}$ which can be seen as follows:

Let $x \in A$, then $x \in \overline{A}$.

Taking $a_n = x, \forall n$, the sequence $\{a_n\}$ is a constant sequence in A and so $a_n \to x$.

Then $g(x) = \lim_{n\to\infty} f(a_n)$

But since $f(a_n) = f(x)$ we get

$$\lim_{n\to\infty} f(a_n) = f(x), \forall x \in A$$

$\therefore f(x) = g(x), \forall x \in A$

Thus g extends f to $\overline{A}$.

(2) Let $\varepsilon > 0$ be given. By uniform continuity of f we can find $\delta > 0$ such that for all $a, b \in A$ we have

$$d_1(a, b) < \delta \Rightarrow d_2(f(a), f(b)) < \varepsilon \quad \text{...(1)}$$

Let x, y be any point in $\overline{A}$ such that

$$d_1(x, y) < \delta$$

there exist sequences $\{a_n\}$ and $\{b_n\}$ in A such that $a_n \to x$ and $b_n \to y$ respectively.

i.e.,
$$d_1(a_n, x) \to 0 \text{ as } n \to \infty, \text{ and } d_1(b_n, y) \to 0 \text{ as } n \to \infty$$

For $r = \dfrac{\delta - d_1(x, y)}{2} > 0$, $\exists$ a positive integer n_0 (depending on r) such that

$$d_1(a_n, x) < r, d_1(b_n, y) < r, \quad \forall n \ge n_0$$

Now
$$d_1(a_n, b_n) \le d_1(a_n, x) + d_1(x, y) + d_1(y, b_n)$$
$$< r + d_1(x, y) + r = \delta, \quad \forall n \ge n_0$$

It follows from (1) that

$$d_2(f(a_n), f(b_n)) < \varepsilon, \quad \forall n \ge n_0 \quad \text{...(2)}$$

By definition of g, we have

$$f(a_n) \to g(x), \text{ and } f(b_n) \to g(y) \text{ as } n \to \infty$$

i.e., for each $\varepsilon > 0$, $\exists$ positive integers m_1, m_2 such that

$$d_2(f(a_n), g(x)) < \varepsilon/3, \quad \forall\, n \geq m_1$$

and

$$d_2(f(b_n), g(x)) < \varepsilon/3, \quad \forall\, n \geq m_2$$

By triangle inequality in Y

$$d_2(g(x), g(y)) \leq d_2(g(x), f(a_n)) + d_2(f(a_n), f(b_n)) + d_2(f(b_n), g(y)).$$

Using (2) and definition of g,

$$d_2(g(x), g(y)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad \forall\, n \geq m = \max(m_1, m_2, n_0)$$

Thus $\quad d_1(x, y) < \delta \Rightarrow d_2(g(x), g(y)) < \varepsilon, \quad \forall\, x, y \in \overline{A}$

Hence g is uniformly continuous.

(3) We shall now show that g is unique. Let, if possible, there be another extension $h : \overline{A} \to Y$ of f to $\overline{A}$ such that h is uniformly continuous.

We have for all $x \in A$

$$g(x) = f(x) = h(x)$$

and for all $x \in \overline{A}$

$$g(x) = f(x) = h(x), \text{ by taking limits}$$

Hence $\quad g(x) = h(x), \quad \forall\, x \in \overline{A}$

This shows that g is unique.

• 4.1 Banach Fixed Point Theorem

Definition. Let (X, d) be a metric space. A mapping $f: X \to X$ is said to be a *contraction mapping*, if there exists a positive real number α with $\alpha < 1$, such that

$$d(f(x), f(y)) \leq \alpha\, d(x, y), \quad \forall\, x, y \in X$$

we observe that, applying f to each of the two points of the space contracts the distance between them. Obviously f is continuous.

Example 34. If $x = \{x_n\} \in l_2$, then $f(x) = \left\{\dfrac{x_n}{2}\right\}$ is a contraction mapping on l_2. For if $y = \{y_n\}$ is any other point of l_2, then

$$d(f(x), f(y)) = \left(\sum_{n=1}^{\infty}\left(\frac{x_n}{2} - \frac{y_n}{2}\right)^2\right)^{1/2} = \frac{1}{2}\, d(x, y)$$

*Example 35.** If $f(x) = x^2, 0 \leq x \leq \dfrac{1}{3}$. Then f is a contraction mapping on $[0, \dfrac{1}{3}]$ with the usual metric d.

$$d(f(x), f(y)) = d(x^2, y^2) = |x^2 - y^2| = |x - y||x + y| < \frac{2}{3}|x - y|$$

$$\therefore \quad d(f(x), f(y) \leq \frac{2}{3} d(x, y)$$

Definition. A point $x \in X$ is called a *fixed point* of the mapping $f : X \to X$, if $f(x) = x$.

Theorem 17. ***Banach fixed point theorem.*** *Any contraction mapping f of a non-empty complete metric space (X, d) into itself has a unique fixed point.*

For all $x, y \in X$, we have $d(f(x), f(y)) \leq \alpha\, d(x, y)$ ...(1)

for some $\alpha, 0 < \alpha < 1$

This implies that f is continuous.

Now choose any point $x_0 \in X$. Let us define a sequence $\{x_n\}$ by

$$x_1 = f(x_0), x_2 = f(x_1), \ldots, x_{n+1} = f(x_n), \ldots$$

Then

$$x_n = f^n(x_0), \quad \forall\, n \in \mathbf{N}$$

We shall show that the sequence $\{x_n\}$ is Cauchy. For each positive integer n we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(f(x_{n-1}), f(x_n)) \leq \alpha\, d(x_{n-1}, x_n) \\ &\leq \alpha^2\, d(x_{n-2}, x_{n-1}) \\ &\leq \alpha^3\, d(x_{n-3}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n\, d(x_0, x_1) \end{aligned} \quad \text{...(2)}$$

By triangle inequality, we have for $n \geq m$

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\leq \alpha^m\, d(x_0, x_1) + \alpha^{m-1}\, d(x_0, x_1) + \ldots + \alpha^{n-1}\, d(x_0, x_1) \\ &= \alpha^m\, [1 + \alpha + \alpha^2 + \ldots + \alpha^{n-m-1}]\, d(x_0, x_1) \\ &= \frac{\alpha^m\,(1 - \alpha^{n-m})}{1 - \alpha}\, d(x_0, x_1) \\ &< \frac{\alpha^m\, d(x_0, x_1)}{1 - \alpha} \to 0, \text{ as } m \to \infty \qquad [\because \alpha < 1] \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence, and X being complete implies $x_n \to x$, for some $x \in X$.

Since f is continuous, therefore we have

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

Hence,

$$f(x) = x$$

To prove uniqueness, suppose $f(y) = y$, for some $y \in X$ then

$$d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y)$$

Since $\alpha < 1$, and $d(x, y) \geq 0$ therefore we must have

$$d(x, y) = 0, \text{ i.e., } x = y.$$

EXERCISE

1. Let $\{x_n^k\}$ be a sequence in H_∞ (H_∞ as defined in Q. 9 Exercise, p. 716). Let $x = \{x_n\} \in H_\infty$, then prove that $\{x^k\}$ converges to x in H_∞ if and only if $\lim_{k\to\infty} x_n^k = x_n, \ \forall n \in \mathrm{N}$

   [*Hint*: Here $d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}$, $|x_n| \leq 1$, $|y_n| \leq 1$, $\forall n \in \mathbf{N}$, suppose $\lim_{k\to\infty} x_n^k = x_n$, $\forall n$.

   Choose a positive integer m such that $\sum_{n=m+1}^{\infty} \frac{2}{2^n} < \varepsilon/2$. Show that $\exists$ a positive integer m_1 such that if

   $$k \geq m_1, \text{ then } \sum_{n=m+1}^{\infty} \frac{|x_n^k - x_n|}{2^n} < \varepsilon/2$$

   Deduce that if $k \geq m_1$

   $$d(x^k, x) < \varepsilon$$

   For the converse note that

   $$d(x^k, x) = \sum_{n=1}^{\infty} \frac{|x_n^k - x_n|}{2^n} < \sum_{n=1}^{\infty} \varepsilon/2^n, \quad \forall n \geq m_1]$$

2. Let $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} y_n = y$, where $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers in the metric space (X, d) and $x, y \in X$. Prove that the sequence of real numbers $\{d(x_n, y_n)\}$ converges to the real number $d(x, y)$.
   [*Hint*: $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$].
3. Show that a Cauchy sequence is convergent $\Leftrightarrow$ It has a convergent subsequence.
4. Prove that if (X, d) is a complete space, and each $x \in X$ is a limit point of X, then X is uncountable.
5. Give an example of a complete metric space (X, d) and a sequence of non-empty closed sets $\{A_n\}$ in X with $A_1 \supseteq A_2 \supseteq A_3 \ldots \supseteq A_n \ldots$ such that $\bigcap_{n=1}^{\infty} A_n = \phi$

   [*Hint*: The space **R** of real numbers with the usual metric is a complete metric space. Consider the sequence of closed sets

   $$A_n = [n, \infty[, n \in \mathbf{N}, \ \bigcap_{n=1}^{\infty} A_n = \phi]$$

6. Give an example of a homeomorphism $f: X \to Y$ and a Cauchy sequence $\{x_n\}$ in X for which $\{f(x_n)\}$ is not Cauchy in Y.

   [*Hint*: Let $X = \{1, \frac{1}{2}, \frac{1}{3} \ldots\}$ with d the usual metric and d' the discrete metric. Then the identity function $I : (X, d) \to (X, d')$ is a homeomorphism, but the Cauchy sequence $\{1/n\}$ in (X, d) has its image $\{1/n\}$ in (X, d') which is not Cauchy].

7. Let X and Y be metric spaces and A be a non-empty subset of X. If f and g are continuous function from X into Y such that $f(x) = g(x)$, for every x in A, show that $f(x) = g(x)$ for every $x \in \overline{A}$.
8. Let f be a continuous real-valued function defined on $\mathbf{R}$ which satisfies the functional equation $f(x+y) = f(x) + f(y)$. Show that the function must have the form $f(x) = mx$ for some real number m.
9. Let $\{x_n\} \in l_2$. Prove that f defined by $f\{x_n\} = \sum_{n=1}^{\infty} a_n x_n$, a_n's are real numbers, is a continuous real-valued function on l_2.
10. Let f be a real-valued function on a metric space X. Prove that f is continuous on X if and only if the following sets $\{x : f(x) < c\}$ and $\{x : f(x) > c\}$ are open in X for every $c \in \mathbf{R}$.
11. Let (X, d) be a metric space such that $d(x, y) \leq 1$, $\forall\, x, y \in X$, and let $\{x_n\}$ be a sequence in X. If $f(x) = \{d(x, x_n)\}$, $x \in X$, then prove that
    (*i*) f is a continuous function from X into H_∞.
    (*ii*) If the range set of $\{x_n\}$ is dense in X, then f is one to one.
12. Give an example of a function $f : X \to Y$ which is one-to-one onto Y and continuous on X but not a homeomorphism.

    [*Hint*: Let $X = [0, 1]$ and let d be the usual metric and d' the discrete metric. Then the identity function $I : (X, d') \to (X, d)$ is one to one continuous but not a homeomorphism.]
13. If (X, d) is a complete metric space and if $\mathbf{F}$ is a family of real-valued continuous functions defined on X such that the set $\{f(x): f \in \mathbf{F}\}$ is bounded for every $x \in X$. Then there is a non-empty open set $G \subseteq X$, and an $M > 0$ such that $|f(x)| \leq M$ for every $x \in G$ and for every $f \in \mathbf{F}$. [This is known as *uniform boundedness principle*].
14. Prove that if $f : X \to Y$ is an isometry from X onto Y, then for every Cauchy sequence $\{x_n\}$ in X, $\{f(x_n)\}$ is a Cauchy sequence in Y.
15. Prove that if the spaces (X, d) and (Y, d') are isometric then either they are both complete or neither is complete.
16. Let (X, d) be a metric space with $x_0 \in X$. Define $f : X \to \mathbf{R}$ by $f(x) = d(x, x_0)$. Prove that f is uniformly continuous on X.
17. Let $\phi : C[0,1] \to C[0,1]$ be defined by $\phi(f) = \alpha \int_0^x t\, f(t)dt$, where α is a constant.
    (*a*) Find α such that ϕ is a contraction mapping.
    (*b*) Also show that for each value of α, ϕ has a unique fixed point.
18. Prove the 'converse' of the Banach's fixed point theorem: if for each non-empty closed subset A of a metric space X, and for each contraction mapping $f : A \to A$, f has a fixed point, then X is complete.

5. COMPACTNESS

The concept of compactness is an abstraction of an important property known as 'Heine Borel Property' posed by subsets of $\mathbf{R}$ which are closed and bounded. Heine Borel theorem states that if $I \subseteq \mathbf{R}$ is a closed interval, any family of open interval in R whose union contains I has a finite subfamily whose Compactness is concerned with covering the sets by open sets. Before defining compactness we need the following definitions.

Definition. Let (X, d) be a metric space. A family of subsets $\{A_\alpha\}$ in X is called a *cover* of any subset A of X if $A \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha A$, Λ is any non-empty index set. If each A_α, $\alpha \in \Lambda$ is open in X, then the cover $\{A_\alpha\}$ is called an *open cover* of A.

A subfamily of the family $\{A_\alpha\}$ which itself is an open cover is called an *open subcover* of A. If the number of members in the subfamily is finite it is called a *finite subcover* of A.

Definition. A subset A of a metric space (X, d) is said to be *compact* if every open cover of A admits of a finite subcover, *i.e.*, for each family of open subsets $\{G_\alpha\}$ of X for which $\bigcup_{\alpha \in \Lambda} G_\alpha \supseteq A$, there exists a finite subfamily say $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $A \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$.

A metric space (X, d) is *compact* if X is itself compact, *i.e.*, for each family of open subsets $\{G_\alpha\}$ of X for which $\bigcup_{\alpha \in \Lambda} G_\alpha = X$, there exists a finite subfamily $\{G_{\alpha_i} : i = 1, 2, \ldots n\}$ such that

$$X = \bigcup_{i=1}^{n} G_{\alpha_i}$$

ILLUSTRATIONS

1. Any closed interval with the usual metric is compact.
2. The discrete space (X, d), where X is a finite set, is compact.
3. The space $(\mathbf{R}, d)$ where $\mathbf{R}$ is the set of reals and d is the usual metric is not compact, for the cover $\{]-n, n[: n \in \mathbf{N}\}$ is such that $\bigcup_{n=1}^{\infty}]-n, n[= \mathbf{R}$, which do not have a finite subcover.

Example 36. Prove that the open interval]0, 1[with the usual metric is not compact.

■ The family of open intervals $\left\{\left]\frac{1}{n}, 1\right[: n = 2, 3 \ldots\right\}$ is such that $\bigcup_{n=2}^{\infty} \left]\frac{1}{n}, 1\right[=]0, 1[$. Therefore $\left\{\left]\frac{1}{n}, 1\right[: n = 2, 3, \ldots\right\}$ is an open cover of]0, 1[, which has no finite subcover.

Example 37. Let X be an infinite set with the discrete metric. Show that (X, d) is not compact.

■ For each $x \in X$, $\{x\}$ is open in X

Also $\bigcup_{x \in X} \{x\} = X$

Therefore $\{\{x\} : x \in X\}$ is an open cover of X and since X is infinite, this open cover has no finite subcover.

Theorem 18. *Every closed subset of a compact metric space is compact.*

Let (X, d) be any compact metric space and F be any non-empty closed subset of X. We shall show that F is compact.

Let $\{G_\alpha : \alpha \in \Lambda\}$ be a family of open sets in X such that

$$\bigcup_{\alpha \in \Lambda} G_\alpha \supseteq F$$

Then $\left(\bigcup_{\alpha \in \Lambda} G_\alpha\right) \cup (X - F)$ is an open cover of X and by compactness of X, it has a finite subcover, say,

$$G_{\alpha_1}, G_{\alpha_2} \ldots G_{\alpha_n}, X - F$$

$\therefore$

$$\bigcup_{i=1}^{n} G_{\alpha_i} \cup (X - F) = X$$

or

$$\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq F$$

Hence F is compact.

Note: This theorem shows that each closed subset of a compact metric space is compact. On the real line the closed set N is not compact in R. Also the closed set $F =]0, 1[$ is not compact in (X, d) where $X =]0, 2]$ and d is the usual metric. Observe that in these examples the space is not compact.

2 **Theorem 19.** *Every compact subset F of a metric space (X, d) is closed.*

Let F be any compact set. To prove that F is closed we shall show that F^c is open.

Let $y \in F^c$ and $x \in F$ then $x \neq y$

$\therefore$ $d(x, y) > 0$; let $d(x, y) = r_x$, Then the open spheres $S_{\frac{1}{2}r_x}(x)$ and $S_{\frac{1}{2}r_x}(y)$ are such that

$$S_{\frac{1}{2}r_x}(x) \cap S_{\frac{1}{2}r_x}(y) = \phi$$

For if z belongs to both $S_{\frac{1}{2}r_x}(x)$ and $S_{\frac{1}{2}r_x}(y)$, then

$$d(z, x) < \frac{1}{2}r_x \text{ and } d(z, y) < \frac{1}{2}r_x$$

and by the triangle inequality

$$d(z, x) \le d(x, z) + d(z, y) < \frac{1}{2}r_x + \frac{1}{2}r_x = r_x$$

which contradicts the fact that $d(x, y) = r_x$.

Now consider the collection $\{S_{\frac{1}{2}r_x}(x) : x \in F\}$ of open spheres of F. This collection is such that

$$\bigcup_{x \in F} S_{\frac{1}{2}r_x}(x) \supseteq F$$

Since F is compact, there exists a finite number of open spheres, say,

$$S_{\frac{1}{2}r_{x_1}}(x_1), S_{\frac{1}{2}r_{x_2}}(x_2), \mathrm{K}, S_{\frac{1}{2}r_{x_n}}(x_n)$$

such that

$$\bigcup_{i=1}^{n} S_{r_{xi}/2}(x_i) \supseteq F$$

Let $A_y = \bigcap_{i=1}^{n} S_{\frac{1}{2}r_{xi}}(y)$. The set A_y is an open set, being the intersection of open spheres, containing y.

Since $S_{\frac{1}{2}r_{xi}}(x) \cap S_{\frac{1}{2}r_{xi}}(y) = \phi$, for each i, therefore we have

$$S_{\frac{1}{2}r_{xi}}(x) \cap A_y = \phi$$

And so $\left(\bigcup_{i=1}^{n} S_{\frac{1}{2}r_{x_i}}(x_i)\right) \cap A_y = \phi$

This implies

$$F \cap A_y = \phi, \text{ or } A_y \subseteq F^c$$

Now $\bigcup_{y \in F^c} A_y = F^c$ and each A_y is open, therefore F^c is open and hence F is closed.

Note: The converse of the above theorem may not be true. For example $[0, \infty[$, is closed but not compact. The family of intervals $\{[0, n[: n \in \mathbb{N}\}$ is such that each set $[0, n[$ is open in $[0, \infty[$ and

$$\bigcup_{n=1}^{\infty} [0, n[= [0, \infty[.$$

This family is an open cover of $[0, \infty[$, which has no finite subcover.

Moreover, if a set in any metric space is not closed then it cannot be compact. This can be seen as follows:

Let A be any non-closed set in any metric space (X, d) and let a be a limit point of A which is not in A. Then the family of sets

$$(X - S_{1/n}(a) : n = 1, 2, 3, \ldots\}$$

is an open cover of A for which there is no finite subcover. Hence A is not compact.

Corollary. *A subset A of a compact metric space (X, d) is itself compact if and only if it is closed in (X, d).*

Theorem 20. *Every compact subset A of a metric space (X, d) is bounded.*

Suppose that A is compact and consider an open cover of A consisting of open spheres of radii–1

i.e.,
$$A \subseteq \bigcup_{x \in A} S_1(x)$$

Since A is compact, there must exists $x_1, x_2 \ldots x_n$ such that

$$A \subseteq \bigcup_{i=1}^{n} S_1(x_i)$$

Now let $M = \max d(x_i, x_j),\ 1 \le i \le j \le n.$

Let $x, y \in A$ be any two elements then there exist elements x_i and x_j such that

$$x \in S_1(x_i), \text{ and } y \in S_1(x_j)$$

∴ By triangle inequality

$$d(x,y) \le d(x, x_i) + d(x_i, x_j) + d(x_j, y) \le 1 + M + 1 = M + 2$$

Hence A is bounded.

Note: Every compact subset of a metric space is closed and bounded but the converse need not be true, which can be seen by the following counter example. Let $X = [0, 1]$, and suppose (X, d) is a discrete space. Then the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is closed because every point of A is a limit point of A, it is also bounded because $d(x, y) < 1$ for all $x, y \in A$ and so $d(A) \le 1$. But the open cover $\left\{S_1\left(\frac{1}{n}\right)\right\} = \left\{\frac{1}{n}\right\}$ does not admit of a finite subcover. Hence A is not compact.

statement only

Theorem 21. *Heine-Borel theorem.* *Every closed and bounded subset of the real line is compact.*

Let A be any closed and bounded subset of $\mathbf{R}$. Since A is bounded there exists real numbers a and b such that $A \subseteq [a, b]$.

A being a closed subset of $\mathbf{R}$, it is a closed subset of $[a, b]$. Now since every closed subset of a compact metric space is compact therefore it is enough to show that $[a, b]$ is compact.

If $a = b$, there is nothing to prove, so we assume $a < b$. If possible, let $[a, b]$ be not compact, then there exists an open cover $\{G_\alpha\}$ of $[a, b]$ which has no finite subcover.

Let $a_1 = a,\ b_1 = b$ and $c_1 = \dfrac{a_1 + b_1}{2}$

So $[a, b] = [a_1, b_1] = [a_1, c_1] \cup [c_1, b_1]$.

At least one of these intervals in the union cannot be covered by a finite subfamily of $\{G_\alpha\}$. Let $[a_2, b_2]$ denote one of these intervals with this property. Thus $[a_2, b_2] \subseteq [a_1, b_1]$, and $b_2 - a_2 = \dfrac{1}{2}(b_1 - a_1)$,

Let $c_2 = \dfrac{a_2 + b_2}{2}$.

Therefore $[a_2, b_2] = [a_2, c_2] \cup [c_2, b_2]$. As before at least one of these intervals in the union cannot be covered by a finite subfamily of $\{G_\alpha\}$. Denote that interval by $[a_3, b_3]$. Then,

$$[a_3, b_3] \subseteq [a_2\ b_2] \subseteq [a, b_1], \text{ and } b_3 - a_3 = \frac{1}{2}(b_2 - a_2) = \frac{1}{2^2}(b_1 - a_1)$$

Continuing this process, we obtain a sequence $\{[a_n, b_n]\}$ of closed intervals such that each interval $[a_n, b_n]$ cannot be covered by a finite subfamily of $\{G_\alpha\}$, and

$$[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n], \quad \forall\, n \in \mathbf{N}$$

with $b_n - a_n = \dfrac{1}{2^{n-1}}(b_1 - a_1)$ which tends to zero as n tends to ∞.

i.e.,
$$d[a_n, b_n] = |a_n - b_n| \to 0, \text{ as } n \to \infty.$$

Since $\mathbf{R}$ is complete, and $\{[a_n, b_n]\}$ is a non-empty decreasing sequence of closed subsets of $\mathbf{R}$ such that

$$d([a_n, b_n]) \to 0, \text{ as } n \to \infty$$

Therefore, by Cantor's intersection theorem

$\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains exactly one element say x.

i.e.,
$$x \in [a_n, b_n], \ \forall\, n, \text{ and } [a_n, b_n] \subseteq [a_1, b_1] \subseteq [a, b], \forall\, n \qquad \ldots(1)$$

And so $x \in [a, b]$. Now since $\{G_\alpha\}$ is an open cover of $[a, b]$, therefore $x \in G_\alpha$ for some α, where

G_α is open in $[a, b]$.

This implies

$$G_\alpha = H \cap [a, b], \text{ where } H \text{ is open in } \mathbf{R}$$

Now $x \in G_\alpha \Rightarrow x \in H$

But H is open in **R**. Therefore there exists an open sphere

$$S_\varepsilon(x) =]\, x - \varepsilon, x + \varepsilon\, [, \varepsilon > 0$$

such that

$$x \in]\, x - \varepsilon, x + \varepsilon\, [\subseteq H \quad ...(2)$$

Since $b_n - a_n = \frac{1}{2^{n-1}}(b_1 - a_1) \to 0$, as $n \to \infty$

Therefore, for above $\varepsilon > 0$, there exists a positive integer n_0 such that

$$b_n - a_n < \varepsilon, \ \forall\, n \geq n_0$$

In particular,

$$b_{n_0} - a_{n_0} < \varepsilon$$

Therefore $[a_{n_0}, b_{n_0}] \subseteq]\, x - \varepsilon, x + \varepsilon\, [\subseteq H$, by (1) and (2)

Now $[a_{n_0}, b_{n_0}] \subseteq [a, b]$, and $[a_{n_0}, b_{n_0}] \subseteq H$

Therefore $[a_{n_0}, b_{n_0}] \subseteq G_\alpha$, *i.e.*, $[a_{n_0}, b_{n_0}]$ is covered by a single set G_α of the cover $\{G_\alpha\}$ which contradicts the fact that $[a_{n_0}, b_{n_0}]$ is not covered by any finite subfamily of $\{G_\alpha\}$. Hence our assumption that $[a, b]$ is not compact is wrong. Thus $[a, b]$ is compact.

Note: Converse of Heine-Borel theorem is also true. If A is a compact subset of **R**, then A is closed and bounded. Let A be a compact subset of **R**, and $\bigcup_{n=1}^{\infty}]-n, n[= \mathbf{R} \supseteq A$. So there exists positive integers $n_1, n_2, ..., n_k$ such that $\bigcup_{i=1}^{k}]-n_i, n_i[\supseteq A$. Take $m = \max(n_1, n_2, ..., n_k)$, then $A \subseteq]-m, m[$.

Example 38. Consider the bounded set $A =]0, 1]$, A is not closed. Since 0 is a limit point of A which does not belong to A. Let $G = \{]1/n, 2[\ ; n \in \mathbf{N}\}$. G is an open cover of A and there is no finite subset of G which is a cover of A.

Example 39. Let $A = [0, \infty]$, A is a closed set, but it is not bounded. Consider the family of the sets

$$G = \{]n - 2, n[\ : n \in \mathbf{N}\}.$$

G is an open even cover of A. But G has no finite subcover for A.

Remarks:

1. Heine-Borel theorem does not hold in a general metric space as can be seen by the following example: Let (X, d) be the discrete space and X is infinite, X is bounded as the distance between any two of its members is at the most one, X being the whole space is closed. Also for $\varepsilon = \frac{1}{2}$,

$$S_\varepsilon(x) = \{x\}, x \in X \text{ so } X = \bigcup_{x \in X} S_\varepsilon(x).$$

Thus, the open cover $\{S_\varepsilon(x) : x \in X\}$ of X admits of no finite subcover.

Hence X is not compact.

2. However, from Theorems 19 and 20 it follows that the converse of Heine-Borel theorem is true for any general metric space.

Theorem 22. *Continuous image of a compact set is compact.*

Let (X, d_1) be a compact metric space and f be a continuous function from X into the metric space (Y, d_2) then $f(X)$, the image of X under f is compact in Y. Let $\{V_\alpha\}$ be any open cover of $f(X)$; which we denote by $Y_1 (Y_1 \subseteq Y)$, *i.e.*, each V_α is open in Y_1, and

$$Y_1 = \bigcup_{\alpha \in \Lambda} V_\alpha, \Lambda \text{ is any index set}$$

$$\therefore \quad X = f^{-1}(Y_1) = f^{-1}(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} f^{-1}(V_\alpha)$$

Since V_α is open in Y_1, and f is continuous.

$\therefore$ $f^{-1}(V_\alpha)$ is open in X. Hence $\{f^{-1}(V_\alpha)\}$ is an open cover of X. But X is compact, therefore there exists a finite subcover, say

$\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$ of the open cover $\{f^{-1}(V_\alpha)\}$ of X such that

$$X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha i})$$

Let $y \in Y_1 = f(X)$. Then there exists an $x \in X$ such that

$$y = f(x).$$

Since $x \in X$ and $X = \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$; $\therefore x \in \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i})$, and hence $x \in f^{-1}(V_{\alpha_i})$, for some i

$$f(x) \in V_{\alpha_i}, \text{ for some } i$$

or

$$y \in V_{\alpha_i}, \text{ for some } i$$

i.e.,

$$\therefore \quad Y_1 = \bigcup_{i=1}^{n} V_{\alpha_i}$$

$\therefore \{V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n}\}$ is a finite subcover of the open cover $\{V_\alpha\}$ of Y_1. Therefore, Y_1 is compact.

If f is onto then $Y_1 = Y$, and so Y is compact.

Example 40. Let A be a non-empty compact subset of a metric space (X, d) and let F be a closed subset of X such that $A \cap F = \phi$, then $d(A, F) > 0$.

■ If possible $d(A, F) = 0$. Since the function $x \to d(x, F)$ is continuous on A, and A being compact implies $d(x, F)$ assumes a minimum value for some $x \in A$, say x_0. And so

$$d(x_0, F) = d(A, F) = 0$$

This implies $x_0 \in \overline{F} = F$

Hence $x_0 \in A \cap F$, *i.e.*, $A \cap F \neq \phi$, which is a contradiction.

5.1 Compactness and Finite-Intersection Property

Definition. A family of subset of a non-empty set X is said to have the *finite-intersection property* (*FIP*) if every finite subfamily has non-empty intersection.

Ex. The family $\{[-n, n] : n \in \mathbf{N}\}$ of closed (intervals) subsets of $\mathbf{R}$ has the FIP.

Theorem 23. *The metric space (X, d) is compact if and only if every family of closed sets in (X, d) with the FIP has non-empty intersection.*

Let (X, d) be compact, and let $\{F_\alpha\}$ be any family of closed sets in (X, d) with FIP. If possible, let $\bigcap_{\alpha\in\Lambda} F_\alpha$ is empty, then on taking complements in X, we get

$$\bigcup_{\alpha\in\Lambda} F_\alpha^c = X$$

Thus the collection $\{F_\alpha^c\}$ of open sets, being complements of closed sets F_α in (X, d), is an open cover of the compact metric space X; which has a finite subcover say $F_{\alpha_1}^c, F_{\alpha_2}^c, \ldots, F_{\alpha_n}^c$

i.e.,
$$\bigcup_{i=1}^{n} F_{\alpha_i}^c = X$$

Taking complements

$$\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$$

which is a contradiction to the fact that $\{F_\alpha\}$ has the FIP. Hence $\bigcap_{\alpha\in\Lambda} F_\alpha$ is non-empty.

Conversely, suppose every family of closed sets in (X, d) with the FIP has non-empty intersection. In order to show that (X, d) is compact, let $\{G_\alpha\}$ be an open cover of X. Then $\bigcup_{\alpha\in\Lambda} G_\alpha = X$

Taking complements

$$\bigcap_{\alpha\in\Lambda} G_\alpha^c = \phi$$

Therefore $\{G_\alpha^c\}$ is a family of closed sets in X, whose intersection is empty. Therefore, by hypothesis this family does not have the FIP and so there exists a finite subfamily, say $G_{\alpha_1}^c, G_{\alpha_2}^c \ldots G_{\alpha_n}^c$ such that

$$\bigcap_{i=1}^{n} G_{\alpha_i}^c = \phi$$

i.e.,
$$\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)^c = \phi, \text{ or } \bigcup_{i=1}^{n} G_{\alpha_i} = X$$

Hence $\{G_{\alpha_i} : i = 1, 2, \ldots, n\}$ is a finite subcover of $\{G_\alpha\}$, and so (X, d) is compact.

Definition. A metric space (X, d) is said to have ***Bolzano-Weierstrass Property*** if every infinite subset of X has a limit point.

The space **R** with the usual metric does not have Bolzano-Weierstrass property for the set $\{1, 2, 3,...,\}$ is an inifinite set in **R** with no limit points.

Definition. A metric space (X, d) is ***sequentially compact*** if every sequence $\{x_n\}$ in X has a convergent subsequence.

Theorem 24. *A metric space (X, d) is sequentially compact if and only if it has the Bolzano-Weierstrass property.*

Let (X, d) be sequentially compact metric space. Let A be an infinite subset of X. We shall show that A has a limit point. Since A is infinite, we can always extract a sequence say $\{a_n\}$ of distinct points from A. Since X is sequentially compact therefore $\{a_n\}$ contains a convergent subsequence $\{a_{n_k}\}$.

Let $\lim_{k\to\infty} a_{nk} = a$. Consider any neighbourhood $S_\varepsilon(a)$ of 'a'. Then $\varepsilon > 0$ implies there exists a positive integer m such that

$$d(a_{n_k}, a) < \varepsilon, \quad \forall\, k \geq m$$

$$\Rightarrow \qquad a_{n_k} \in S_\varepsilon(a), \quad \forall\, k \geq m$$

This implies a is a limit point of A.

Conversely, suppose (X, d) has the Bolzano-Weierstrass property. Let $\{x_n\}$ be an arbitrary sequence in (X, d) and $S = \{x_n : n \in \mathbf{N}\}$ be its range. There are two possibilities—either S is finite or infinite.

(*i*) If S is finite, then there must exist at least one number $x \in S$ such that $x_n = x$ for infinitely many values of n and so the sequence $\{x_n\}$ has a constant subsequence and hence convergent.

(*ii*) When S is infinite, then by hypothesis S has a limit point say x_0. Therefore for each $\varepsilon > 0$, the set $S \cap S_\varepsilon(x_0)$ is infinite. Choose

$x_{n_1} \in S \cap S_1(x_0)$, $x_{n_2} \in S \cap S_{1/2}(x_0)$, $x_{n_3} \in S \cap S_{1/3}(x_0)$,.... and so on.

Having chosen $x_{n_1}, x_{n_2}, x_{n_3},..., x_{n_k}$ choose $x_{n_{k+1}} \in S \cap S_{1/k+1}(x_0)$ with $n_{k+1} > n_k$. The subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x_0 since $x_{n_k} \in S \cap S_{1/k}(x_0)$ which implies $d(x_{n_k}, x_0)\ 1/k$ and hence (X, d) is sequentially compact.

Theorem 25. *Every compact metric space (X, d) is sequentially compact.*

Suppose A is an infinite subset of S which has no limit point in X. Then for each $x \in A$, there is an $\varepsilon_x > 0$ such that $S_{\varepsilon_x}(x) \cap A = \{x\}$. Otherwise x would be a limit point of A. Clearly the family of sets, $\{S_{\varepsilon_x}(x) : x \in \cap A\} \cup \{X - A\}$ is an open cover of X which admits no finite subcover, this contradicts the compactness of X. Hence A must have a limit point in X. Therefore by the above theorem (X, d) is sequentially compact.

5.2 Relative Compactness, ε-Nets and Totally Bounded Sets

Definition. A subset A of a metric space (X, d) is said to be ***relatively compact*** if $\overline{A}$ is compact.

We have seen that compact sets are always closed, so we can say that compact sets are relatively compact.

ε-Net

Let A be a subset of the metric space (X, d). Let ε be a positive real number. Then by an ε*-net* of A we mean a non-empty subset B of A such that for any $a \in A$, there exists a point $x \in B$ with $d(a, x) < \varepsilon$.

In other words each point in A comes within ε-distance of one of the points in the set B.

For example suppose $A = \mathbf{R}^2$. Then the set $B = \{(m, n): m, n = 0, \pm 1, \pm 2, \pm 3...\}$ constitute an ε*-net* for $\mathbf{R}^2$ provided $\varepsilon > \frac{\sqrt{2}}{2}$.

It is easy to see that a set is bounded if and only if it has an ε*-net*.

Definition. A non-empty subset 'A' of a metric space (X, d) is said to be totally bounded if for any $\varepsilon > 0$ there exists a finite ε*-net* for A, *i.e.*, if for every $\varepsilon > 0$, there is a finite number of open spheres of radius ε whose union is A.

i.e.,
$$A = \bigcup_{x \in B} S_\varepsilon(x),$$

where B is a finite ε*-net* for A. Clearly total boundedness implies boundedness. Since a totally bounded set is the union of a finite number of bounded sets (open spheres). But the converse is not always true. In the case of Euclidean spaces the converse also holds. In general this is not so as can be seen by the following examples.

Example 41. Infinite discrete space X is bounded but not totally bounded, for it has no finite $\frac{1}{2}$-net, since

$$S_{1/2}(x) = \{x\}, x \in X \text{ and } X \text{ is infinite.}$$

Example 42. Consider the space l_2 consisting of sequences $\{x_n\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

and the metric is defined by

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{1/2}, \text{ where } x = \{x_n\}, y = \{y_n\} \in l_2$$

- Let A be a subset of l_2 consisting of sequences

$$e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, \ldots, 0), \quad e_3 = (0, 0, 1, 0, 0, \ldots, 0)$$

since $d(e_i, e_j) = \sqrt{2}, \forall i \neq j$, therefore A is bounded, we shall show that A is not totally bounded. Observe that A has no finite $\frac{1}{\sqrt{2}}$-net, for if it has, then there exists a finite set B of X such that

$$d(e_i, x) < \frac{1}{\sqrt{2}}, \text{ and } d(e_j, y) < \frac{1}{\sqrt{2}}, \text{ for } i \neq j, \text{ and } x, y \text{ in } B$$

Clearly $x \neq y$, for $x = y$ implies by triangle inequality

$$\sqrt{2} = d(e_1, e_j) \leq d(e_i, x) + d(x, e_j) < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

So for each e_i in A there is an x in B with the above property. Thus there corresponds an infinite set B, which is a contradiction to the fact that B is finite.

Ex. Every subset of a totally bounded set is totally bounded.

Note: Every compact metric space (X, d) is totally bounded, since for each $\varepsilon > 0$, the open cover $\{S_\varepsilon(x) : x \in X\}$ of X has a finite subcover.

Note that infinite discrete space and the set A in the above examples are not compact.

Recall that a metric space is separable if it has a countable dense subset. We have the following :

Theorem 26. *Every totally bounded metric space (X, d) is separable.*

Since X is totally bounded, therefore for each positive integer n it has a finite $1/n$ – net, say A_n. Then A_n is a finite set, and

$$X = \bigcup_{a \in A_n} S_{1/n}(a)$$

Let $A = \bigcup_{n=1}^{\infty} A_n$. A is a countable subset of X, being a countable union of finite sets A_n. We shall now prove that A is dense in X, *i.e.*, $\overline{A} = X$. For this, let x be any element of X. Let $S_\varepsilon(x)$ be any open sphere centered at x. Choose a positive integer n such that $\frac{1}{n} < \varepsilon$. Since A_n is $\frac{1}{n}$-net, therefore $x \in X = \bigcup_{a \in A_n} S_{1/n}(a)$ implies $x \in S_{1/n}(a)$ for some $a \in A_n$. This implies

$$d(x, a) < \frac{1}{n} < \varepsilon$$

i.e.,
$$d(x, a) < \varepsilon, \text{ and so } a \in S_\varepsilon(x)$$

Therefore $S_\varepsilon(x) \cap A_n \neq \phi$, and hence $S_\varepsilon(x) \cap A \neq \phi$

$\therefore$ $x \in \overline{A}$. This shows that A is a dense subset of X.

Corollary. *Every compact metric space is separable.*

Note: However, the converse may not be true. For example $(\mathbb{R}, d)$ is a separable but not compact.

The next theorem characterizes total boundedness in terms of sequences in the space, but first we need the following lemma.

Lemma. *If A is an infinite subset of a totally bounded metric space (X, d), then for each $\varepsilon > 0$, there exists an infinite set $B \subseteq A$, such that $d(B) < \varepsilon$.*

Let $\varepsilon > 0$ be given. For $\varepsilon/3 > 0$, (X, d) being totally bounded, has a finite $\varepsilon/3$ net, say $\{x_1, x_2, \ldots, x_n\}$. Then

$$X = \bigcup_{i=1}^{n} S_{\varepsilon/3}(x_i)$$

$$\therefore \quad A = \bigcup_{i=1}^{n} A \cap S_{\varepsilon/3}(x_i) \qquad [\because A \subseteq X]$$

This implies at least one of the sets $A \cap S_{\varepsilon/3}(x_i)$ is infinite ($\because$ A is infinite) call it B. Clearly $B \subseteq A$, and $d(B) < \dfrac{2\varepsilon}{3} < \varepsilon$.

Theorem 27. *A metric space (X, d) is totally bounded if and only if every sequence in X contains a Cauchy subsequence.*

Suppose (X, d) is totally bounded and let $\{x_n\}$ be any sequence in X. Let A denote the range set of the sequence. If A is finite then there is nothing to prove. Otherwise by the above Lemma $\exists$ an infinite set $B_1 \subseteq A$, such that

$$d(B_1) < 1$$

Choose a positive integer n_1 such that $x_{n_1} \in B_1$. Again by the same argument $\exists$ an infinite set $B_2 \subseteq B_1$ such that

$$d(B_2) < \frac{1}{2}$$

Choose a positive integer $n_2 > n_1$ such that $x_{n_2} \in B_2$. Continuing in this manner we obtain a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $x_{nk} \in B_k$ with $d(B_k) < \dfrac{1}{k}$ and $B_{k+1} \subseteq B_k$, $k = 1, 2, 3 \ldots$

We shall now prove that $\{x_{nk}\}$ is Cauchy.

Let $\varepsilon > 0$ be given. Choose a positive integer k_0 such that $\dfrac{1}{k_0} < \varepsilon$

$\therefore$ For $k, m \geq k_0$ we have by our construction of B_k's; $x_{n_k}, x_{n_m} \in B_{k_0}$, and $d(B_{k_0}) < \dfrac{1}{k_0}$

$$\Rightarrow \qquad d(x_{n_k}, x_{n_m}) < \frac{1}{k_0} < \varepsilon$$

$\Rightarrow$ $\{x_{nk}\}$ is Cauchy. Hence every sequence in (X, d) has a Cauchy subsequence.

Conversely, suppose (X, d) is not totally bounded.

Then there exists $\varepsilon_0 > 0$, for which there is no finite ε_0-net for X.

Let $x_1 \in X$ be arbitrary. Then $S_{\varepsilon_0}(x_1) \neq X$ (for otherwise $\{x_1\}$ is an ε_0-net for X).

This implies $\exists\, x_2 \in X$ such that

$$x_2 \notin S_{\varepsilon_0}(x_1), \text{ i.e., } d(x_2, x_1) \geq \varepsilon_0$$

Again we have $S_{\varepsilon_0}(x_1) \cup S_{\varepsilon_0}(x_2) \neq X$ (for otherwise $\{x_1, x_2\}$ is an ε_0-net for X), therefore there exists $x_3 \in X$ such that

$$x_3 \notin S_{\varepsilon_0}(x_1) \cup S_{\varepsilon_0}(x_2) \text{ i.e., } d(x_3, x_1) \geq \varepsilon_0, \text{ and } d(x_3, x_2) \geq \varepsilon_0$$

Continuing in this manner we obtain a sequence $\{x_n\}$ in X such that $d(x_n, x_m) \geq \varepsilon_0$, for $n \neq m$. This implies the sequence $\{x_n\}$ is not Cauchy and so it has no Cauchy subsequence.

Note: In the metric space, totally boundedness is the property that complements completeness to guarantee sequential compactness (and hence compactness) as proved in the following theorem.

Theorem 28. *A metric space (X, d) is sequentially compact if and only if it is complete and totally bounded.*

Let (X, d) be sequentially compact, then every Cauchy sequence $\{x_n\}$ in X has a convergent subsequence and hence it must itself converge. Therefore, (X, d) is complete.

Again if $\{x_n\}$ is any sequence in X then it has a convergent subsequence and so by the above theorem (X, d) is totally bounded.

Conversely, suppose (X, d) is complete and totally bounded. Let $\{x_n\}$ be any sequence in X then totally boundedness of X implies $\{x_n\}$ has a Cauchy subsequence say $\{x_{n_k}\}$. But since (X, d) is complete, therefore the sequence $\{x_{n_k}\}$ must converge and hence (X, d) is sequentially compact.

Example 43. The subspace $X =]0, 1[$ of the real line is totally bounded but certainly not sequentially compact, for consider the sequence $\{1/n\}$, which has no convergent subsequence.

Note than X is not complete, since it is not closed.

* **Example 44.** A subset A of a metric space (X, d) is totally bounded if and only if $\overline{A}$ is totally bounded.

- Let A be totally bounded. To show that $\overline{A}$ is totally bounded, it is enough to show that every sequence in $\overline{A}$ contains a Cauchy subsequence. Let $\{x_n\}$ be any sequence in $\overline{A}$. Let $\varepsilon > 0$ be given. Then $x_n \in \overline{A}$ implies

$$S_{\varepsilon/3}(x_n) \cap A \neq \phi$$

i.e., $\exists\, a_n \in A$ such that $d(a_n, x_n) < \varepsilon/3$ ...(1)

So we obtain a sequence $\{a_n\}$ in A, and A being totally bounded implies $\{a_n\}$ contains a Cauchy subsequence say $\{a_{n_k}\}$. Therefore for $\varepsilon > 0$, $\exists$ a positive integer m such that

$$d(a_{n_j}, a_{n_k}) < \varepsilon/3, \quad \forall\, n_j, n_k \geq m \qquad \text{...(2)}$$

By using triangle inequality and from (1) and (2), we have

$$\begin{aligned} d(x_{n_j}, x_{n_k}) &\leq d(x_{n_j}, a_{n_j}) + d(a_{n_j}, a_{n_k}) + d(a_{n_k}, x_{n_k}) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \quad \forall\, n_j, n_k \geq m \end{aligned}$$

Hence, $\{x_{n_k}\}$ is a Cauchy subsequence of $\{x_n\}$.

Therefore, $\overline{A}$ is totally bounded.

The converse is obvious since A, being a subset of a totally bounded set $\overline{A}$, is itself totally bounded.

In order to show that sequential compactness implies compactness we need the notion of Lebsegue number for an open cover.

Lebesgue number for covers

Let $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover of a metric space (X, d). A real number $\lambda > 0$ is said to be a *Lebesgue number* for the open cover $\{G_\alpha\}$ if for each subset A of X with $d(A) < \lambda$, there is at least one set G_α which contains A.

Note: Not every open cover of a metric space has a Lebesgue number. For example, let $X =]0, 1[$ be a subspace of the real line and $\{]1/n, 1[: n = 2, 3, 4, \ldots\}$ be an open cover of $]0, 1[$. For arbitrary $\lambda > 0$ the set $A =]0, \lambda/2[$ is such that $d(A) < \lambda$. But A is not contained in any of the members of the cover. Note that this space is not sequentially compact.

Theorem 29. ***Lebesgue covering lemma.*** *Every open cover of a sequentially compact metric space (X, d) has a Lebesgue number.*

Let $\{G_\alpha : \alpha \in \Lambda\}$ be any open cover of X. Assume that it has no Lebesgue number. Then for each natural number n there is a non-empty set $A_n \subseteq X$ with $d(A_n) < \dfrac{1}{n}$ such that

$$A_n \not\subset G_\alpha, \text{ for every } \alpha \in \Lambda$$

For each $n \in \mathbf{N}$, choose a point $a_n \in A_n$. Since X is sequentially compact, the sequence $\{a_n\}$ contains a convergent subsequence, say $\{a_{nk}\}$.

Let $\lim\limits_{k \to \infty} a_{n_k} = x$

Now since $x \in X = \bigcup\limits_{\alpha \in \Lambda} G_\alpha$ implies $x \in G_\alpha$, for some $\alpha \in \Lambda$. G_α being open, therefore there is an $\varepsilon > 0$ such that

$$S_\varepsilon(x) \subseteq G_\alpha.$$

For the above $\varepsilon > 0$, $a_{n_k} \to x$, and $d(A_{n_k}) \to 0$, as $k \to \infty$ implies there exists a positive integer k_0, such that

$$d(a_{n_{k_0}}, x) < \varepsilon/2, \text{ and } d(A_{n_{k_0}}) < \varepsilon/2 \quad \ldots(1)$$

Let y be any element of $A_{n_{k_0}}$, then by using triangle inequality, and (1) we get

$$\begin{aligned} d(y, x) &\le d(y, a_{n_{k_0}}) + d(a_{n_{k_0}}, x) \\ &\le d(A_{n_{k_0}}) + d(a_{n_{k_0}}, x) && [\because a_{n_{k_0}} \in A_{n_{k_0}}] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

This implies that $y \in S_\varepsilon(x) \subseteq G_\alpha$. Hence $A_{n_{k_0}} \subseteq G_\alpha$, which contradicts the fact that for each natural number n, $A_n \not\subset G_\alpha$. Hence, $\{G_\alpha\}$ must have a Lebesgue number.

We are now in a position to prove the converse of the Theorem 25, which will establish the equivalence of compactness and sequential compactness in metric spaces.

Theorem 30. *Every sequentially compact metric space (X, d) is compact.*

Let $\{G_\alpha\}$ be any open cover of X. Since (X, d) is sequentially compact therefore by above lemma $\{G_\alpha\}$ has a Lebesgue number say $\lambda > 0$. Also (X, d) being sequentially compact is totally bounded and so it has a finite $\frac{\lambda}{3}$-net, say $\{x_1, x_2 \ldots x_n\}$.

Then
$$X = \bigcup_{i=1}^{n} S_{\lambda/3}(x_i)$$

Now for each i, $1 \le i \le n$ we have $d(S_{\lambda/3}(x_i)) \le \frac{2\lambda}{3} < \lambda$, and so by definition of Lebesgue number there exists at least one G_{α_i} such that

$$S_{\lambda/3}(x_i) \subseteq G_{\alpha_i}, i = 1, 2, \ldots n.$$

This implies
$$\bigcup_{i=1}^{n} S_{\lambda/3}(x_i) \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

i.e.,
$$X \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$$

Hence $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ is a finite subcover of $\{G_\alpha\}$ and so (X, d) is compact.

Corollary. *A metric space is compact if and only if it is sequentially compact.*

Theorem 31. *A metric space is compact if and only if it is complete and totally bounded.*

Follows from theorem 28 and the above Corollary.

Theorem 32. *A closed subspace of a complete metric space is compact if and only if it is totally bounded.*

Since a closed subspace of a complete metric space is itself complete, result follows from the above theorem.

We have seen that compactness is another name of Heine-Borel Property. Our results so far establish the following equivalence in a metric space.

(1) Bolzano-Weierstrass Property

(2) Compactness

(3) Sequential compactness

(4) Completeness and totally boundedness.

As a consequence of the Lebesgue covering lemma and the above corollary, we have the following useful result.

Theorem 33. *Let f be a continuous function from a compact metric space (X, d_1) into a metric space (Y, d_2). Then f is uniformly continuous.*

Let $\varepsilon > 0$ be given. For each x in X, $f^{-1}(S_{\varepsilon/2}(f(x)))$ is an open subset of X containing x, being an inverse image of an open sphere $S_{\varepsilon/2}(f(x))$ in Y under the continuous function $f : X \to Y$.

Therefore, the collection $\{f^{-1}(S_{\varepsilon/2}(f(x)))\}$ is an open cover of X. Since X is compact, therefore by the Lebesgue covering lemma and above corollary, this open cover has a Lebesgue number, say, $\delta > 0$. Let x, y be any two elements of X with $d_1(x, y) < \delta$, then the set $\{x, y\}$ is a set in X with diameter less than δ and so by the definition of Lebesgue number $x, y \in f^{-1}(S_{\varepsilon/2}(x'))$ for some $x' \in X$

i.e., $$f(x), f(y) \in S_{\varepsilon/2}(f(x')),$$

$$\Rightarrow \quad d_2(f(x), f(x')) < \varepsilon/2, \text{ and } d_2(f(y), f(x')) < \varepsilon/2$$

By triangle inequality,

$$d_2(f(x), f(y)) \leq d_2(f(x), f(x')) + d_2(f(x'), f(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence f is uniformly continuous.

Aliter.

Let $\varepsilon > 0$ be given. f being continuous on X, implies for each $x \in X$; there exists $\delta_x > 0$ (depending on ε and x) such that for $x' \in X$

$$d_1(x', x) < \delta_x \Rightarrow d_2(f(x'), f(x)) < \varepsilon/2 \qquad \text{...(1)}$$

The collection $\{S_{\delta_x/2}(x) : x \in X\}$ of open spheres forms an open cover of the compact space X, therefore there exists a finite subcover $\{S_{\delta_i/2}(x_i) : x_i \in X, 1 \leq i \leq n\}$ such that

$$X = \bigcup_{i=1}^{n} S_{\delta_i/2}(x_i) \qquad \text{...(2)}$$

Choose $\delta = \min\left\{\frac{\delta_1}{2}, \frac{\delta_2}{2}, \ldots, \frac{\delta_n}{2}\right\}$, then $\delta > 0$; since each $\delta_i > 0$.

Let x, y be any two elements of X, with $d_1(x, y) < \delta$. Then by (2),

$$x \in S_{\delta_i/2}(x_i) \text{ for some } i.$$

Therefore, using (1) we have

$$d_1(x, x_i) < \frac{\delta_i}{2} < \delta_i \Rightarrow d_2(f(x), f(x_i)) < \varepsilon/2 \qquad \text{...(3)}$$

Also, by triangle inequality, we have

$$d_1(x_i, y) \leq d_1(x_i, x) + d_1(x, y) < \frac{\delta_i}{2} + \delta < \frac{\delta_i}{2} + \frac{\delta_i}{2} = \delta_i$$

(by the choice of δ)

and so, again from (1)

$$d_2(f(x_i), f(y)) < \varepsilon/2 \qquad \text{...(4)}$$

From equations (3) and (4), using triangle inequality, we obtain

$$d_2(f(x), f(y)) < d_2(f(x), f(x_i)) + d_2(f(x_i), f(y))$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence f is uniformly continuous.

Definition. A collection **F** of functions from the metric space (X, d_1) to the metric space (Y, d_2) is called *equicontinuous*, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$d_2(f(x), f(y)) < \varepsilon, \text{ whenever } d_1(x, y) < \delta, \text{ for all } x, y \in X, \text{ and all } f \in \mathrm{F}.$$

According to this definition, the functions belonging to the equicontinuous collection are uniformly continuous.

EXERCISE

1. If A and B are two compact subsets of a metric space (X, d). Prove that $A \cup B$ and $A \cap B$ are compact.
2. If A and B are non-empty subsets of a metric space (X, d) and B is compact. Prove that $d(A, B) = 0$ if and only if $\overline{A} \cap B$ is non-empty.
3. Let (X, d) be any metric space and let A and B are subsets of X. Prove that if A is closed and B is compact and $d(A, B) = 0$, then $A \cap B \neq \phi$.
4. If A is a compact set of diameter $d(A)$. Prove that there exists a pair of points x and y of A such that $d(x, y) = d(A)$.
5. If A and B are disjoint compact sets in a metric space (X, d), then prove that $d(A, B) > 0$. Show also that there exists disjoint open sets G_1 and G_2 such that $A \subseteq G_1, B \subseteq G_2$.
6. Prove that a metric space (X, d) is compact if and only if every family of closed sets with an empty intersection has a finite subfamily with empty intersection.
7. If $\{F_\alpha : \alpha \in \Lambda\}$ is an infinite family of closed sets with the finite intersection property, and one of the sets of the family is compact. Prove that $\bigcap_{\alpha \in \Lambda} F_\alpha$ is not empty.
8. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, and let d be the Euclidean metric. Show that the set $\{1, \frac{1}{3}, \frac{1}{5}, ...\}$, is closed and bounded set in (X, d) but not compact. Explain why this does not contradict Heine-Borel Theorem.
9. Show that a subspace of $\mathbf{R}^n$ is bounded if and only if it is totally bounded.
10. If A is a subspace of a complete metric space, show that $\overline{A}$ is compact if and only if A is totally bounded.
11. Prove *Ascoli's theorem*: A closed subspace **F** of $C[0, 1]$ is compact if and only if **F** is equicontinuous and uniformly bounded.
12. Show that a closed subspace of a complete metric space is compact if and only if it is totally bounded.
13. Prove that a metric space (X, d) is bounded if and only if it has an ε-net, for some $\varepsilon > 0$.
14. Prove that boundedness and total boundedness are equivalent in Euclidean spaces.
15. Let (X, d) be a compact metric space and let (Y, d') be any metric space. Prove that if $f: X \to Y$ is one to one continuous and onto, then f is a homeomorphism.
16. Prove that the set of all functions which are coninuous and nowhere differentiable on [0, 1] is a set of the second category in the space $C[0, 1]$.
17. Prove that from any infinite open cover of a separable metric space one can extract a countable open cover.
18. Prove that a separable metric space is compact if from every countable open cover, one can extract a finite open cover.
19. Let $A = \mathbf{N} \times \mathbf{N}$, and set

$$F_{(m,n)} = \{(x, y) : x, y \in \mathbf{R}, \text{ and } |x| > m, |y| > n\}.$$

Show that $\{F_{(m,n)}\}$ has the finite intersection property, and further show that $\cap \{F_{(m,n)}\} = \phi$.

20. Show that a metric space X is compact if and only if every real-valued continuous function on X is bounded.
21. Let $X = \{x: 0 < d(0, x) \leq 1, \text{ and } x \in \mathbf{R}^2\}$, where $0 = (0, 0)$, and d is usual metric. Show that X is closed and bounded, but not compact. Also show that X is not totally bounded.

upto thm 39 & Ex45

6. CONNECTEDNESS

So far, we have discussed the three important C's in metric spaces viz. the continuity, completeness, and compactness. The fourth important C which plays the vital role as regards the separation or connection between the subsets of a metric space in *connectedness*. The word connected means not separated so let us first define what we mean by separated sets.

Separated Sets

Two sets A and B in a metric space (X, d) are said to be *separated* if neither has a point in common with the closure of the other.

i.e., $$A \cap \overline{B} = \phi, \text{ and } \overline{A} \cap B = \phi$$

Note that if A and B are separated then they are disjoint since

$$A \cap B \subseteq A \cap \overline{B} = \phi,$$

but two disjoint sets are not necessarily spearated. For example if,

$$A = \{x : -\infty < x < 0\},\ B = \{x : 0 \leq x < \infty\},$$

then A and B are disjoint but not separated. Clearly subsets of two separated sets are themselves separated. Two closed sets (open sets) are separated if and only if they are disjoint.

If the union of two separated sets is a closed set (open set) then the sets are themselves closed (open). For if A and B are two separated sets such that the $A \cup B$ is closed, then

$$\overline{A} = \overline{A} \cap (\overline{A} \cup \overline{B}) = \overline{A} \cap \overline{(A \cup B)} = \overline{A} \cap (A \cup B) = A \cup \phi = A$$

Definition. A subset A of a metric space (X, d) is said to be *connected* if it cannot be expressed as the union of two non-empty separated sets. If A is not connected, then it is said to be *disconnected.*

Any discrete space with more than one point is disconnected. Equivalent definitions for connectedness are contained in the following theorem.

Theorem 34. *Let Y be a subset of a metric space (X, d), then the following are equivalent :*

(*i*) *Y is connected.*

(*ii*) *Y cannot be expressed as disjoint union of two non-empty closed sets in Y.*

(*iii*) *ϕ and Y are the only sets which are both open and closed in Y.*

(*i*) $\Rightarrow$ (*ii*). If possible, let $Y = A \cup B$, where A and B are closed in Y and $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$ We claim that A and B are separated.

i.e., $$A \cap \overline{B} = \phi, \text{ and } \overline{A} \cap B = \phi.$$

Clearly $Y \cap \overline{A} = A$, and $Y \cap \overline{B} = B$ ($\because$ A and B are closed in Y).

$\therefore$ $$A \cap \overline{B} = (\overline{A} \cap Y) \cap \overline{B} = \overline{A} \cap (Y \cap \overline{B}) = \overline{A} \cap B.$$

If possible, let $A \cap \overline{B} \neq \phi$, therefore there exists

$$y \in A \cap \overline{B}$$

$\Rightarrow$ $$y \in A, \text{ and } y \in \overline{B}$$

But since $A = Y \cap \overline{A}$

$\therefore$ $$y \in Y$$

Now $y \in \overline{B}, y \in Y$ implies every neighbourhood of y in Y intersects B and so $y \in B$ ($\because$ B is closed in Y).

Thus $y \in A \cap B$, implies $A \cap B \neq \phi$, which is a contradiction.

Hence $A \cap \overline{B} = \phi$.

Thus Y is the union of two non-empty separated sets. This implies that Y is disconnected, which is a contradiction to the given hypothesis. Hence (*ii*) is true.

(*ii*) $\Rightarrow$ (*i*). If possible, let Y be disconnected, then $Y = A \cup B$, where A and B are two non-empty subsets of Y such that

$$A \cap \overline{B} = \overline{A} \cap B = \phi$$

Clearly $A \cap B = \phi$ $\qquad (\because\ A \cap B \subseteq A \cap \overline{B} = \phi)$

Now $$Y \cap \overline{A} = (A \cup B) \cap \overline{A} = (A \cap \overline{A}) \cup (B \cap \overline{A}) = A \cup \phi = A$$

$$[\because\ B \cap \overline{A} = \phi]$$

Therefore A, being the intersection of Y and the closed set $\overline{A}$, is closed in Y.

Similarly, $Y \cap \overline{B} = B$ implies B is closed in Y.

This gives that Y is a disjoint union of two non-empty closed sets A and B in Y, which is a contradiction to (*ii*). Hence Y is connected.

(*ii*) $\Rightarrow$ (*iii*). If possible, let there exist a non-empty proper subset A of Y which is both open and closed in Y. Then its complement $B = Y - A$ in Y is both closed and open in Y and $Y = A \cup B$, $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$; which contradicts (*ii*). Hence (*iii*) must be true.

(*iii*) $\Rightarrow$ (*ii*). Obvious.

Note: The above theorem still holds if the closed sets in (*ii*) are replaced by open sets, that is, Y is connected if and only if it cannot be expressed as the disjoint union of two non-empty open sets in Y.

Theorem 35. *A subset Y of a metric space X is disconnected if and only if $Y \subseteq G_1 \cup G_2$, where G_1 and G_2 are open sets in X such that $Y \cap G_1 \neq \phi$, $Y \cap G_2 \neq \phi$, $G_1 \cap G_2 \cap Y = \phi$.*

Let Y be disconnected, then there exists a non-empty proper subset A of Y which is both open and closed in Y. This implies that its complement $B = Y - A$ is a non-empty proper subset of Y which is both closed and open in Y.

Since A and B are open in Y, therefore, there exists two open sets G_1 and G_2 in Y such that

$$A = G_1 \cap Y,\ B = G_2 \cap Y$$

$\therefore$ $$Y = A \cup B \subseteq G_1 \cup G_2$$

Also $(G_1 \cap G_2) \cap Y = (G_1 \cap Y) \cap (G_2 \cap Y) = A \cap B = \phi$

Clearly

$$G_1 \cap Y \neq \phi, G_2 \cap Y \neq \phi. \qquad [\because A \neq \phi, B \neq \phi]$$

Conversely, if there exists two open subsets G_1 and G_2 of X such that

$$Y \subseteq G_1 \cup G_2$$

and $(G_1 \cap G_2) \cap Y = \phi, G_1 \cap Y \neq \phi, G_2 \cap Y \neq \phi$

Then $Y = Y \cap (G_1 \cap G_2) = (Y \cap G_1) \cup (Y \cap G_2)$

Let $A = G_1 \cap Y,$ and $B = G_2 \cap Y,$

then A and B are open in Y.

So that $Y = A \cup B.$

Moreover $A \cap B = (G_1 \cap Y) \cap (G_2 \cap Y) = \phi$

Thus $A = Y - B$ is both open and closed in Y. Clearly A is a non-empty proper subset of Y. $(\because A \neq \phi, B \neq \phi)$

Hence Y is disconnected.

Note: Corresponding result also holds using closed sets, *i.e.*, Y is disconnected if and only if $Y \subseteq F_1 \cup F_2$, where F_1 and F_2 are closed sets in X such that

$$Y \cap F_1 \neq \phi, Y \cap F_2 \neq \phi \text{ and } F_1 \cap F_2 \cap Y = \phi$$

Remark: By the above theorem $Y \subseteq X$ is disconnected if there exists two open subsets G_1 and G_2 of X such that $Y = (G_1 \cap Y) \cup (G_2 \cap Y), G_1 \cap G_2 \subseteq Y^c, G_1 \cap Y \neq \phi$

and $G_2 \cap Y \neq \phi.$

Then we say that $\{G_1 \cap Y, G_2 \cap Y\}$ is a disconnection of Y. Note that this disconnection is not unique. It follows that Y is disconnected if and only if it has a disconnection. Similar remark holds for closed sets.

• **Theorem 36.** *Let A be a connected subset of a metric space X, and let B be a subset of X such that* $A \subseteq B \subseteq \overline{A}$, *then B is also connected.*

If possible, B be disconnected, then there exist two open subsets G and H of X such that

$$B \subseteq G \cup H, G \cap H \cap B = \phi, G \cap B \neq \phi, \text{ and } H \cap B \neq \phi$$

Now $A \subseteq B$ implies that $A \subseteq G \cup H,$ and $G \cap H \subseteq B^c \subseteq A^c$

i.e.,

$$G \cap H \cap A = \phi$$

Also $G \cap A \neq \phi,$ for it not, then $A \subseteq G^c.$

This implies that $\overline{A} \subseteq G^c$ $\qquad [\because G^c$ is closed being complement of open set $G]$

i.e.,

$$B \subseteq G^c$$

or

$$B \cap G = \phi, \text{ which is not possible.}$$

Similarly, $H \cap A \neq \phi$

It follows that $\{G \cap A, H \cap A\}$ is a disconnection of A which is a contradiction, since A is connected.

Hence B must be connected. In particular, $\overline{A}$ is also connected.

Note: Like compactness, connectedness is preserved under continuous functions.

Theorem 37. *Continuous image of a connected set is connected.*

Let $f : X \to Y$ be a continuous function from a metric space X to a metric space Y. Let A be a connected subset of X. If $A = \phi$, then there is nothing to prove. Let $A \neq \phi$. We have to show that $f(A)$ is connected. If possible, let $f(A)$ be disconnected. Then Y contains open subsets G_1, G_2 which intersect $f(A)$ and are such that

$$f(A) \subseteq G_1 \cup G_2 \text{ and } G_1 \cap G_2 \cap f(A) = \phi$$

This implies

$$A \subseteq f^{-1}(G_1 \cup G_2) = f^{-1}(G_1) \cup f^{-1}(G_2), \text{ and}$$
$$f^{-1}(G_1 \cap G_2 \cap f(A)) = f^{-1}(\phi) = \phi$$

or
$$A \subseteq f^{-1}(G_1) \cup f^{-1}(G_2), \text{ and } f^{-1}(G_1) \cap f^{-1}(G_2) \cap A = \phi$$
$$(\because f^{-1}(f(A)) = A)$$

But $f^{-1}(G_1)$ and $f^{-1}(G_2)$ are both open in X, being inverse images of open sets G_1 and G_2 under the continuous function f.

Also
$$f^{-1}(G_1) \cap A = f^{-1}(G_1) \cap f^{-1}(f(A)) = f^{-1}(G_1 \cap f(A)) \neq \phi$$
$$[\because G \cap f(A) \neq \phi]$$

Similarly $f^{-1}(G_2) \cap A \neq \phi$

This implies that A is disconnected, which is a contradiction.

Hence $f(A)$ must be connected.

Theorem 38. *The union of two connected sets, having non-empty intersection, is connected.*

Let A and B be any two connected subsets of a metric space X, such that

$$A \cap B \neq \phi.$$

Let $Y = A \cup B$. To show Y is connected, let D be any non-empty subset of Y which is both open and closed in Y. Then $D \subseteq Y = A \cup B$ implies D must intersect A or B or both. Suppose it intersects A, *i.e.*, $D \cap A \neq \phi$. Then $D \cap A$ is a non-empty subset of A which is both open and closed in A. Therefore $D \cap A = A$, *i.e.*, $A \subseteq D$, since A is connected. Now $A \cap B \subseteq D \cap B$, and $A \cap B \neq \phi$, so that $D \cap B$ is a non-empty subset of B which is both open and closed in B and so $D \cap B = B$, *i.e.*, $B \subseteq D$, since B is connected.

Thus,
$$Y = A \cup B \subseteq D \cup D = D$$

Hence, $Y = D$. It follows that Y is connected.

Theorem 39. *An arbitrary union of connected sets, with non-empty intersection, is connected.*

Let $\{A_\alpha\}$ be a family of connected sets having non-empty intersection *i.e.*, $\underset{\alpha}{\cap} A_\alpha \neq \phi$. Taking $Y = \underset{\alpha}{\cup} A_\alpha$, the proof follows by the same argument as given in the above theorem.

Ex. 1. If $\{A_n\}$ is a sequence of connected subsets of a metric space X, each of which intersects its successor, *i.e.*, $A_n \cap A_{n+1} \neq \phi$, $\forall\, n \in \mathrm{N}$, then show that $\overset{\infty}{\underset{n=1}{\cup}} A_n$ is connected.

[*Hint*: Taking $B_n = A_1 \cup A_2 \cup A_3 \ldots \cup A_n$, we have

$$\overset{\infty}{\underset{n=1}{\cap}} B_n \supseteq A_1 \text{ and } \overset{\infty}{\underset{n=1}{\cup}} B_n = \overset{\infty}{\underset{n=1}{\cup}} A_n.]$$

Ex. 2. Show that the union of any non-empty—family $\{A_\alpha\}$ of connected subsets of a metric space X, each pair of which intersects, is connected.

[*Hint*. Fix α_0, taking $B_\alpha = A_{\alpha_0} \cup A_\alpha$ for each α, we have

$$\underset{\alpha}{\cap} B_\alpha \supseteq A_{\alpha_0} \text{ and } \underset{\alpha}{\cup} B_\alpha = \underset{\alpha}{\cup} A_\alpha.]$$

Example 45. Discuss the connectedness of the following subsets of the Euclidean space $\mathbf{R}^2$.

(*i*) $D = \{(x, y) : x \neq 0, \text{ and } y = \sin 1/x\}$

(*ii*) $E = \{(x, y) : x = 0, \text{ and } -1 \leq y \leq 1\} \cup D$

■ (*i*) $D \subseteq \mathbf{R}^2$, where $D = \{(x, y) : x \neq 0, \text{ and } y = \sin 1/x\}$

Let $A = \{(x, y) : x > 0, \text{ and } y = \sin 1/x\}$, and

$$B = \{(x, y) : x < 0, \text{ and } y = \sin 1/x\}$$

Then $\quad D = A \cup B, \text{ and } A \cap B = \phi$

Since $\quad A = D \cap \{x, y) : x > 0\}, \text{ and}$

$$B = D \cap \{x, y) : x < 0\}$$

The sets $\{(x, y) : x > 0\}$ and $\{(x, y) : x < 0\}$ are pairwise disjoint open subsets in $\mathbf{R}^2$. Obviously A and B are open in D and they are also non-empty, *i.e.*,

$$A \neq \phi, \ B \neq \phi$$

Therefore, $\{A, B\}$ is disconnection of D.

Hence, D is a disconnected subset of $\mathbf{R}^2$.

(*ii*) Next, we have $E \subseteq \mathbf{R}^2$, given by

$$E = D \cup \{(0, y) : -1 \leq y \leq 1\}$$

Let $F = \{(0, y) : -1 \leq y \leq 1\}$,

then $\quad E = A \cup B \cup F$

From the graph of $y = \sin 1/x$, it is easy to verify that

$$\overline{A} = A \cup F, \text{ and } \overline{B} = B \cup F$$

Therefore

$$E = \overline{A} \cup \overline{B}$$

Also $$\overline{A} \cap \overline{B} = F \neq \phi$$

We now define a function $f :]0, \infty[\to \mathbf{R}^2$ by

$$f(x) = (x, \sin 1/x)$$

The function f is continuous and the set A, being the continuous image of the connected set $]0, \infty[$, is connected. So $\overline{A}$ is connected. Similarly the function $g :]-\infty, 0[\to \mathbf{R}^2$ defined by $g(x) = (x, \sin 1/x)$ is continuous. By the same argument $\overline{B}$ is connected.

Hence, E is a connected subset of $\mathbf{R}^2$.

Theorem 40. *A non-empty subset X of* **R** *(with usual metric) is connected if and only if X is an interval or a singleton.*

Let X be a non-empty connected subset of **R** containing at least two elements. If possible, let X be not an interval, then there exists $a, b, c \in \mathbf{R}$ such that

$$a < c < b, a, b \in X, \text{ but } c \notin X.$$

Then $G_1 =]-\infty, c[$ and $G_2 =]c, \infty[$ are two disjoint open sets in **R** which intersect X (since $a \in G_1 \cap X$ and $b \in G_2 \cap X$) such that

$$X = (X \cap G_1) \cup (X \cap G_2) \subseteq G_1 \cup G_2$$

This shows that X is disconnected, which is a contradiction. Hence X must be an interval.

Conversely, if X is a singleton set, then there is nothing to prove. If possible, let X be disconnected, then there exist two open subsets G_1 and G_2 of **R** such that

$$X \subseteq G_1 \cup G_2, G_1 \cap G_2 \cap X = \phi, G_1 \cap X \neq \phi, G_2 \cap X \neq \phi.$$

Let $a \in G_1 \cap X, b \in G_2 \cap X$. Assume that $a < b$. The set $[a, b] \cap G_1$ is non-empty and bounded above (by b) and so it has a supremum say ξ. Clearly $a < \xi \leq b$. Now $\xi \notin G_2$, for if it were in the open set G_2, then there exist $\varepsilon > 0$ such that

$$]\xi - \varepsilon, \xi + \varepsilon[\subseteq G_2$$

This implies $\xi - \varepsilon$ is an upper bound of $[a, b] \cap G_1$ which contradicts the choice of ξ as the least upper bound.

Similarly $\xi \notin G_1$, for if $\xi \in G_1$, G_1 being open, then there exists $\delta > 0$ such that

$$]\xi - \delta, \xi + \delta[\subseteq G_1.$$

Now we have $\xi < b$ and $]\xi, b[\cap G_2 \neq \phi$ so that ξ cannot be an upper bound of $[a, b] \cap G_1$, which contradicts the choice of ξ.

Hence $a, b \in X, a < \xi < b$ and $\xi \notin X$, it follows that X is not an interval.

Corollary. *The real line* **R** *is connected.*

Since **R** is an interval so it is connected by the above theorem.

Aliter. Let, if possible **R** be disconnected. Then there exist two non-empty disjoint closed sets A and B in **R** such that $\mathbf{R} = A \cup B$.

Let $a_1 \in A$ and $b_1 \in B$, $a_1 \neq b_1$ so either $a_1 < b_1$ or $a_1 > b_1$. Without loss of generality we may assume $a_1 < b_1$. Let $I_1 = [a_1, b_1]$. The mid-point $\frac{a_1 + b_1}{2}$ of $[a_1, b_1]$ being a point of **R** belongs to A or to B (but not to both). In case it belongs to A, take the interval $\left[\frac{a_1 + b_1}{2}, b_1\right]$ and name it $I_2 = [a_2, b_2]$; otherwise we call the interval $\left[a_1, \frac{a_1 + b_1}{2}\right]$ as $I_2 = [a_2, b_2]$, where $I_2 \subseteq I_1$ and $b_2 - a_2 = \frac{1}{2}(b_1 - a_1)$. Now bisecting I_2 as before and continuing the process indefinitely we get a sequence $\{I_n\} = \{[a_n, b_n]\}$ or closed intervals such that

$$I_1 \supseteq I_2 \supseteq I_3 \ldots \supseteq I_n \ldots$$

with the property that $a_n \in A, b_n \in B, \forall\, n \in \mathbf{N}$; and $l(I_n) = \frac{1}{2^{n-1}}(b_1 - a_1)$ which tends to zero as n tends to infinity.

Therefore by Cantor's intersection theorem,

$$\bigcap_{n=1}^{\infty} I_n = \{c\}$$

i.e.,
$$c \in I_n \quad \forall\, n \in \mathbf{N}$$

Clearly c is a limit point of both A and B, for if $\varepsilon > 0$ is any arbitrary real number then there exists a positive integer m_0 such that

$$I_n \subseteq \,]c - \varepsilon, c + \varepsilon[, \quad \forall\, n \geq m_0$$

This implies $]c - \varepsilon, c + \varepsilon[$ contains infinite number of points $a_{m_0}, a_{m_0+1} \cdots$ of A as well as infinite number of points $b_{m_0}, b_{m_0+1}, \cdots$ of B. So c is a limit point of both A and B. But A and B are closed subsets of **R**. Therefore, c belongs to both A and B, which is a contradiction to the fact that $A \cap B = \phi$. Hence **R** is connected.

Definition. A real-valued function is said to have an *intermediate value property* if it assumes every value between any two of its values.

Theorem 41. ***Generalized Intermediate-Value Theorem.*** *Every real-valued continuous function f defined on a connected metric space X has the intermediate-value property.*

Continuity of the function $f: X \to \mathbf{R}$ and connectedness of X implies $f(X)$ is a connected subset of **R**. Then from Theorem 40, $f(X)$ is either a singleton or an interval.

If $f(X)$ is singleton then there is nothing to prove. Let $f(X)$ be an interval.

Let $f(x) \neq f(y)$, $x, y \in X$ be any two values of $f(X)$. Then either $f(x) < f(y)$ or $f(x) > f(y)$ Without loss of generality we may assume that $f(x) < f(y)$.

Let A be any real number lying between $f(x)$ and $f(y)$

i.e.,
$$f(x) < A < f(y).$$

Then, $A \in f(X)$ [$\because$ $f(X)$ is an interval]

$$A = f(a), \text{ for some } x \in X.$$

i.e.,

Hence f has the intermediate value property.

Corollary (*Intermediate-value theorem*). *If a real-valued function f is continuous on the closed interval [a, b]. Then f has the intermediate value property, i.e., f assumes every value between f (a) and f (b).*

This follows from the above theorem. Since the interval $[a, b]$ is a connected subset of R.

Theorem 42. *A metric space X is connected if and only if every real-valued continuous function f has the intermediate value property.*

The necessary part has been proved in the above theorem.

For the sufficient part we shall show that if X is not connected then there exists a real-valued continuous function which does not possess the Intermediate value property. Let X be disconnected, then there exist two disjoint non-empty open sets G and H in X such that

$$X = G \cup H$$

The function $f: X \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in G \\ 1, & \text{if } x \in H \end{cases}$$

does not take the intermediate value $\frac{1}{2}$ at any point of X.

So f does not possess the intermediate value property. However, f is continuous. Since the range $f(X)$ of f is the discrete space $\{0, 1\}$, Therefore, a complete collection of open subsets of $f(X)$ is given by ϕ, $\{0, 1\}$, $\{0\}$, $\{1\}$. By definition of f the inverse image of these sets are ϕ, X, G, H respectively, all of which are open in X.

Corollary. *A metric space X is disconnected if and only if there exists a continuous function from X onto the discrete metric space* $\{0, 1\}$.

This follows from above theorem and the fact that the discrete metric space with more than one point is always disconnected.

6.1 Components of a Metric Space

Definition. A subset A of a metric space X is said to be a *component* of X if it is the maximal connected subset of X, *i.e.*, if A is connected but not contained properly in any larger connected subset of X.

Thus if X is connected, then X is the only component of itself. Components always exist, since singleton sets are connected. Moreover, in a discrete metric space X singletons are the only connected subsets of X, since a subset A of X containing more than one element is obviously disconnected and so the components of X are singletons.

Properties:

(*i*) Components are closed sets

Let X be any metric space and A be its component. By definition, A is a maximal connected subset of X. Suppose A is not closed, then $\overline{A} \neq A$, *i.e.*, $A \subset \overline{A}$. Now $\overline{A}$, being the closure of the connected set A is itself connected which properly contains A, this contradicts the maximality of A. Hence, A must bc closed.

(ii) Components need not be open sets

Consider the metric space of rationals $(\mathbf{Q}, d)$ with the usual metric. The components of $\mathbf{Q}$ are singletons which are not open in $\mathbf{Q}$. Let Y be any subset of $\mathbf{Q}$ containing more than one element. Choose y_1, y_2 in Y with $y_1 < y_2$, then there exists an irrational number ξ such that $y_1 < \xi < y_2$. Clearly $\{]-\infty, \xi[,]\xi, \infty[\}$ is a disconnection of Y, and so Y is disconnected.

Hence, the only connected subsets of $\mathbf{Q}$ are singletons.

(iii) Components of a metric space are either identical or pairwise disjoint

Let A and B be any two components of a metric space X. Then either $A \cap B = \phi$ or $A \cap B \neq \phi$. If $A \cap B = \phi$, then there is nothing to prove. If $A \cap B \neq \phi$ then $A \cup B$ is a connected subset of X. Also $A \subseteq A \cup B$ and $B \subseteq A \cup B$, the definition of components implies $A \cup B = A$, and $A \cup B = B$ and hence $A = B$.

Theorem 43. *Every connected subset of a metric space is contained in a component of X.*

Let A be any connected subset of X. Consider the collection $\{A_\alpha\}$ of all connected subsets of X containing A.

$\{A_\alpha\} \neq \phi$, since A itself is a connected subset of X containing A. Let $Y = \underset{\alpha}{\cup} A_\alpha$.

Then Y is a connected subset of X containing A, since each A_α is a connected subset of X containing A and $\underset{\alpha}{\cap} A_\alpha \neq \phi$ $(\because A \subseteq \underset{\alpha}{\cap} A_\alpha)$.

Moreover, Y is a maximal connected subset of X, for if $Y \subseteq B$, where B is a connected subset of X, then $B \in \{A_\alpha\}$ $(\because A \subseteq Y \Rightarrow A \subseteq B)$ and so $B \subseteq \underset{\alpha}{\cup} A_\alpha = Y$. Thus $B = Y$. This shows that Y is a maximal connected subset of X containing A; and hence Y is the required component of X containing A. Also components are pairwise disjoint, therefore Y is the only component of X containing A.

Corollary 1. *Each element of a metric space X is contained in exactly one component.*

This follows directly from the above theorem since each singleton is connected.

Corollary 2. *A non-empty connected subset of a metric space X is a component, if it is both open and closed.*

Let A be a non-empty connected subset of X, which is both open and closed. Then by the above theorem, A is contained in some component, say, B of X. To prove that A is a component of X, we shall show that $A = B$. If possible let $A \neq B$, then A is a proper subset of B which is both open and closed in B. This implies $A = \phi$ $(\because B$ being component, is connected), which is a contradiction. Hence $A = B$.

Ex. Show that the metric space if $(\mathbf{R}^n, d)$ where

$$d(x, y) = \max_{1 \le i \le n} |x_i - y_i| \quad x = (x_1, x_2, \ldots, x_n),$$

$$y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$$

is connected.

[*Hint*: Every non-empty proper subset of $\mathbf{R}^n$ has non-empty boundary].

B.A/B.Sc. T.Y. Semester-V
DSEM-5, Section-A
Paper XII: Metric Spaces

No. of periods: 60 **Max. Marks: 50** **Credits: 2**

Unit I:

Definitions and examples:
Definition of Metric Space, Examples of Metric Space, Diameter of a nonempty Set.

Open and Closed sets:
Open and Closed Spheres, Neighbourhood of a Point, Open Sets, Limit Points, Closed Sets, Subspaces, Closure of a Set.

Unit II:

Convergence and Completeness:
Definition, Cauchy Sequence, Cantor's Intersection Theorem, Baire's Category Theorem.

Continuity and Uniform Continuity:
Definitions, Examples, Theorems on Continuity and Uniform Continuity, Banach Fixed Point Theorem.

Unit III:

Compactness:
Definitions and Theorems on Compactness, Heine-Borel Theorem, Compactness and Finite Intersection Property, Relative Compactness, Є-Nets and Totally Bounded Sets, Lebesgue Number for Covers.

Connectedness:
Separated Sets, Definition and Theorems on Connectedness.

Text Book: S.C. Malik and Savita Arora, "Mathematical Analysis", New Edge International (P) Limited Publisher, New Delhi (Fourth Edition).

Scope:

Unit I : Chapter 19:- Art. 1, 2, 2.1, 2.2, 2.3 (Theorem 1 only), 2.4, 2.5, 2.6, 2.7.
Unit II : Chapter 19:- Art. 3, 4 (Theorem 16 statement only), 4.1.
Unit III: Chapter 19:- Art. 5 (Theorem 21statement only), 5.1, 5.2(Theorems 26 to 33 Statements only) , Art. 6. (up to Theor em 39 and Example 45).

Reference Books:

1. Somasundaram & Chaudhary "A First Course in Mathematical Analysis", Narosa Pub. House New Delhi.
2. R. Goldberg, "Methods of Real Analysis", Oxford & IBH Pub. Co.PVT Ltd Shantinarayan & M.D. Raisinghania, "Elements of Real Analysis", S. Chand. Co. Ltd.
3. E. T. Copson "Metric Spaces", Cambridge University Press. Universal Book Co. New Delhi.
4. T. M. Apostol "Mathematical Analysis", Narosa Pub. House New Delhi.
5. T. M. Karade, "Lecturers on Analysis", Sonu Nilu Pub. Nagpur.