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Unit II

* Conjugacy & G- sets :-

Let G be a group and X a set. Then G is said to act on X. if there is a mapping $\phi : G \times X \longrightarrow X$ with $\phi(a, x)$ written $a*x$ such that $a, b \in G$, $x \in X$.

i) $a*(b*x) = (ab)*x$
ii) $e*x = x$

The mapping ϕ is called the action of G on X & X is said to be G-set.

Examples -

① Let G be a group Define $\phi : G \times G \rightarrow G$ given by $\phi(a, x) = a*x = ax$ $a \in G$, $x \in G$
TO show that G is G-set

→ Let G be a group & define

$$a*x = ax$$

Let $a, b \in G$

i) $a*(b*x) = a*bx = a(b.x) = (ab)x$ $\therefore = ab*x$
ii) $e*x = ex = x$ $(\because e \in G)$

$\therefore$ G is a G-set

② Let G be a group. Define $\phi : G \times G \rightarrow G$ given by $\phi(a, x) = a*x = axa^{-1}$, $a \in G$, $x \in G$.
We show that G is G-set

→ Let $a, b \in G$ then

i) $a*(b*x) = a*(bxb^{-1}) =$
$= abxb^{-1}a^{-1}$
$= abx(ab)^{-1}$ $(\because (ab)^{-1} = b^{-1}a^{-1})$
$= (ab)*x$

ii) $e * x = exe^{-1} = x$ $\quad (\because e \in G.$

$\therefore$ G is G-set.

③ Let G be a group and $H < G$ then the set $\frac{G}{H}$ is G-set we define $a * xH = axH$ $a \in G$ & $xH \in G/H$.

→ Let G be a group & $H < G$.
and consider $a, b \in G$, $xH \in G/H$.

i) $a * (b * xH) = a * (bxH)$
$= (abxH)$
$= ab * xH$.

ii) $e * xH = exH$
$= xH$ $\quad (\because ex = x)$.

$\therefore$ The set $\frac{G}{H}$ is G-set

④ Let G be a group & $H \triangle G$ then the set $\frac{G}{H}$ is a G-set if we define

$a * xH = axa^{-1}H \quad \forall a \in G \quad xH \in G/H$

→ Let $a, b \in G$, $xH \in G/H$.

i) $a * (b * xH) = a * (bxb^{-1})H$.
$= abxb^{-1}a^{-1}H$
$= abx(ab)^{-1}H$
$= ab * xH$.

ii) $e * xH = exe^{-1}H = xH$.

$\therefore$ The set $\frac{G}{H}$ is G-set.

* Theorem -

Let G be a group and let X be a set

i) If X is a G-set then then action of G on X induces a homomorphism

$$\phi : G \longrightarrow S_X$$

ii) Any homomorphism $\phi : G \longrightarrow S_X$ induces as action of G onto X.

→ Let G is a group and X is nonempty set

$$S_X = A(X) = \{ F \mid F : X \longrightarrow X \text{ is a bijection} \}$$

is group of permutation on X under composition of function.

i) Let X be a G-set i.e $G \times X \longrightarrow X$. is action such that

i) $a*(b*x) = ab*x$ ii) $e*x = x$ $\forall a,b \in G$, $x \in X$

for each $a \in G$ define $\phi(a) : X \longrightarrow X$ by

$$\phi(a)(x) = a*x \quad \forall x \in X \quad \text{—(1)}$$

i) $\phi(a)$ is one-one :-

Since $\phi(a)(x) = \phi(a)(y)$

$\Rightarrow a*x = a*y$

$\Rightarrow a^{-1}*(a*x) = a^{-1}*(a*y)$ $\quad (\because a \in G,\ a^{-1} \in G)$

$\Rightarrow (a^{-1}.a)*x = (a^{-1}.a)*y$

$\Rightarrow e*x = e*y$ $\quad (\because a*(b*x) = (ab)*x)$

$\Rightarrow x = y$ $\quad (\because e*x = x)$

$\therefore \phi(a)$ is one-one.

② $\phi(a)$ is onto :-

Since $\forall\, y \in X$ = codomain of $\phi(a)$

i.e $y = a * x$

$\Rightarrow a^{-1} * y = a^{-1} * (a * x)$

$\Rightarrow a^{-1} * y = (a^{-1} * a) * x$

$\Rightarrow a^{-1} * y = e * x$ $\quad (\because a*(b*x) = ab*x)$

$\Rightarrow a^{-1} * y = x$ $\quad (\because e*x = x)$

So,

$$\phi(a)(x) = \phi(a)(a^{-1} * y)$$
$$= a * (a^{-1} * y)$$
$$= (a \cdot a^{-1}) * y \quad (\because a*(b*x) = (ab)*x$$
$$= e * y \quad (e*x = x)$$
$$= y = \text{codomain of } \phi(a)$$

$\therefore$ Range of $\phi(a)$ = codomain of $\phi(a)$

$\therefore$ $\phi(a)$ is onto.

③ $\phi : G \longrightarrow S_X$ is a function.

For any $a, b \in G$ we have.

& $\Rightarrow ab \in G$

$$\phi(ab)(x) = (ab) * x$$
$$= a * (b * x) \quad (\because \text{By } 1.$$
$$= a * (\phi(b)(x))$$
$$= \phi(a).(\phi(b)(x))$$

$$\phi(ab)(x) = [\phi(a) \circ \phi(b)](x) \quad \forall\, x \in X.$$

$$\Rightarrow \phi(ab) = \phi(a) \circ \phi(b) \quad \forall\, a, b \in G.$$

$\therefore$ ϕ is a homomorphism.

ii) Let $\phi : G \longrightarrow S_X$ be a group homomorphism
Define $a * x = \phi(a)(x) \quad \forall\, a \in G,\ x \in X$ — (2)
Now
$\forall\, a, b \in G$ & $x \in X$ we have

i) $a*(b*x) = a*(\phi(b)(x))$.

$= \phi(a)(\phi(b)(x))$.

$= [\phi(a) \circ \phi(b)](x)$
$= \phi(ab)(x)$ ($\because \phi$ is homo.)
$= ab * x$

ii) $e * x = \phi(e) \cdot x$

As $\phi : G \longrightarrow S_X$ is a homomorphism & $e \in G$ is the identity so $\phi(e)$ is the identity of S_X
i.e $\phi(e) : X \longrightarrow X$ is an identity function
So, $\phi(e)(x) = x \ \forall\, x \in X$.

$\therefore\ e * x = \phi(e)x = x \quad \forall\, x \in X$

$\therefore$ $*$ gives action of G on X.
which is induced by the homomorphism $\phi : G \longrightarrow S_X$.

* Theorem - [Cayley's Theorem].

Let G be a group then G is isomorphic into the symmetric group S_G.

→ Let G be a group. then G is a G-set then the action of G onto G induces a homomorphism

$$\phi : G \longrightarrow S_G \text{ given by}$$

$$\phi(a)(x) = a * x = ax \quad a \in G,\ x \in G.$$

i) $\phi(a)$ is one-one :-

Since $\forall\ a, b \in G$

$$\Rightarrow \phi(a)x = \phi(a)y$$

$$\Rightarrow \phi(a)(x) = \phi(a)(y) \quad \forall x \in G.$$

$$\Rightarrow a * x = a * y$$

$$\Rightarrow ax = ay.$$

$$\Rightarrow a^{-1}.ax = a^{-1}.ay$$

$$\Rightarrow x = y \qquad (\because a^{-1}.a = e$$

Thus, $\phi : G \longrightarrow S_G$ is a one-one, homomorphism.

$$\phi : G \longrightarrow S_G \text{ is an isomorphism.}$$

$$\therefore G \cong Im(\phi) \ \& \ Im(\phi) < S_G$$

$\Rightarrow$ G is isomorphism into the symmetric group S_G.

*** Theorem :-**

Let G be a group and $H < G$ of finite index n then there is a homomorphism $\phi : G \longrightarrow S_n$ such that $Ker\phi = \bigcap_{x \in G} xHx^{-1}$.

Solⁿ- Let G be a group and $H < G$ of finite index n i.e $[G:H] = n \in \mathbb{N}$

$\frac{G}{H} = \{xH \mid x \in G\}$ has cardinality n.

i.e $\left|\frac{G}{H}\right| = [G:H] = n$.

we have S_n by $S_{\frac{G}{H}}$ $(S_n \cong S_{G/H})$

Consider the action of G on $\frac{G}{H}$ ($\frac{G}{H}$ is a G-set) given by $a * xH = axH \quad \forall a \in G, xH \in \frac{G}{H}$.

This action induces a homomorphism.

$\phi : G \longrightarrow S_{\frac{G}{H}}$ given by

$$\phi(a)(xH) = a * xH = axH \quad \forall a \in G, xH \in \frac{G}{H}$$

$\therefore \operatorname{Ker}\phi = \{a \in G \mid \phi(a) = e$, identity permutation on $G/H\}$

$= \{a \in G \mid \phi(a)\,xH = e \cdot xH\}$

$= \{a \in G \mid \phi(a)(xH) = xH\}$

$= \{a \in G \mid a * xH = xH\} \quad \forall x \in G$

$= \{a \in G \mid axH = xH\}$

$\Rightarrow \{a \in G \mid x^{-1}axH = H\} \quad \forall x \in G.$

$x^{-1}ax \in H$

$= \{a \in G \mid \cancel{a \in x^{-1}ax} \quad \forall x \in G\}$

$= \{a \in G \mid a \in xHx^{-1} \quad \forall x \in G\}$

$$\operatorname{Ker}\phi = \bigcap_{x \in G} xHx^{-1}$$

* <u>corollary</u>:-

Let G be a group with a normal subgroup H of index n then G/H is isomorphic into S_n.

→ Let G be a group with a normal subgroup H of G & $\left|\frac{G}{H}\right| = [G:H] = n \in \mathbb{N}$

$\therefore S_n \cong S_{G/H}$ then by theorem.

hence

$\phi : G \longrightarrow S_n$ is homomorphism.

Such that $\ker\phi = \bigcap_{x \in G} xHx^{-1}$

Now, $H \triangleleft G$ So, $xH = Hx \quad \forall x \in G$.

$\Rightarrow xHx^{-1} = Hxx^{-1} = He \qquad (\because xx^{-1} = e)$

$$\therefore \ker\phi = \bigcap_{x \in G} Hxx^{-1} = \bigcap_{x \in G} He = H \quad \forall x \in G$$

Here $\ker\phi = H$.

$\therefore$ we have $\phi : G \rightarrow S_n$ as a homomorphism with $\ker\phi = H$

$\therefore$ By first isomorphism theorem.

$$\frac{G}{\ker\phi} \cong \mathrm{Im}\,\phi$$

$$\Rightarrow \frac{G}{\ker\phi} \cong S_n \Rightarrow \frac{G}{H} \cong S_n$$

$\therefore \frac{G}{H}$ is isomorphic into S_n.

* Corollary :-

Let G be simple group with a subgroup $\neq G$ of finite index n then G is isomorphic into S_n

→ Let G be a simple group. So, it has only two normal subgroup $\{e\}$ & G itself

Let H be a subgroup of G, $H \neq G$.
& $[G:H] = n \in \mathbb{N}$

By theorem there is a homomorphism
$\phi : G \to S_n$ such that $\ker\phi = \bigcap_{x \in G} xHx^{-1}$

& We know that $\ker\phi \trianglelefteq G$.
Now
$xHx^{-1} \subseteq \ker\phi \quad \forall\, x \in G$.

$\ker\phi \subseteq eHe^{-1} = H \subsetneq G$
$\Rightarrow \ker\phi \neq G$. $\therefore \ker\phi = \{e\}$

$\phi : G \to S_n$ is one-one, homomorphism.
$\frac{G}{\ker\phi} \cong S_n$
$\Rightarrow \frac{G}{\{e\}} \cong S_n$
i.e G is isomorphic to S_n.

Def^n - Stabilizer :-
Let G be a group acting on a set X and let $x \in X$ Then the set
$G_x = \{g \in G \mid gx = x\}$

Which can be easily shown to be a Subgroup is called the stabilizer group of x in G.

.W - To show that G_x is subgroup of G.

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Def^n - orbit of an element :-

Let X be a G-set and $x \in X$ Then the set $Gx = \{ax \mid a \in G\}$ is called the orbit of x in G.

* The action of the group G is by Conjugation, the orbit of $x \in G$ is

$$Gx = \{axa^{-1} \mid a \in G\}$$ called the Conjugate class of x & denoted by $C(x)$.

* Theorem -

Let G be a group acting on a set X Then the set of all orbits in X under G is a partition of X for any $x \in X$ there is a bijection.

$$Gx \longrightarrow \frac{G}{G_x} \text{ \& hence.}$$

$$|Gx| = [G : G_x]$$

$\therefore$ If X is a finite set $|X| = \sum_{x \in C} [G : G_x]$

Where C is a subset of X containing exactly one element from each orbit.

proof $\rightarrow$ i) Let G be a group acting on a set X action of G on X is given

$$g * x \in X \quad \forall g \in G, x \in X$$

i) $a*(b*x) = ab * x$ ii) $e * x = x$ $\forall a, b \in G$, $\forall x \in X$

Stabilizer group of $x \in X$ is

$G_x = \{g \in G \mid g * x = x\}$ orbit of $x \in X$ is $Gx = \{g * x \mid g \in G\} \subseteq X$

for $x, y \in X$ we define a relation

$x \sim y$ means $x = g * y$ for some $g \in G$

i) Reflexive :

Since $e \in G$, $x \in X$.

$\Rightarrow x = e * x$

$\Rightarrow x \sim x \quad \forall x \in X$

ii) Symmetric :

Let $x \sim y \Rightarrow x = g * y$ for some $g \in G$

$\Rightarrow g^{-1} * x = g^{-1} * g * y$

$\Rightarrow g^{-1} * x = (g^{-1} g) * y$

$\Rightarrow g^{-1} * x = e * y = y$

$\Rightarrow y = g^{-1} * x$ for some $g^{-1} \in G$

$\Rightarrow y \sim x$

iii) Transitive :

Let $x \sim y$ and $y \sim z$ $x, y, z \in X$.

$\Rightarrow x = g_1 * y$ & $y = g_2 * z$ for some $g_1, g_2 \in G$

$\Rightarrow x = g_1 * (g_2 * z) =$

$\Rightarrow x = (g_1 g_2) * z$

$\Rightarrow x \sim z$ for some $g_1 g_2 \in G$.

Hence $\sim$ is an equivalence relation on the set X and hence X is partition into equivalence classes.

$x \sim y$ iff equivalence classes of x = equivalence classes of y

$C(x)$ = Equivalence class of x

$= \{y \in G \mid y \sim x\}$

$= \{a \in G \mid a * x\} = Gx.$

$\therefore$ Equivalence classes of x is the orbit Gx of x in X.

for $x \in G$ $\quad Gx = \{g \in G \mid g * x\} \subseteq X$ define

$$\phi : Gx \longrightarrow \frac{G}{G_x} \text{ by}$$

$$\phi(a * x) = a G_x \quad \forall a \in G.$$

① ϕ is well defined :-

Since $a * x = b * x$

$\Rightarrow b^{-1} * (a * x) = b^{-1} * (b * x)$

$\Rightarrow (b^{-1} a) * x = (b^{-1} \cdot b) * x$

$\Rightarrow b^{-1} a * x = e * x$

$\Rightarrow b^{-1} a * x = x$

$\Rightarrow b^{-1} a \in G_x$

$\Rightarrow b^{-1} a G_x = G_x \qquad (\because xH = H \Leftrightarrow x \in H)$

$\Rightarrow b b^{-1} a G_x = b G_x$

$\Rightarrow e \cdot a G_x = b G_x \Rightarrow a G_x = b G_x$

$\Rightarrow \phi(a * x) = \phi(b * x)$

$\therefore \phi$ is well defined.

② ϕ is one-one :-

Since $\phi(a * x) = \phi(b * x)$

$\Rightarrow a G_x = b G_x$

$\Rightarrow b^{-1} a G_x = b^{-1} b G_x$

$\Rightarrow b^{-1} a G_x = e G_x = G_x$

$\Rightarrow b^{-1} a \in G_x$

$\Rightarrow b^{-1}a \in G_x$

$\Rightarrow b^{-1}a * x = x$

$\Rightarrow b^{-1} * (a * x) = x$

$\Rightarrow b * (b^{-1} * (a * x)) = b * x$

$\Rightarrow (bb^{-1}) * (a * x) = b * x$

$\Rightarrow e * (a * x) = b * x$

$\Rightarrow a * x = b * x \Rightarrow \phi$ is one-one

3 ϕ is onto:-

Sin ~~$\phi(a*$~~ $y \in \frac{G}{G_x}$ = Codomain of ϕ.

$\Rightarrow y = aG_x$

$\Rightarrow y = \phi(a * x)$

$\Rightarrow \phi(a * x) = y =$ codomain of ϕ.

$\Rightarrow$ Range of ϕ = codomain of ϕ.

$\Rightarrow \phi$ is onto.

Thus, $\phi : Gx \longrightarrow \frac{G}{G_x}$ is a bijection function

i.e $|Gx| = \left|\frac{G}{G_x}\right| = [G : G_x] \quad \forall x \in X$.

iii) X is partition into equivalence classes Equivalence classes are orbits i.e Gx, $x \in X$

$$\therefore X = \bigcup_{x \in X} Gx$$

Any two orbits are either disjoint or identical as they are equivalence classes.

Let c be a subset of x such that c contains exactly one element from each orbit then

$$x = \bigcup_{x \in C} Gx \Rightarrow |x| = \left| \bigcup_{x \in C} Gx \right|$$

$$\Rightarrow |x| = \sum_{x \in C} |Gx| = \sum_{x \in C} [G : G_x]$$

* <u>Theorem</u> -

Let G be a group. Then the following are true.

i) The set of conjugate classes of G is a partition of G.

ii) $|c(a)| = [G : N(a)]$

iii) If G is finite, $|G| = \sum [G : N(a)]$, a runing over exactly one element from each conjugate class.

→ Let G be a group with identity e.

i) Define a relation '$\sim$' on G by for $a, b \in G$, $a \sim b$ means $b = xax^{-1}$ for some $x \in G$

i) Reflexive :- Since $e \in G$ So, $a = eae^{-1}$

$\therefore$ $a \sim a$ $\forall a \in G$

ii) Symmetric :-

Let $a \sim b$ means $b = xax^{-1}$ for some $x \in G$

$\Rightarrow x^{-1}bx = x^{-1}xax^{-1}x$ for som $x^{-1} \in G$

$\Rightarrow x^{-1}bx = eae$

$\Rightarrow a = x^{-1}bx$ for some $x^{-1} \in G$

$\therefore b \sim a$

③ Transitive :

Let $a \sim b$ and $b \sim c$

$\Rightarrow b = xax^{-1}$ & $c = yby^{-1}$ for some $x, y \in G$

$\Rightarrow c = yby^{-1}$

$\Rightarrow c = yxax^{-1}y^{-1}$

$\Rightarrow c = (yx)a(yx)^{-1}$ $\quad (\because (yx)^{-1} = x^{-1}y^{-1})$

for some $yx \in G$.

$\Rightarrow a \sim c$.

Thus '$\sim$' is an equivalence relation on G.
& Hence G is partition into conjugate classes.

conjugate classes of a in G is

$$C(a) = \{ b \in G \mid a \sim b \}$$
$$= \{ b \in G \mid b = xax^{-1}, x \in G \}$$
$$= \{ xax^{-1} \mid x \in G \}$$

$$\therefore G = \cancel{\bigcup_{x \in G} xax^{-1}} = \bigcup_{x \in G} C(a)$$

Let C be a subset of G which contain exactly one element from each conjugate class in G then.

$$G = \bigcup_{x \in C} C(x)$$

$$\Rightarrow |G| = \left| \bigcup_{x \in C} C(x) \right| \Rightarrow |G| = \sum_{x \in C} |C(x)| \quad \text{---} (1)$$

ii) for $a \in G$

$C(a) = \{ xax^{-1} \mid x \in G \}$ is a conjugate class of a in G.

$N(a) = \{x \in G \mid xa = ax\}$ is normalizer of a in G hence $N(a) < G$.

& $\frac{G}{N(a)} = \{xN(a) \mid x \in G\}$

Now Define $\phi : C(a) \longrightarrow \frac{G}{N(a)}$ given by.

$\phi(xax^{-1}) = xN(a) \quad \forall x \in G$ — (2)

(1) ϕ is well defined :

since $xax^{-1} = yay^{-1}$

$\Rightarrow y^{-1}xax^{-1}x = y^{-1}yay^{-1}x$

$\Rightarrow y^{-1}xae = eay^{-1}x$

$\Rightarrow y^{-1}xa = ay^{-1}x$

$\Rightarrow y^{-1}x \in N(a)$ $\quad$ ($\because x \in H$ iff $xH = H$)

$\Rightarrow y^{-1}xN(a) = N(a)$

$\Rightarrow yy^{-1}xN(a) = yN(a)$

$\Rightarrow e.xN(a) = yN(a)$

$\Rightarrow xN(a) = yN(a)$

$\Rightarrow \phi(xax^{-1}) = \phi(yay^{-1})$

(2) clearly, ϕ is onto. By eqn (2)

(3) ϕ is one-one :

Since $\phi(xax^{-1}) = \phi(yay^{-1})$

$\Rightarrow xN(a) = yN(a)$

$\Rightarrow y^{-1}xN(a) = y^{-1}yN(a)$

$\Rightarrow y^{-1}xN(a) = N(a)$ $\quad$ ($\because y^{-1}y = e$).

$\Rightarrow y^{-1}x \in N(a)$ $\quad$ ($\because xH = H$ iff $x \in H$)

$\Rightarrow y^{-1}xa = ay^{-1}x$

$\Rightarrow yy^{-1}xax^{-1} = yay^{-1}xx^{-1}$ $\quad$ ($\because xx^{-1} = e$, $yy^{-1} = e$)

$\Rightarrow xax^{-1} = yay^{-1}$

$\therefore \phi$ is ~~well~~ one-one

Thus $\phi : C(a) \longrightarrow \dfrac{G}{N(a)}$ is one-one, onto

$$\therefore \quad |C(a)| = \left|\frac{G}{N(a)}\right| = [G : N(a)]$$

By (1) $\therefore |G| = \sum_{a \in C} |C(a)| = \sum_{x \in C} [G : N(x)]$

$\therefore x = a \in C$

$$\therefore \quad |G| = \sum_{a \in C} [G : N(a)]$$

Note - If G is group $Z(G) \triangle G$, $Z(G) \subseteq N(a) < G$ $\forall a \in G$

$G \neq \{e\} \Rightarrow |N(a)| > 1$.

$\therefore a \in Z(G)$ iff $N(a) = G$.

iff $C(a) = \{xax^{-1} \mid x \in G\} = \{a\}$

iff $|C(a)| = \left|\dfrac{G}{N(a)}\right| = [G : N(a)] = 1$

$a \notin Z(G)$ iff $N(a)$ is a proper Subgroup of G

iff $|C(a)| = [G : N(a)] > 1$.

Definition - Let S & T be two subsets of a group G then T is said to be Conjugate to S if $\exists\, x \in G$ s.t

$$T = xSx^{-1}.$$

Theorem - Let G be a group. then for any Subset S of G

$$|C(S)| = [G : N(S)]$$

proof - Let G be group with identity e.

Now,

$$C(S) = \{xSx^{-1} \mid x \in G\}$$
$$= \{x \in G \mid xS = Sx \; \forall x \in G\}.$$

We know that

$$N(S) = \{x \in G \mid x^{-1}Sx = S\}.$$
$$= \{x \in G \mid xS = Sx\}$$

& $$\frac{G}{N(S)} = \{xN(S) \mid x \in G\}$$

Define a mapping $\phi : C(S) \longrightarrow \frac{G}{N(S)}$

given by $\phi(xSx^{-1}) = xN(S)$. $\forall x \in G$.

i) ϕ is well define & one-one :

Since, we have $xSx^{-1} = ySy^{-1}$

premultipling y^{-1} on both side

$\Leftrightarrow \quad y^{-1}xSx^{-1} = y^{-1}ySy^{-1}$

$\Leftrightarrow \quad y^{-1}xSx^{-1} = Sy^{-1} \qquad (\because y^{-1}y = e)$

Post multipling x on both side

$\Leftrightarrow \quad y^{-1}xSx^{-1}x = Sy^{-1}x$

$\Leftrightarrow \quad y^{-1}xS = Sy^{-1}x \qquad (\because x^{-1}x = e)$

$\Leftrightarrow \quad y^{-1}x \in N(S)$

$\Leftrightarrow \quad y^{-1}xN(S) = N(S)$.

$\Leftrightarrow \quad yy^{-1}xN(S) = yN(S)$

$\quad xN(S) = yN(S) \quad (\because yy^{-1} = e)$

$\Leftrightarrow \quad \phi(xSx^{-1}) = \phi(ySy^{-1})$

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ii) ϕ is onto :-

consider $y \in \frac{G}{N(S)}$ = codomain of ϕ.

$\Rightarrow y = xN(S) \quad \forall x \in G.$

$\Rightarrow y = \phi(xgx^{-1})$

$\Rightarrow \phi(xSx^{-1}) = y$ = Codomain of ϕ.

$\Rightarrow$ Range of ϕ = codomain of ϕ.

$\therefore \phi$ is onto.

$\therefore \phi : C(S) \rightarrow \frac{G}{N(S)}$ is one-one, & onto.

$$\therefore |C(S)| = \left|\frac{G}{N(S)}\right|.$$

Note - class equation of finite group G is

$$|G| = |Z(G)| + \sum_{x \in C} [G : N(x)]$$

Where C contains exactly one element from each conjugate class containing at least two element.

* Theorem :-

Let G be a finite group order of p^n where p is prime & $n > 0$ then.

i) G has a nontrivial center Z.

ii) $Z \cap N$ is nontrivial for any nontrivial normal Subgroup N of G.

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iii) If H is a proper subgroup of G then H is properly contained in $N(H)$. Hence if H is a subgroup of order p^{n-1} then $H \trianglelefteq G$.

proof:- Let G be finite group of order p^n where p is prime & $n \in \mathbb{N}$
(i.e G is a p-group).
$Z = Z(G)$ be centre of G.

If G is abelian then $Z(G) = G$.
Consider G as a nonabelian group.

(i) class equation of G is

$$|G| = |Z| + \sum_{x \in C} [G : N(x)] \quad \text{—— (1)}$$

Where C contains exactly one element from each conjugate class containing more than one element.

for $x \in C$ $\quad Z(G) \subsetneq N(x) \subsetneq G$ so
$\therefore x \notin Z(G)$ $\quad |N(x)| > 1$.

& $|N(x)| < p^n = |G|$
$N(x)$ is subgroup of G.
By lagrange's theorem, $|N(x)| \,\big|\, |G|$
i.e $|N(x)| \,\big|\, p^n$.
$\Rightarrow |N(x)| = p^i$ for some i $1 \leq i \leq n-1$.

$$\therefore [G : N(x)] = \left|\frac{G}{N(x)}\right| = \frac{|G|}{|N(x)|} = \frac{p^n}{p^i} = p^{n-i}$$

is a multiple of p ($\because n-i \in \mathbb{N}$) $\forall x \in C$

$\therefore \sum_{x \in C} [G : N(x)]$ is a multiple of p.

Now eq^n ①

$$|Z| = |G| - \sum_{x \in C} [G : N(x)]$$

$$= p^n - \sum_{x \in C} [G : N(x)] \text{ is a multiple of } p.$$

i.e $p \mid |Z|$. i.e $|Z|$ is a multiple of p.

i.e $|Z|$ has atleast p element

$$\Rightarrow Z = Z(G) \neq \{e\}$$

i.e centre of G is nontrivial.

(ii) Let N be a nontrivial normal subgroup of G

i.e $\{e\} \subsetneq N \subsetneq G$.

we have class ~~ple~~ equation of G.

$$G = Z \cup \left(\bigcup_{x \in G} C(x) \right)$$

$$G \cap N = \left[Z \cup \left[\bigcup_{x \in G} \{C(x)\} \right] \right] \cap N$$

$$= (Z \cap N) \cup \left[\bigcup_{x \in C} C(x) \cap N \right]$$

$$\Rightarrow N = (Z \cap N) \cup \left[\bigcup_{x \in C} C(x) \cap N \right]$$

$$\therefore |N| = |Z \cap N| + \sum_{x \in C} |C(x) \cap N| \quad \text{②}$$

& $|C(x)| = [G : N(x)]$ is a multiple of p $\forall x \in C$

If $x \in N$ then $gxg^{-1} \in N$ $\forall g \in G$. $(N \Delta G)$.

AS

$C(x) = \{gxg^{-1} \mid g \in G\}$ we have $C(x) \subseteq N$

$C(x) \cap N = C(x)$

~~ie~~

we have $C(x) \cap N = \phi$ if $x \notin N$.

Suppose $C(x) \cap N \neq \phi$. ~~for some $x \in N$~~.

~~is wrong~~.

then $\exists\ y \in G$ such that

$y \in C(x) \cap N$.

$\Rightarrow y \in C(x)$ & $y \in N$.

$\Rightarrow y = gxg^{-1}$ for some $g \in G$.

$\Rightarrow x = g^{-1} y (g^{-1})^{-1} \in N$ as $y \in N \Delta G$

$\Rightarrow$ $x \in N$. which is contradiction.

$\therefore$ ~~our~~ our supposition is wrong

$\therefore$ $C(x) \cap N \neq \phi$ for some $x \in N$ is wrong.

Thus $C(x) \cap N = \phi$. for some $x \notin N$.

for $x \in C$, $C(x) \cap N = C(x)$ or ϕ.

$\therefore$ $|C(x) \cap N| = |C(x)|$ or 0 a multiple of p.

Now $|N| > 1$ By lagrange's Theorem

$\Rightarrow$ $|N| / |G|$ i.e $|N| / p^n$.

i.e $|N|$ is multiple of p.

By eq^n 2

$|Z \cap N| = |N| - \sum_{x \in C} |C(x) \cap N|$. is multiple of p.

$\therefore$ $|Z \cap N|$ is multiple of p i.e $Z \cap N$.

Contain atleast p elements.

$\therefore$ $Z \cap N \neq \{e\}$.

i.e $Z \cap N$ is a nontrivial subgroup of G

iii) Let H be a proper subgroup of G.
Let K be a maximal normal subgroup of G contained in H

As $K \trianglelefteq G$ so $\frac{G}{K}$ is a quotient group.
Now

$$|G| = p^n \quad \therefore \left|\frac{G}{K}\right| = \frac{|G|}{|K|}.$$

By ① $\frac{G}{K}$ has a nontrivial centre say $\frac{L}{K}$

$$\frac{L}{K} = Z\left(\frac{G}{K}\right) \trianglelefteq \frac{G}{K}.$$

$$\text{i.e } \frac{L}{K} \trianglelefteq \frac{G}{K} \quad \& \quad \frac{L}{K} \neq \{K\}.$$

$$\text{Then } L \trianglelefteq G. \quad K \subsetneq L.$$

L cannot be contained in H
i.e. $L \not\subseteq H$.

Let $l \in L$ & $h \in H$ any then $hK, lK \in G/K$.

$$lK \in \frac{L}{K} = Z\left(\frac{G}{K}\right)$$

Commute with every element of $\frac{G}{K}$

$$\therefore hK \cdot lK = lK \cdot hK$$

$\therefore hl = he \cdot le \in hK \cdot lK$ i.e $hl \in lK \cdot hK$

$\Rightarrow hl = lk_1 \cdot hk_2$ for some $k_1, k_2 \in K$.

$\Rightarrow l^{-1}hl = k_1hk_2$ & $k_1h \in Kh = hK$ $(K \trianglelefteq G)$
i.e $k_1h = hk_3 \in hK \subseteq H$

$\Rightarrow l^{-1}hl \in H \quad \forall h \in H \quad \forall l \in L.$

$$\therefore l^{-1}Hl \subseteq H \quad \forall l \in L.$$

Replaing l by l^{-1}

$\Rightarrow (l^{-1})^{-1} H l^{-1} \subseteq H \quad \forall l \in L.$

$\Rightarrow \quad lHl^{-1} \subseteq H \quad \forall l \in L.$

$\Rightarrow \quad lHl^{-1} = H \quad \forall l \in L.$

$\Rightarrow \quad l \in N(H) \quad \forall l \in L.$

i.e $\quad L \subseteq N(H).$

we have. $H \subseteq N(H)$ always $\forall H < G.$

[N(H) is the largest subgroup of G in which H is normal]

Suppose $H = N(H)$ Then

$L \subseteq N(H) = H \Rightarrow L \subseteq H \rightarrow \leftarrow$. to $L \not\subseteq H$.

Hence $H \neq N(H)$ i.e

H is proper subgroup of N(H)

In particular for $|H| = p^{n-1}$ (H is a proper subgroup of G. $|G| = p^n$).

we have

$$H \subsetneq N(H) < G.$$

$\therefore \quad |N(H)| > |H| = p^{n-1} \;\&\; |N(H)| < |G| = p^n$

$\therefore$ By lagrange's theorem,

$|N(H)| \,\big|\, |G|$ i.e $|N(H)| \,\big|\, p^n$.

$\Rightarrow \quad |N(H)| = p^n = |G| \;\&\; N(H) \leq G.$

$\Rightarrow \quad N(H) = G.$

$\therefore H \trianglelefteq N(H)$ gives $H \trianglelefteq G.$

* Corollary –

Every group of order p^2 (p-prime) is abelian.

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Proof - Let G be a group of order p^2 where p is prime. Then G has a nontrivial center $Z(G)$ & $Z(G) \Delta G$.

$|Z(G)| > 1$ & $Z(G) < G$.

By lagrange's Theorem,

$$|Z(G)| \,\big|\, |G|$$

i.e $|Z(G)| \,\big|\, p^2$.

$\Rightarrow |Z(G)| = p$ or $|Z(G)| = p^2$.

Suppose, $|Z(G)| \neq p^2$ i.e $|Z(G)| = p$.

$\Rightarrow Z(G)$ is proper Subgroup of G.

consider, any $a \in G - Z(G)$

$\Rightarrow a \in G$, $a \notin Z(G)$

Then

$$N(a) = \{x \in G \mid ax = xa\} < G.$$

& $Z(G) \lneq N(a)$

$\Rightarrow |N(a)| > |Z(G)| = p$.

$$|N(a)| < p^2 = |G|.$$

& $|N(a)| \,\big|\, |G|$ $\Rightarrow |N(a)| \,\big|\, p^2$

$\Rightarrow |N(a)| = p^2 = |G|$.

$\Rightarrow N(a) = G$.

i.e $ax = xa$ $\forall x \in G$

$\Rightarrow a \in Z(G)$.

which is contradiction to $a \notin Z(G)$.

$\therefore$ supposition $|Z(G)| \neq p^2$ is wrong.

$\therefore |Z(G)| = p^2 = |G|$. i.e $Z(G) = G$.

$\therefore G$ is abelian group.

Defn -

Let a group G act on the set X. for $g\in G$ we define X_g by

$$X_g = \{x\in X \mid g*x = x\}.$$

* Theorem - (Burnside)

Let G be a finite group acting on finite set X then the number k of obtain in X under G is

$$k = \frac{1}{|G|}\sum_{g\in G}|X_g|.$$

proof - Let G be a finite group and G act on a finite set X Let k be the number of orbit's in X under G.

i.e $|C| = k$

Where C is a subset of X containing exactly one element from each orbit in X under G.

orbit generated by $x\in G$

$$Gx = \{g*x \mid g\in G\}$$

for $g\in G$ $X_g = \{x\in X \mid g*x = x\}$

for $x\in X$ $G_x = \{g\in G \mid g*x = x\}$

Consider the set $S = \{(g,x)\in G\times X \mid g*x = x\}$

for any fixed $g\in G$.

The number of ordered pair (g,x) in S is exactly $|X_g|$ -

$$\therefore S = \sum_{g\in G}|X_g|$$

for any fixed $x \in X$. The number of ordered pairs (g, x) in S is exactly $|G_x|$

$$\therefore |S| = \sum_{x \in X} |G_x|$$

Hence

$$|S| = \sum_{g \in G} |X_g| = \sum_{x \in X} |G_x| \quad \text{—— } 1$$

By theorem

$$\cancel{|S| = \sum_{g \in G} |X_g| =} \sum_{x \in X} |Gx| = [G : G_x]$$

$$\Rightarrow \quad |Gx| = \left| \frac{G}{G_x} \right|$$

$$\Rightarrow |G_x| = \frac{|G|}{|Gx|} \quad \forall x \in X$$

$$\Rightarrow \quad \sum_{x \in X} |G_x| = |G| \sum_{x \in X} \frac{1}{|Gx|} \quad \text{—— } 2$$

Now we know that

$$X = \bigcup_{a \in C} Ga$$

$$\sum_{x \in X} \frac{1}{|Gx|} = \sum_{x \in \bigcup_{a \in C} Ga} \frac{1}{|Gx|} = \sum_{a \in C} \left(\sum_{x \in Ga} \frac{1}{|Ga|} \right) \quad \text{—— } 3$$

Ga = orbit containing a = equivalence class generated by a, $x \in Ga$ iff $Gx = Ga$

$$\therefore \sum_{x \in Ga} \frac{1}{Ga} = 1.$$

$\therefore$ By eqⁿ ③ becomes

$$\sum_{x \in X} \frac{1}{|Gx|} = \sum_{a \in C} 1 = |C| = K$$

$\therefore$ eqⁿ ② becomes

$$\sum_{x \in X} |G_x| = |G|\, K$$

$$\Rightarrow \quad K = \frac{1}{|G|} \sum_{x \in X} |G_x|$$

$$\therefore K = \frac{1}{|G|} \sum_{g \in G} |X_g| .$$

* <u>Example</u> -

① Let H be a subgroup of a group G
Let $A, B \in \mathcal{P}(G)$ the power set G.
Define A be conjugate to B with respect to H if $B = hAh^{-1}$ for some $h \in H$ then

i) Conjugacy define on $\mathcal{P}(G)$ is an equivalence relation

ii) If $C_H(A)$ is a Equivalence classes of $A \in \mathcal{P}(G)$ then $|C_H(A)| = [H : H \cap N(A)]$

<u>Proof</u> - Let H is a subgroup of a finite group G $\quad \mathcal{P}(G) = \{A \mid A \subseteq G\}$
for $A, B \in \mathcal{P}(G)$, then the relation

$$A \sim B \text{ iff } B = hAh^{-1} \text{ for some } h \in H$$

ⓘ <u>Reflexive</u> -

Let $e \in H \quad \therefore A = eAe^{-1} \quad \forall A \in \mathcal{P}(G)$

$$\Rightarrow \quad A \sim A$$

ii) Symmetric :

Let $A \sim B$ iff $B = hAh^{-1}$ for some $h \in H$

$\Rightarrow h^{-1}Bh = A$, for some $h^{-1} \in H$

$\Rightarrow A = h^{-1}B(h^{-1})^{-1}$ for some $h^{-1} \in H$

$\Rightarrow B \sim A$.

iii) Transitive :

Consider $A \sim B$ & $B \sim C$.

$\Rightarrow B = h_1 A h_1^{-1}$ & $C = h_2 B h_2^{-1}$ for some $h_1, h_2 \in H$

$\Rightarrow C = h_2 B h_2^{-1}$

$\Rightarrow C = h_2 h_1 A h_1^{-1} h_2^{-1}$

$\Rightarrow C = h_2 h_1 A (h_2 h_1)^{-1}$ for some $h_2 h_1 \in H$

$\Rightarrow A \sim C$.

Hence conjugacy relation '$\sim$' is an equivalence relation

we have

$C_H(A)$ = Equivalence classes $A \in P(G)$

$= \{ B \in P(G) \mid A \sim B \}$

$= \{ B \mid B = hAh^{-1}$ for some $h \in H \}$

$C_H(A) = \{ hAh^{-1} \mid h \in H \}$.

We know that $H < G$, $N(A) < G$.

$\Rightarrow H \cap N(A) < H$

& $\frac{H}{H \cap N(A)} = \{ x(H \cap N(A)) \mid x \in H \}$ be a quotient group

Define a mapping $\phi : C_H(A) \longrightarrow \frac{H}{H \cap N(A)}$

$\phi(hAh^{-1}) = h(H \cap N(A))$ $\forall h \in H$ —— (1)

i) ϕ is well defined & one-one :-

Since

$$xAx^{-1} = yAy^{-1} \qquad x, y \in H$$

$$\Leftrightarrow y^{-1}xAx^{-1} = Ay^{-1} \qquad (\because y^{-1}y = e)$$

$$\Leftrightarrow y^{-1}xAx^{-1}y = A$$

$$\Leftrightarrow y^{-1}xA(y^{-1}x)^{-1} = A$$

$$\Leftrightarrow y^{-1}x \in H \cap N(A).$$

$$\Leftrightarrow y^{-1}x\,(H \cap N(A)) = H \cap N(A).$$

$$\Leftrightarrow x(H \cap N(A)) = y\,(H \cap N(A)).$$

$$\Leftrightarrow \phi(xAx^{-1}) = \phi(yAy^{-1})$$

$\therefore$ ϕ is well defined & one-one

ii) clearly, ϕ is onto :-

Thus $\phi : C_H(A) \longrightarrow \dfrac{H}{H \cap N(A)}$ is one-one & onto Function.

$$\therefore \ |C_H(A)| = \left| \frac{H}{H \cap N(A)} \right|$$

$$\Rightarrow |C_H(A)| = [H : H \cap N(A)].$$

* Normal series :-

Let G be a group. A sequence $(G_0, G_1, G_2 \cdots G_t$ of subgroups of group G is called a normal series of G if

$$\{e\} = G_0 \triangleleft G_1 \triangleleft G_2 \cdots \triangleleft G_t = G.$$

The factors of a normal series are the quotient $\dfrac{G_i}{G_{i-1}}$, $1 \le i \le t$.

* Definition :-

A composition series of a group G is a normal series $(G_0, G_1, G_2 \cdots G_t)$ with repetition whose factors $\frac{G_i}{G_{i-1}}$ are all simple groups.

The factors $\frac{G_i}{G_{i-1}}$ are called composition factor of G.

Example -

(1) For any group G. $\{e\} = G_0 \subset G_1 = G$ is trivially a normal series of G. If G is a simple group then $\{e\} \subset G$ is the only composition series.

(2) Consider the group $(\mathbb{Z}, +)$
we have following normal series for $\mathbb{Z}$

$$\{e\} \subset \mathbb{Z}, \ \{0\} \subset [2] = 2\mathbb{Z} \subset \mathbb{Z}, \ \{0\} \subset [3] = 3\mathbb{Z} \subset \mathbb{Z}$$

$$\{0\} \subseteq [4] = 4\mathbb{Z} \subset \mathbb{Z} \text{ etc.}$$

[$(\mathbb{Z}, +)$ is abelian, so every subgroup of $\mathbb{Z}$ is normal]

$\mathbb{Z}$ has no composition series.

* Lemma - Every finite group has a composition series.

proof - Let G be a finite group. If $G = \{e\}$ trivial then it has the composition series G without factors.

If G is simple group then G has the composition series $\{e\} \subset G$.

Consider $|G| > 1$ and G is not simple i.e G has a normal subgroup other than

$\{e\} \neq G$. Assume lemma is true for any group whose order is $< |G|$.

Let H be a maximal normal subgroup of G.

i.e $\{e\} \underset{\neq}{\triangleleft} H \underset{\neq}{\triangleleft} G$ & $\frac{G}{H}$ is simple group

Now $|H| < |G|$ by induction hypothesis H is a composition series say

$$\{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \cdots \triangleleft H_s = H$$

Where $\frac{H_i}{H_{i-1}}$ is simple group $\forall i = 1, 2 \cdots s$.

$$\Rightarrow \{e\} = H_0 \triangleleft H_1 \triangleleft H_2 \cdots \triangleleft H_s = H \triangleleft G.$$

is a composition series for the group G.

* Example -

① Consider the quaternion group

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2$ etc

$\cancel{ij = jk = ki = 1}$

Find composition series for Q_8

→ we have

$$\{1\} \subset \{1, -1\} \subset \{\pm 1, \pm i\} \subset Q_8$$

$$\{1\} = G_0 \subset \{\pm 1\} = G_1 \subset \{\pm 1, \pm i\} = G_2 \subset Q_8 = G.$$

$\therefore$

$\therefore \frac{G_1}{G_0} = \frac{\{\pm 1\}}{\{1\}}$ $\therefore \left|\frac{G_1}{G_0}\right| = \frac{2}{1} = 2.$

$\therefore \frac{G_1}{G_0}$ is simple group. & $G_0 \triangleleft G_1$

$\Rightarrow \left|\frac{G_2}{G_1}\right| = \frac{4}{2} = 2$ $\therefore \frac{G_2}{G_1}$ is simple group & $G_1 \triangleleft G_2$.

& $\left|\frac{G}{G_2}\right| = \left|\frac{Q_8}{\{\pm 1, \pm i\}}\right| = \frac{8}{4} = 2$ $\therefore \frac{G}{G_2}$ is simple group $G_2 \triangleleft G$

$$\Rightarrow \{1\} = G_0 \triangleleft \{\pm 1\} = G_1 \triangleleft G_2 \triangleleft Q_8 = G.$$

$\therefore Q_8$ has composition series.

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Similarly,

$\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm j\} \subset Q_8$

& $\{1\} \subset \{\pm 1\} \subset \{\pm 1, \pm k\} \subset Q_8$

is also a composition series.

(2) Find the composition series for S_3

→ We know that

$S_3 = \{e, (123), (132), (12), (23), (13)\}$

Composition series for S_3

$\{e\} \subset \{e, (123), (132)\} \subset S_3$

$\Rightarrow \{e\} \subset A_3 \subset S_3$.

$\therefore \frac{A_3}{\{e\}} = \frac{|A_3|}{|\{e\}|} = \frac{3!}{2} = \frac{\cancel{6}}{\cancel{2}}\ 3$

$\therefore \frac{A_3}{\{e\}}$ is simple group $\{e\} \triangle A_3$.

$\therefore \left|\frac{S_3}{A_3}\right| = \frac{3!}{3!/2} = 2 \quad \therefore \frac{S_3}{A_3}$ is simple group

$\therefore A_3 \triangle S_3$.

$\therefore \{e\} \triangle A_3 \triangle S_3$

$\therefore S_3$ has a composition series.

* Definition -

The normal series $S = (G_0, G_1, G_2 \ldots G_r)$ & $S' = (G_0', G_1', \ldots G_r')$ of G are said to be equivalent written $S \sim S'$ if the factor of one series are isomorphic to the factors of the factors of the orther after some permutation

$$\frac{G_i'}{G_{i-1}'} \simeq \frac{G_{\sigma(i)}}{G_{\sigma(i)-1}} \quad i = 1, 2 \ldots r \text{ for some } \sigma \in S_r$$

Theorem - (Jordan Hölder)

Any two composition series of a finite group are equivalent.

proof - Let G be a finite group. If $|G| = 1$ then it is simple. it has composition series.

Let $S_1 = \{G_0, G_1, G_2 - - - G_{r-1}, G_r\}$
& $S_2 = \{H_0, H_1, H_2 - - - - - H_{s-1}, H_s\}$
be any two composition series of the group G.

$S_1 : \{e\} = G_0 \Delta G_1 \Delta G_2 - - - - \Delta G_{r-1} \Delta G_r = G$
$\frac{G_i}{G_{i-1}}$ is a simple group $\forall\ i = 1, 2 - - r$

$S_2 = \{e\} = H_0 \Delta H_1 \Delta H_2 - - - - \Delta H_{s-1} \Delta H_s = G$
$\frac{H_j}{H_{j-1}}$ is a simple group $\forall\ j = 1, 2 - - - s$

$\therefore G_{r-1}$ & H_{s-1} then by maximal subgroup of G.

Case I :-
If $G_{r-1} = H_{s-1}$ then by induction hypothesis
Any two composition series of G_{r-1} are equivalent.

Also $\frac{G_r}{G_{r-1}} \cong \frac{H_s}{H_{s-1}} = \frac{G}{H_{s-1}}$

So, composition series of G which are S_1 & S_2 and they are equivalent i.e $S_1 \sim S_2$.

Case II :- Consider $G_{r-1} \neq H_{s-1}$

& G_{t-1} & H_{s-1} are maximal subgroup of G.

$G_{t-1}, H_{s-1} \triangle G$ & $G_{t-1} \circ H_{s-1} = G$.

Also, $K = G_{t-1} \cap H_{s-1}$ is a maximal normal subgroup of both G_{t-1} & H_{s-1}

Here $|K| < |G_{t-1}| < |G| \Rightarrow |K| < |G|$

K is finite group so it has a composition series

$$\{e\} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{j-1} \subset K_j = K.$$

Now we have composition series for G_{t-1} & H_{s-1} as $(K_0, K_1, K_2 \cdots K_j, G_{t-1})$ & $(K_0, K_1, K_2 \cdots K_j, H_{s-1})$

consider composition series for G obtained from above.

$S_3 = (K_0, K_1, K_2 \cdots K_j = K, G_{t-1}, G)$

& $S_4 = (K_0, K_1, K_2 \cdots K_j = K, H_{s-1})$.

By second isomorphism Theorem

$$\frac{G_{t-1} \cdot H_{s-1}}{H_{s-1}} \cong \frac{G_{t-1}}{G_{t-1} \cap H_{s-1}} \quad \& \quad \frac{G_{t-1} \cdot H_{s-1}}{G_{t-1}} \cong \frac{H_{s-1}}{G_{t-1} \cap H_{s-1}}$$

i.e $\frac{G}{H_{s-1}} \cong \frac{G_{t-1}}{K}$ & $\frac{G}{G_{t-1}} \cong \frac{H_{s-1}}{K}$.

using above fact we see that $S_3 \sim S_4$

$S_1 = (G_0, G_1, \cdots G_t)$

From Composition series S_1 & S_3 using the fact any two composition series of G_{t-1} are equivalent we have $S_1 \sim S_3$

Similarly, we have $S_2 \sim S_4$
$S_1 \sim S_3$, $S_3 \sim S_4$, $S_4 \sim S_2$ $\Rightarrow S_1 \sim S_2$

$\Rightarrow$ Any two composition series of finite group is equivalent.

Example (1) An abelian group G has a composition series

$$\{e\} = G_0 \subset G_1 \subset G_2 \cdots \subset G_r = G$$

In this case each factor group $\frac{G_i}{G_{i-1}}$ is abelian & simple $\forall i = 1, 2 \cdots r$

Thus gives $\left|\frac{G_i}{G_{i-1}}\right| = P_i$ a prime number for $i = 1, 2 \cdots r$

$\therefore P_1 = \frac{|G_1|}{|G_0|} = \frac{|G_1|}{1} = |G_1|$

$P_2 = \left|\frac{G_2}{G_1}\right| = \frac{|G_2|}{|G_1|} = \frac{|G_2|}{P_1} \Rightarrow |G_2| = P_1 P_2$.

$\vdots$

& $\Rightarrow |G_{r-1}| = P_1 P_2 \cdots P_{r-1}$

$P_r = \left|\frac{G_r}{G_{r-1}}\right| = \frac{|G_r|}{|G_{r-1}|} = \frac{|G|}{P_1 P_2 \cdots P_{r-1}}$

$\Rightarrow |G| = P_1 P_2 \cdots P_{r-1} P_r$

& $\therefore$ An abelian group G has a composition series iff $|G| = P_1 P_2 \cdots P_r$ for some prime $P_1, P_2 \cdots P_r$

i.e $|G|$ is a finite.

$\Rightarrow G$ is a finite group

composition factors of G are determined by the prime factor of $|G|$.

Note- Let G be a group. for $a, b \in G$, $aba^{-1}b^{-1}$ is called a commutator in G subgroup generated by $\{aba^{-1}b^{-1} \mid a, b \in G\}$ by G' or $G^{(1)}$ & $G^{(1)} \triangleleft G$ & $\frac{G}{G'}$ is abelian

We define the n^{th} derived group of G written $G^{(n)}$

$$G^{(1)} = G' \quad \& \quad G^{n} = G^{(n-1)'} \quad n > 1$$

* Definition :-

A Group G is said to be solvable if $G^{(K)} = \{e\}$ for some positive integer k

It is obvious that If G is abelian then $G' = \{e\}$ thus trivially. Every abelian group is solvable.

* Theorem -

Let G be a group. If G is solvable then Every subgroup of G & every homomorphic image of G are solvable conversaly, If N is a normal subgroup of G such that N & $\frac{G}{N}$ are solvable then G is solvable.

proof- i) Let G be a solvable group

To prove - Every subgroup of solvable Group G is solvable.

Let H be any subgroup of G i.e $H < G$

As $\{aba^{-1}b^{-1} \mid a,b \in H\} \subseteq \{aba^{-1}b^{-1} \mid a,b \in G\} < G$

i.e $H' \subseteq G' \Rightarrow H' < G'$

$H < G \Rightarrow H' < G'$

& $H' < G' \Rightarrow (H')' < (G')'$ i.e $H^{(2)} < G^{(2)}$

In General $H^{(i)} < G^{(i)}$ $\forall\ i = 1, 2, 3 \ldots$

In particular, for $i = k$ we have

$$H^{(k)} < G^{(k)} = \{e\} \text{ i.e } H^{k} \subseteq \{e\}$$

$$\text{i.e } \{e\} \subseteq H^{(k)} \subseteq G^{(k)} = \{e\}$$

$$\Rightarrow H^{(k)} = \{e\} \text{ where } k \in \mathbb{N}$$

Hence subgroup H is solvable.

$\therefore$ Every subgroup of solvable group G is solvable.

ii) Let G be a solvable group i.e $G^{(k)} = \{e\}$ $\forall\ k \in \mathbb{N}$.

<u>To prove</u> - Any homomorphic image $\bar{G}$ is solvable.

Let $\bar{G}$ be a homomorphic image of the Group G then $\exists$ an onto group homomorphism

$$\phi : G \longrightarrow \bar{G} \quad [\phi(G) = \bar{G}]$$

for any commutator $aba^{-1}b^{-1}$ in G $(a, b \in G)$

$$\phi(aba^{-1}b^{-1}) = \phi(a) \cdot \phi(b)\, \phi(a^{-1})\, \phi(b^{-1})$$
$$= \phi(a) \cdot \phi(b)\, [\phi(a)]^{-1} [\phi(b)]^{-1}$$ is a commutator in $\bar{G}$ ($\because \phi$ is homomorphism)

Conversaly, Let $a'b'a'^{-1}b'^{-1}$ be any commutator in $\bar{G} = \phi(G)$ $(a', b' \in \bar{G})$

$\because \phi$ is onto $\exists\ a, b \in G$ such that

$$\phi(a) = a' \quad \& \quad \phi(b) = b'$$

Then

$$a'b'a'^{-1}b'^{-1} = \phi(a)\,\phi(b)\,\phi(a)^{-1}\,\phi(b)^{-1}$$

$$= \phi(aba^{-1}b^{-1})$$

$\because \phi$ is a homomorphism.

i.e commutator $\bar{G} = \phi$ (a commutator in G)

$$\Rightarrow \phi[\{aba^{-1}b^{-1} \mid a, b \in G\}] = \{a'b'a'^{-1}b'^{-1} \mid a', b' \in \bar{G}\}$$

$$\phi[\{aba^{-1}b^{-1} \mid a, b \in G\}] = \bar{G}'$$

i.e $\phi(G^{(1)}) = \bar{G}'$ Similarly, $\phi(G^{(1)})' = [\bar{G}^{(1)}]'$

i.e $\phi(G^{(2)}) = \bar{G}^{(2)}$

In general $\phi(G^{(i)}) = \bar{G}^{(i)} \quad \forall i = 1, 2, 3 \ldots$

In particular for $i = k$ we have.

$$\phi(G^{(k)}) = \bar{G}^{(k)} \quad \text{i.e } \phi(e) = \bar{G}^{(k)}$$

i.e $\bar{G}^{(k)} = \{\phi(e)\} = \{e'\}$ where $e' \in \bar{G}$ is the identity.

Hence $\bar{G}$ is a solvable group.

Conside the onto natural homomorphism $\phi : G \to \frac{G}{N}$ given by $\phi(x) = xN \quad \forall x \in G$

Then $\phi(G^{(i)}) = \frac{G}{N}^{(i)} \quad \forall i = 1, 2, 3 \ldots$

for $i = k$ we have

$$\phi(G^{(k)}) = \left(\frac{G}{N}\right)^{(k)} = \{N\}$$

$$\Rightarrow \quad G^{(k)}N = \{N\}$$

$\forall\, a \in G^{(k)} \Rightarrow a = ae \in G^{(k)} N = N.$

$\therefore G^{k} \subseteq N$

From this we have $(G^{(k)})^{(l)} \subseteq N^{(l)}$

i.e $G^{(k+l)} \subseteq N^{(l)} = \{e\}$

$\Rightarrow G^{(k+l)} = \{e\}$ Where $k+l \in \mathbb{N}$

$\therefore G$ is a Solvable group.

* <u>Theorem</u> - A group G is solvable iff G has a normal series with abelian factors. Further a finite group is solvable iff it's composition factors are cyclic group of prime factors.

<u>proof</u> - I) Let G be a solvable group. ~~then~~ i.e $G^{(k)} = \{e\}$ for some $k \in \mathbb{N}$

we have a normal series $k \in \mathbb{N}$

$$\{e\} = G^{(k)} \triangle G^{(k-1)} \triangle G^{(k-2)} \triangle \cdots\cdots \triangle G^{(1)} \triangle G^{0} = G$$

& $\dfrac{G^{(i)}}{G^{(i+1)}}$ is an abelian group $\forall\, i = 0, 2 -- k-1$

$\left[G' \triangle G \text{ \& } \dfrac{G}{G'} \text{ is abelian} \right]$

$\therefore G$ has a normal series with factors as abelian group. conversaly,
Let G be a group having a normal series with abelian factors.
Let the series be $\{e\} = G_l \triangle G_{l-1} \triangle \cdots \triangle G$
with $\dfrac{G_i}{G_{i+1}}$ is abelian group $\forall\, i = 0,1,2 \cdots$

Let $\phi : G_i \longrightarrow \frac{G_i}{G_{i+1}}$ be the onto natural homomorphism

$$\phi(x) = x G_{i+1} \quad \forall x \in G_i \quad 0 \le i \le l-1$$

Now $\frac{G_i}{G_{i+1}}$ is an abelian group

So $\left(\frac{G_i}{G_{i+1}}\right)' =$ Trivial subgroup of $\frac{G_i}{G_{i+1}}$

$$= \{G_{i+1}\}$$

As ϕ is onto homomorphism we have.

$$\phi\left(G_i^{(n)}\right) = \left(\frac{G_i}{G_{i+1}}\right)^n \quad \forall n = 1, 2, 3 \ldots$$

Taking $n = 1$ we have $\phi(G_i^{(1)}) = \left(\frac{G_i}{G_{i+1}}\right)' = \{G_{i+1}\}$

i.e $G_i^{(1)} G_{i+1} = G_{i+1}$

$$\Rightarrow G_i^{(1)} \subseteq G_{i+1} \quad \forall i = 0, 1, 2 \ldots l-1$$

Thus

$$G' = G_0^{(1)} = G_1 \qquad G^2 = (G^{(1)})' = \subseteq G_1^{(1)} \subseteq G_2$$

....

In general. $G^{(k)} \subseteq G_k \quad \forall k = 1, 2 \ldots$.

In particul are $k = l$ $\{e\} \subseteq G^{(l)} \subseteq G_l = \{e\}$

$$\Rightarrow G^{(l)} = \{e\} \quad l \in N$$

$\Rightarrow G$ is solvable group.

II) Let G be a finite solvable group

$\Rightarrow G$ has a normal series with abelian factors. Let the normal series of G.

$$\{e\} = G_l \triangleleft G_{l-1} \triangleleft \cdots \triangleleft G_1 \triangleleft G_0 = G.$$

The factors $\frac{G_i}{G_{i+1}} = F_i$ are abelian groups $\forall\, i = 0, 1, 2 \ldots l-1$.

F_i has a composition series.

$$\{e\} = F_{i,0} \triangleleft F_{i,1} \triangleleft \cdots \triangleleft F_{i,k} = F_i$$

Here $F_{i,0} = \{e_i\}$ is a trivial subgroup of F_i
i.e $e_i = G_{i+1}$ i.e $F_{i,0} = \frac{G_{i+1}}{G_{i+1}} = \{G_{i+1}\} = \{e\}$

$\therefore F_{i,k-1} \triangleleft F_{i,k} = \frac{G_i}{G_{i+1}}$ So $F_{i,k-1}$ is the type $\frac{G_{i,k-1}}{G_{i+1}}$.

as $F_{i,k-1} \triangleleft F_{i,k}$.

we have $G_{i,k-1} \triangleleft G_i$ Also $\Rightarrow \frac{F_i}{F_{i,k+1}} = \frac{G_i | G_{i+1}}{\frac{G_{i,k+1}}{G_{i+1}}}$

is Simple group

& $\frac{F_i}{F_{i,k-1}} \cong \frac{G_i}{G_{i,k-1}}$ (Third isomorphism Theorem)

So, $\frac{G_i}{G_{i,k-1}}$ is a simple group for $i = 0, 1, 2 \ldots l-1$

Thus $G_{i,0} \triangleleft G_{i,1} \cdots \triangleleft G_{i,k-1} \triangleleft G_i$ is a series having factors as simple group for $i = 0, 1, 2 \ldots l-1$

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Now, $\frac{G_i}{G_{i+1}}$ is abelian. So, factors in the above series are abelian & as they are simple are cyclic group of prime order. this we get a composition series from $G_l = \{e\}$ to $G_0 = G$ having factors which are cyclic group of prime order.

conversely, Let G be a finite group having composition series with factors as cyclic as prime order

We know that Every composition series for G. is a normal series & every cyclic group is abelian.

Thus the composition series of G is normal series of G with factors as abelian group

$\therefore$ G is solvable.

ample - ① show that S_3 is a solvable

→ Let $S_3 = \{(e), (123), (132), (23), (12), (13)\}$ is finite group & S_3 has unique composition series

$$\{e\} \Delta \{e, (123), (132)\} \Delta S_3$$

i.e $\{e\} \Delta A_3 \Delta S_3$

with factors $\frac{A_3}{\{e\}} = A_3 \cong Z_3$

& $\frac{S_3}{A_3} = \frac{3!}{3!/2} = 2$

$\therefore$ $\frac{A_3}{\{e\}}$ & $\frac{S_3}{A_3}$ are cyclic group

$\therefore$ S_3 is solvable

(2) Nilpotent group :-

A group G is said to be nilpotent if $Z_m(G) = G$ for some m. The smallest m such that $Z_m(G) = G$ is called the class of nilpotency of G.

Note - Consider a group G. Denote Z_1 as the center of the group G i.e $Z_1 = Z_1(G) = Z(G)$ then $Z_1 \triangleleft G$.

Consider the group $\frac{G}{Z_1}$ & it center $Z\left(\frac{G}{Z_1}\right)$

Now $Z\left(\frac{G}{Z_1}\right) \triangleleft \frac{G}{Z_1}$ & $Z\left(\frac{G}{Z_1}\right) = \frac{Z_2}{Z_1}$ where $Z_2 \triangleleft G$

Also, $Z_1 \triangleleft Z_2$ $\left(\because \frac{Z_2}{Z_1} \triangleleft \frac{G}{Z_1}\right)$

$\therefore Z_1(G) \triangleleft Z_2(G) \triangleleft G$.

Consider the group $\frac{G}{Z_2}$ & it's center $Z\left(\frac{G}{Z_2}\right)$

Let $Z\left(\frac{G}{Z_2}\right) = \frac{Z_3}{Z_2} \triangleleft \frac{G}{Z_2}$ so $Z_2 \triangleleft Z_3 \triangleleft G$

$\Rightarrow Z_1(G) \triangleleft Z_2(G) \triangleleft Z_3(G) \triangleleft G$ & so on.

$\Rightarrow Z_n \cancel{(G)} = Z_n(G)$ is known as n^{th} center of G.

Note (2) If G is abelian then $Z_1(G) = Z(G) = G$

$\therefore Z_1(G) = G$ $\therefore G$ is nilpotent group

(3) Every ~~nilpotent~~ abelian group is nilpotent

(4) $Z_n(G) = \{x \in G \mid xyx^{-1}y^{-1} \in Z_{n-1}(G) \ \forall y \in G\}$

$(Z_n(G))' \subset Z_{n-1}(G)$ The ascending series

$\{e\} \subset Z_0(G) \subset Z_1(G) \ --- \subset Z_n(G) \subset ---$

of Subgroup of G.

Th^m - A group of order p^n (p-prime) is nilpotent.

proof - Let G be a group with $|G| = p^n$ Where p is prime & $n \in \mathbb{N}$

To prove :- G is nilpotent

If G is abelian then $Z_1(G) = G$

$\Rightarrow$ G is nilpotent.

Consider $G \neq Z(G)$. i.e G is a nonabelian p-group.

we have $Z(G) \neq \{e\}$ i.e $\{e\} \lneq Z(G) \lneq G$.

i.e $|\{e\}| < |Z(G)| < |G| \Rightarrow 1 < |Z(G)| < p^n = |G|$

By lagrange's Theorem.

$|Z(G)| \,/\, p^n$ i.e $|Z(G)| = p^r$ $1 \le r \le n-1$

$Z(G) \trianglelefteq G$. so $\frac{G}{Z(G)}$ is a group.

& $\left|\frac{G}{Z(G)}\right| = \frac{|G|}{|Z(G)|} = \frac{p^n}{p^r} = p^{n-r}$ $n-r \in \mathbb{N}$

so, $\frac{G}{Z(G)}$ is also a p-group, so, it's nontrivial.

i.e $Z\left(\frac{G}{Z(G)}\right) \neq \{Z(G)\}$ ($Z(G)$ is identity in $G/Z(G)$)

$\frac{Z_2(G)}{Z_1(G)} = Z\left(\frac{G}{Z_1(G)}\right) \trianglelefteq \frac{G}{Z(G)}$

$\Rightarrow Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq G$.

i.e $\{e\} \trianglelefteq Z_1(G) = Z(G) \trianglelefteq Z_2(G) \trianglelefteq G$

$\{e\} < Z_1(G) < Z_2(G) < G$

$\Rightarrow |\{e\}| < |Z_1(G)| < |Z_2(G)| < |G|$

$$1 < |Z_1(G)| < |Z_2(G)| < |G| = p^n$$

again we get $Z_2(G) < G$, By lagranges Theorem

$$|Z_2(G)| / |G| \Rightarrow |Z_2(G)| \,|\, p^n \Rightarrow |Z_2(G)| = p^{r_1}$$

$$1 \le r_1 \le n-1$$

$$\left|\frac{G}{Z_2(G)}\right| = \frac{|G|}{|Z_2(G)|} = \frac{p^n}{p^{r_1}} = p^{n-r_1}, \quad n-r_1 \in N$$

$\frac{G}{Z_2(G)}$ is also p-group So, it has nontrivial center

i.e $Z\left(\frac{G}{Z_2(G)}\right) \neq \{Z_2(G)\}$

$$\Rightarrow \frac{Z_3(G)}{Z_2(G)} = Z\left(\frac{G}{Z_2(G)}\right) \triangleleft \frac{G}{Z_2(G)} \Rightarrow Z_2(G) \triangleleft Z_3(G)$$

$$\Rightarrow \{e\} \triangleleft Z_1(G) \triangleleft Z_2(G) \triangleleft Z_3(G) \triangleleft G$$

Continuing in this manner we get

$$\{e\} \triangleleft Z_1(G) \triangleleft Z_2(G) \text{-----} \triangleleft G$$

Hence $\exists\ m \in N$ such that $Z_m(G) = G$.

$\therefore$ G is abelian.

Theorem - A group G is nilpotent iff G has a normal series $\{e\} = G_0 \subset G_1 \text{---} \subset G_m = G$ such that $\frac{G_i}{G_{i-1}} \subset Z\left(\frac{G}{G_{i-1}}\right)$ $\forall\ i = 1, 2 \text{--} m$

proof - I) Let G be nilpotent group So, $Z_m(G) = G$ for some $m \in N$

Then we have a normal series of G as

$$\{e\} = Z_0(G) \triangleleft Z_1(G) \text{---} \triangleleft Z_m(G) = G$$

Where $\frac{Z_i(G)}{Z_{i-1}(G)} = Z\left(\frac{G}{Z_{i-1}(G)}\right)$ $\forall\ i = 1, 2 \text{--} m$

i.e $\{e\} = Z_0(G) = G_0 \subset G_1 = Z_1(G) \subset G_2 = Z_2(G) \text{---} \subset G_m = Z_m(G)$

Where $\dfrac{G_i}{G_{i-1}} = \dfrac{Z_i(G)}{Z_{i-1}(G)} \cong Z\left(\dfrac{G}{Z_{i-1}(G)}\right) = Z\left(\dfrac{G}{G_{i-1}}\right)$

$\forall\, i = 1, \ldots m$

i.e $\dfrac{Z_i(G)}{Z_{i-1}(G)} = \dfrac{G_i}{G_{i-1}}$ factors of the series are all abelian groups.

II) conversaly, Let the Group G has normal series $\{e\} = G_0 \Delta G_1 \Delta G_2 \ldots \Delta G_m = G$ with $\dfrac{G_i}{G_{i-1}} \subset Z\left(\dfrac{G}{G_{i-1}}\right)$ $\forall\, i = 1, 2 \ldots m$.

To prove - G is nilpotent.

AS $\dfrac{G_i}{G_{i-1}} \subset Z\left(\dfrac{G}{G_{i-1}}\right) \Rightarrow \dfrac{G_i}{G_{i-1}} < Z\left(\dfrac{G}{G_{i-1}}\right)$

we get $\dfrac{G_i}{G_{i-1}} \Delta Z\left(\dfrac{G}{G_{i-1}}\right)$ $\forall\, i = 1, 2 \ldots m$

for $i = 1$

$G_1 \cong \dfrac{G_1}{G_0} \Delta Z\left(\dfrac{G}{G_0}\right) \cong Z(G) = Z_1(G)$

Thus $G_1 \Delta Z(G) = Z_1(G) \Rightarrow G_1 \subseteq Z_1(G)$

for $i = 2$

$\dfrac{G_2}{G_1} \Delta Z\left(\dfrac{G}{G_1}\right) =$ $\because x \in G_2 \Rightarrow x G_1 \in \dfrac{G_2}{G_1}$

i.e $x G_1 \in Z\left(\dfrac{G_2}{G_1}\right)$

$\Rightarrow x G_1\, y G_1 = y G_1\, x G_1$ $\forall\, y \in G$

$\Rightarrow (xy) G_1 = (yx) G_1$ $\quad ((xN)(yN) = (xy)N)$

$\Rightarrow (xy)(yx)^{-1} G_1 = G_1$

$\Rightarrow xyx^{-1}y^{-1} \in G_1$ $\forall\, y \in G$

$\Rightarrow \quad xyx^{-1}y^{-1} \in Z_1(G) \quad \forall\, y \in G.$

$\Rightarrow \quad x \in Z_2(G). \Rightarrow G_2 \subseteq Z_2(G)$

Continuing in this process we get

$G_n \subset Z_n(G)$

Taking $n=m$ we get $G_m \subset Z_m(G) \subset G$.

i.e $G \subset Z_m(G) \subset G \Rightarrow Z_m(G) = G$
$m \in N$

Hence G is nilpotent.

* Corollary - Every nilpotent group is solvable.

→ Let G be any nilpotent group so $Z_m(G) = G$ for some $m \in N$ then

G has a normal series

$\Rightarrow \{e\} = Z_0(G) \,\Delta\, Z_1(G) \,\Delta\, \cdots \,\Delta\, Z_m(G) = G$

Where $\dfrac{Z_i}{Z_{i-1}} = Z\left(\dfrac{G}{Z_{i-1}}\right) \quad \forall\, i = 1, 2 \cdots m$.

i.e Factor of the normal series of G are all abelian.

$\Leftrightarrow$ G is solvable.