(4) Partial differentiation :

- Definition of partial differentiation
- Order or successive differentiation
- total differentiation and chain rule
- change of variables from cartesian to polar coordinates
- condition for maxima and minima (without proof)
- Linear homogenous partial differential equations with constant coefficients.
- Rules of finding the complementary function.

Questions from previous years papers -

Oct-16 (1) for a function $f(x,y)$, x & y are cartesian coordinates write down these coordinates in polar form.

(2) Explain maxima and minima.

Mar-15 (1) Explain chain rule.

Nov-14 (1) Explain condition for maxima and minima.

Partial differentiation:

Definition: Let $z = f(x, y)$ be function of two variables x and y.

If we keep y constant and x variable then z becomes a function of y only.

The derivative of z with respect to x, keeping y constant is called partial derivative of z wrt x and is denoted by,

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y)$$

Then $$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}.$$

Similarly,

The partial derivative of z wrt y keeping x is constant is denoted by,

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y)$$

then $$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}.$$

Partial differentiation of higher orders (successive differentiation)

Let $z = f(x, y)$ then $\partial z/\partial x$ and $\partial z/\partial y$ being the functions of x and y can ~~be~~ further be differentiated partially with respect to x and y

Then symbolically,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} \text{ or } \frac{\partial^2 f}{\partial y\, \partial x} \text{ or } f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y} \text{ or } \frac{\partial^2 f}{\partial x\, \partial y} \text{ or } f_{xy}$$

also $$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}.$$

Ex: $f(x,y) = x^2 y$

$\frac{\partial f}{\partial x} = 2xy$ (keeping y constant)

$\frac{\partial f}{\partial y} = x^2$ (keeping x constant).

$\frac{\partial^2 f}{\partial x^2} = 2y$ $\quad$ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^2) = 2x$

$\frac{\partial^2 f}{\partial y^2} = 0$ $\quad$ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy) = 2x.$

Total differentiation and chain rule -

IMP

We consider a function $f \cdot z = f(x,y)$ — ①

ie a function z of two variables.

let (x,y), $(x+dx, y+dy)$ be any two points so that dx, dy are the changes in the independent variables x and y.

let dz be the consequent change in z.

$\therefore$ we can write $z + dz = f(x+dx, y+dy)$. — ②

from equation ① and ②.

(substract eqn ① from eqn ②)

$\therefore z + dz - z = f(x+dx, y+dy) - f(x,y)$

$dz = f(x+dx, y+dy) - f(x,y).$

$dz = f(x+dx,\ y+dy) - f(x,y)$

add and subtract $f(x+dx, y)$

$\therefore dz = f(x+dx\,,\ y+dy) - f(x+dx, y) + f(x+dx, y) - f(x,y)$ —Ⓐ

OR method

Lagrange's mean value theorem

$$\frac{f(a+h) - f(a)}{h} = f'(a+\theta h). \qquad \frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\therefore \frac{f(x+dx, y+dy) - f(x+dx,y)}{dy} = f_y'(x+dx\,,\ y+\theta dy) \quad \text{at constant } x+\Delta x$$

$$\frac{f(x+dx,y) - f(x,y)}{dx} = f_x'(x+\theta dx\,, y) \quad \text{at constant } y.$$

$\therefore f(x+dx, y+dy) - f(x+dx,y) = dy \cdot f_y(x+dx, y+\theta dy)$

& $f(x+dx, y) - f(x,y) = dx \cdot f_x(x+\theta dx, y)$.

$\therefore$ eqn Ⓐ becomes.

$dz = dy \cdot f_y(x+dx, y+\theta dy) + dx \cdot f_x(x+\theta dx,\ y)$

$= f_y\, dy + f_x\, dx$

$$dz = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial x} dx$$

$$\therefore \boxed{dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

$$\therefore dz = \frac{f(x+dx, y+dy) - f(x+dx,y)}{dy} dy + \frac{f(x+dx,y) - f(x,y)}{dx} dx$$

$= \frac{\partial f}{\partial y}$ on taking limits as $dx \to 0$ & $dy \to 0$

$$\therefore dz = \lim_{dy\to 0} \frac{f(x+dx, y+dy) - f(x+dx,y)}{dy} dy + \lim_{dx\to 0} \frac{f(x+dx,y) - f(x,y)}{dx} \cdot dx$$

$$\boxed{dz = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial x} dx}$$ This is the expression for partial differentiation

Imp

Composite function (Chain rule)

Let $z = f(x, y)$ and let $x = \phi(t)$ and $y = \psi(t)$

so that x and y are themselves functions of a third variable t.

let $z(x, y)$ possess continuous partial derivatives and let $x = \phi(t)$ and $y = \psi(t)$ also possess continuous derivatives.

Then $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$

$x = \phi(t)$ $\qquad$ $y = \psi(t)$

$x + dx = \phi(t + dt)$ $\qquad$ $y + dy = \psi(t + dt)$

$z + dz = f(x + dx, y + dy)$ & $z = f(x, y)$ (subtracting)

$\therefore dz = f(x + dx, y + dy) - f(x, y)$

$dz = f(x + dx, y + dy) - f(x, y + dy) + f(x, y + dy) - f(x, y)$

$= \dfrac{f(x + dx, y + dy) - f(x, y + dy)}{dx} \cdot dx + \dfrac{f(x, y + dy) - f(x, y)}{dy} dy.$

$= \dfrac{\partial f}{\partial x} dx + \dfrac{\partial f}{\partial y} dy$ by taking limits as $dx \to 0$ and $dy \to 0$.

$$\boxed{\therefore \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.}$$

Q. Find $\frac{dz}{dt}$ when $z = xy^2 + x^2y$ when $x = at^2$ and $y = 2at$.

→ Given $z = xy^2 + x^2y$ and $x = at^2$ $y = 2at$.

to find out $\frac{dz}{dt}$, by chain rule we can write

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \quad \text{--- (1)}$$

$\therefore$ $x = at^2$ and $y = 2at$

$$\frac{dx}{dt} = 2at \qquad \frac{dy}{dt} = 2a.$$

$$\therefore \frac{\partial z}{\partial x} = y^2 + 2xy \quad \text{and also}$$

$$\frac{\partial z}{\partial y} = 2xy + x^2.$$

$\therefore$ eqn (1) becomes,

$$\frac{dz}{dt} = (y^2 + 2xy) \cdot 2at + (2xy + x^2) \cdot 2a.$$

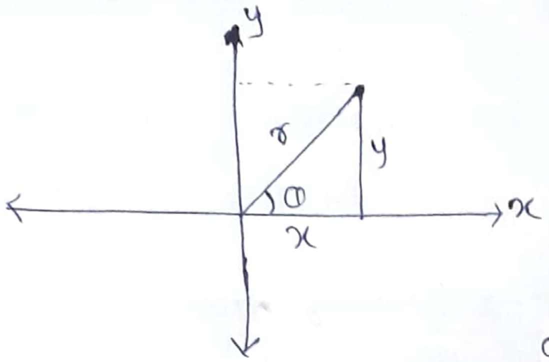
$$= 2aty^2 + 4xyat + 4xya + 2x^2a.$$

$$= 2at(2at)^2 + 4(at^2)(2at)at + 4(at^2)(2at)a + 2(at^2)^2 a$$

$$\boxed{\frac{dz}{dt} = a^3(16t^3 + 10t^4)}.$$

Change of variables from cartesian to polar coordinates :

V.V. Imp



from diagram,

$\sin\theta = \frac{y}{r}$ & $\cos\theta = \frac{x}{r}$

$\therefore y = r\sin\theta$ — ① & $x = r\cos\theta$ — ②

and also squaring and adding eqⁿ ① and ② we get

$$x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta)$$

$$x^2+y^2 = r^2$$

$$\therefore r = \sqrt{x^2+y^2}.$$

and dividing eqⁿ ① by ② we get

$$\tan\theta = y/x$$

$$\therefore \theta = \tan^{-1}(y/x).$$

$x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\qquad$ $\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta} = -r\sin\theta$ $\qquad$ $\frac{\partial y}{\partial \theta} = r\cos\theta.$

$r = \sqrt{x^2+y^2}.$

$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta.$ $\qquad$ $\frac{\partial r}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}. = \frac{r\sin\theta}{r} = \sin\theta$

$\theta = \tan^{-1}(y/x).$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\left(-y/x^2\right) = \frac{-y/x^2}{1+y^2/x^2} = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\left(\frac{1}{x}\right)$$

$$= \frac{1}{\frac{(x^2+y^2)}{x^2}}\cdot\frac{1}{x}$$

$$= \frac{x^2 \times 1/x}{x^2+y^2} = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2}$$

$$= \frac{\cos\theta}{r}.$$

$V(r,\theta)$.

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial V}{\partial r}(\cos\theta) + \frac{\partial V}{\partial \theta}\cdot\left(-\frac{\sin\theta}{r}\right).$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial V}{\partial r}\cdot(\sin\theta) + \frac{\partial V}{\partial \theta}\left(\frac{\cos\theta}{r}\right).$$

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial V}{\partial x}\cdot(\cos\theta) + \frac{\partial V}{\partial y}(\sin\theta)$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial V}{\partial x}(-r\sin\theta) + \frac{\partial V}{\partial y}(r\cos\theta).$$

$$\therefore \boxed{\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}(\cos\theta) + \frac{\partial}{\partial \theta}\cdot\left(-\frac{\sin\theta}{r}\right). \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}(\sin\theta) + \frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r}\right).\end{aligned}}$$

These two equations gives the transformation of $f(x,y)$ into $f(r,\theta)$ ie from cartesian coordinate system to polar coordinate system.

Condition for maxima and minima (without proof)

- The term extreme is used for both maximum and minimum.

✓ The necessary conditions for maxima and minima of a function is that its first derivative must be equal to zero.

ie $f'(x) = 0$.

Here we assume that the function $f(x)$ is differentiable wrt x.

✓ The fun $f(x)$ is a Minimum ~~maximum~~ value of the fun f if

$f'(x) = 0$ and $f''(x) > 0$.

✓ $f(x)$ is a maximum value of the function f if

$f'(x) = 0$ and $f''(x) < 0$.

Ex: Find maximum and minimum value of the given function (f).

$$f(x) = 3x^2 + 5x^3$$

$$f'(x) = 6x + 15x^2$$

$$\therefore 6x + 15x^2 = 0$$

$$2x + 5x^2 = 0$$

$$2x = -5x^2$$

$$x = -2/5$$

$$f'(x) = 6x + 15x^2$$

$$f''(x) = 6 + 30x$$

$$f''(-2/5) = 6 + 30 \times (-2/5)$$

$$= 6 - 12$$

$$= -6 < 0$$

$\therefore f(-2/5)$ is a maximum value.

Linear Homogenous Partial differential equations with constant coefficients :-

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y)$$

is called a homogenous linear partial differential equation of n^{th} order with constant coefficients.

It is called as homogenous because all the terms contain derivatives of the same order.

putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$.

$$\therefore (a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n) z = F(x,y)$$

or $F(D, D')z = F(x,y)$.

Rules of finding the complementary function.

Consider the equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

or $(a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0$

1st step: Put $D = m$ and $D' = 1$

$$a_0 m^2 + a_1 m + a_2 = 0.$$

This is called as auxillary equation.

2^{nd} step: Solve the auxillary equation

Case I: If the roots of the auxillary eqn are real and different. say m_1 and m_2.

Then complentary function

$CF = f_1(y+m_1x) + f_2(y+m_2x)$

Case II: If the roots are equal, say m

Then $CF = f_1(y+mx) + f_2(y+mx)$.

Solve: ① $D^3 - 4D^2D' + 3DD'^2 = 0$

$m^3 - 4m^2 + 3m = 0$

$m(m^2 - 4m + 3) = 0$

$m(m-1)(m-3) = 0$

$m = 0 \quad m = 1 \quad m = 3$.

$\therefore$ the required soln is,

$z = f_1(y) + f_2(y+x) + f_3(y+3x)$

② $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = 0$

$(D^2 - 4DD' + 4D'^2)z = 0$

$D = m$ and $D' = 1$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \quad \Rightarrow m = 2, 2$

$\therefore$ the required soln is $z = f_1(y+2x) + x f_2(y+2x)$.

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