Unit 2

Vector Analysis

* **Vectors :**

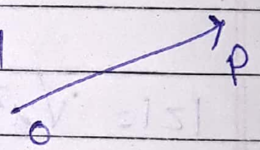
A quantity having both magnitude as well as direction is known as vector
Ex. Velocity, displacement, acceleration

* **Scalars :**

A quantity having only magnitude is called as scalars.
Ex. mass, length.

* **Representation of vectors :**

The vector OP is represented by a line. ~~from~~ as O is the initial point P is the terminal (final) point and is denoted by $\overrightarrow{OP}$



- The magnitude of vector is given by $|\overrightarrow{OP}|$. (also known as modulus).

* **Types of vectors.**

1) Equal vectors : The given two vectors are said to be equal vectors if they have same magnitude & same direction.

2) Null vector : A vector having the initial and terminal points coincident is termed as a zero vector or a null vector. The null vector has modulus zero.

Scanned by CamScanner

3) Unit vector : A vector having its modulus is unity is called as unit vector.

$$\hat{a} = \frac{a}{|a|}$$

4) Collinear vectors: The vectors parallel to the same line, regardless of their magnitudes and dir^ns are termed as collinear vectors.

5) Like vectors : The vectors which are collinear and have same direction

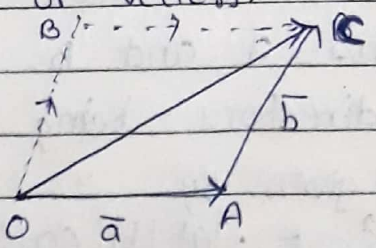
6) Unlike vectors : The vectors which are collinear but have opposite sense of direction are called as unlike vectors.

* Addition of vectors -

Let, the two vectors $\bar{a}$ & $\bar{b}$ are acting at a point O as shown

let $\overrightarrow{OA} = a$ $\overrightarrow{OB} = b$.

The geometrical construction utilised to find the vector sum of two vectors a & b as known as the parallelogram law of addition of vectors.



$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

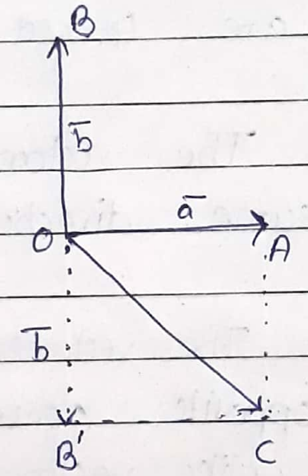
$= a + b$.

* **Subtraction of two vectors:**

If $\bar{a}$ and $\bar{b}$ are two vectors then subtraction of these two vectors is given by

$$\vec{a} - \vec{b} \equiv \vec{a} + (-\vec{b})$$

ie subtraction of $\vec{b}$ from $\vec{a}$ may be regarded as the addition of $-\vec{b}$ and $\vec{a}$.



$$\therefore \bar{a} - \bar{b} \equiv \bar{a} + (-\bar{b}) \equiv \overline{OA} + \overline{OB'} = \overline{OC}.$$

* **Modulus of a vector.**

if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then

$$|r| = \sqrt{x^2 + y^2 + z^2}.$$

* **The scalar or dot product of two vectors :-**

The dot product of two vectors a & b with modulus a and b respectively and their directions being inclined at angle θ is given by

$$\vec{a} \cdot \vec{b} = |a| \cdot |b| \cdot \cos\theta$$

$\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ $\qquad$ $\bar{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$.

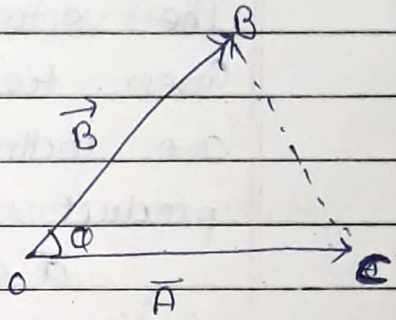
$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = \frac{(a_x\hat{i}+a_y\hat{j}+a_z\hat{k})(b_x\hat{i}+b_y\hat{j}+b_z\hat{k})}{\sqrt{a_x^2+a_y^2+a_z^2}\cdot\sqrt{b_x^2+b_y^2+b_z^2}}$$

or

Definition : The scalar product of two vectors A & B is defined as the product of magnitudes of vectors and cosine of the angle between their directions.

Thus

$\vec{A}\cdot\vec{B} = |A|\cdot|B|\cos\theta$
$= OA\cdot OB\cdot\cos\theta$
$= OA\cdot OC.$



* Characteristics of dot product.

1) The dot product of two vectors $\bar{a}$ & $\bar{b}$ is independent of order.
ie $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a} = ab\cos\theta$
(follows commutative law).

2) If $\bar{a}\cdot\bar{b}=0$ then either of two vectors is a null vector or the vectors a & b are mutually perpendicular
ie $\vec{a}=0$ or $\bar{b}=0$ or $\theta=\pi/2$.

ie $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ then $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular vectors.

3) The vectors $\bar{a}$ & $\bar{b}$ are parallel if $\theta=0$ or π

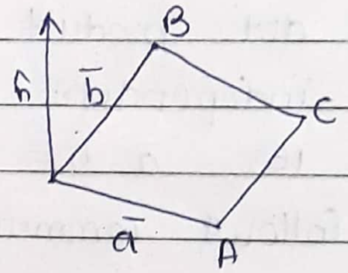
4) The scalar product of two equal vectors $\vec{a}, \vec{a}$ is given by,
$\vec{a}\cdot\vec{a} = a\cdot a\cos\theta = a^2 \quad \because \theta=0$
angle betn two same vectors is 0

In general $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

* The vector or cross product of two vectors:-
Given two vectors $\bar{a}$ & $\bar{b}$ whose directions are inclined at an angle θ their vector product is defined as

$$\bar{a} \times \bar{b} = |a|.|b| \sin\theta \; \hat{n}.$$

$\hat{n}$ is a unit vector perpendicular to the plane of a and b.



1) The vector product is not commutative ie by reversing the order of vectors, the sign of product is reversed.

$$\bar{b} \times \bar{a} = |b||a| \sin(-\theta)\, \hat{n}$$
$$= -|b|\,|a| \sin\theta\, \hat{n}$$

$$\bar{a} \times \bar{b} = |a|.|b| \sin\theta\, \hat{n}.$$

2) The vector a and b are parallel, if the angle bet^n $\bar{a}$ & $\bar{b}$ bet^n their dir^{ns} is 0 or π.

so that $\bar{a} \times \bar{b} = 0$ as $\sin\theta = 0$.
for $\theta = 0$ & π.

3) The vector product of two equal vectors

$\bar{a} \times \bar{a} = 0.$

In particular, $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the principal axis then

$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

& $i \times j = -\hat{j} \times \hat{i} = \hat{k}$

4) The vector product of two unit vectors $\vec{a}$ & $\vec{b}$ is ($\hat{a}$ & $\hat{b}$)

$\hat{a} \times \hat{b} = \sin\theta \ \hat{n}$ as $|\hat{a}| = |\hat{b}| = 1.$

Vector product

$\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\bar{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\hat{i} \times \hat{j} = -(\hat{j} \times \hat{i}) = \hat{k}$

$\hat{j} \times \hat{k} = -(\hat{k} \times \hat{j}) = \hat{i}$

$\hat{k} \times \hat{i} = -(\hat{i} \times \hat{k}) = \hat{j}$

* The cross product of two vectors gives area of parallelogram.

Solve : 1) If $V = 2\hat{i} + 4\hat{j}$ $w = \hat{i} + 5\hat{j}$ then $V \cdot w = ?$ ~~recall~~.

$$V \cdot w = (2\hat{i} + 4\hat{j}) \cdot (\hat{i} + 5\hat{j})$$
$$= 2(1) + 4(5)$$
$$= 2 + 20 = 22.$$

2) Find the angle between $V = 2\hat{i} + 3\hat{j} + \hat{k}$ and $w = 4\hat{i} + \hat{j} + 2\hat{k}$.

→

$$V = 2\hat{i} + 3\hat{j} + \hat{k}$$
$$w = 4\hat{i} + \hat{j} + 2\hat{k}$$

$$\cos\theta = \frac{V \cdot w}{|V| \cdot |w|}$$

$$|V| = \sqrt{(2)^2 + (3)^2 + (1)^2} = \sqrt{14}$$
$$|w| = \sqrt{(4)^2 + (1)^2 + (2)^2} = \sqrt{21}$$

$$V \cdot w = (2\hat{i} + 3\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 2\hat{k})$$
$$= 2(4) + 3(1) + 1(2)$$
$$= 8 + 3 + 2 = 13$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{\sqrt{14} \cdot \sqrt{21}}\right)$$

3) find the cross product ~~of~~ (u×v) if
$u = 2\hat{i} + \hat{j} - 3\hat{k}$ ~~...~~ $v = 4\hat{i} + 5\hat{j}$

→
$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 4 & 5 & 0 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & -3 \\ 5 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -3 \\ 4 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}$$

$$= \hat{i}(0+15) - \hat{j}(0+12) + \hat{k}(10-4)$$
$$= 15\hat{i} - 12\hat{j} + 6\hat{k}.$$

∴ area of parallelogram
$$= \sqrt{(15)^2 + (12)^2 + (6)^2}$$
$$= \sqrt{225 + 144 + 36}$$
$$= \sqrt{405}$$

4) If $a = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $b = \hat{i} + 2\hat{j} + 6\hat{k}$
then calallate -
a) module of each
b) scalar product a.b
c) Vector sum & difference [(a+b), (a-b)]

→ $|a| = \sqrt{9+16+25} = \sqrt{50}$
$|b| = \sqrt{1+4+36} = \sqrt{41}$.
$a \cdot b = (3)(-1) + 4(2) + (-5)(6) = -25$
$a+b = 2\hat{i} + 6\hat{j} + \hat{k}$
$a-b = 4\hat{i} + 2\hat{j} - 11\hat{k}$.

* Vector product is expressed as determinant

If $\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

$\bar{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$

$\bar{a} \times \bar{b} = (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \times (b_x\hat{i} + b_y\hat{j} + b_z\hat{k})$

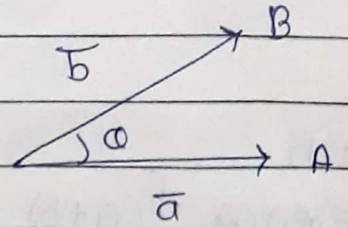
$= a_x b_x(\hat{i}\times\hat{i}) + a_x b_y(\hat{i}\times\hat{j}) + a_x b_z(\hat{i}\times\hat{k})$
$+ a_y b_x(\hat{j}\times\hat{i}) + a_y b_y(\hat{j}\times\hat{j}) + a_y b_z(\hat{j}\times\hat{k})$
$+ a_z b_x(\hat{k}\times\hat{i}) + a_z b_y(\hat{k}\times\hat{j}) + a_z b_z(\hat{k}\times\hat{k})$

$= a_x b_y\hat{k} - a_x b_z\hat{j} - a_y b_x\hat{k} + a_y b_z\hat{i}$
$+ a_z b_x\hat{j} - a_z b_y\hat{i}$

$= (a_y b_z - a_z b_y)\hat{i} - (a_x b_z - a_z b_x)\hat{j}$
$+ (a_x b_y - a_y b_x)\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

* Graphically : dot product of two vectors



$\bar{a}\cdot\bar{b} = |a|\cdot|b|\cdot\cos\theta$

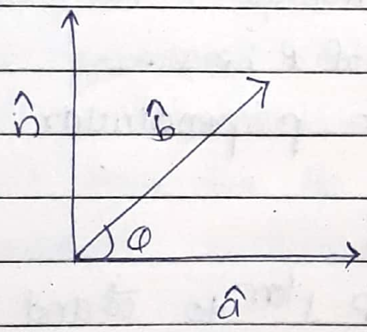
* Vector product (cross product)

1) The vector or cross product of two vectors $\bar{a}$ & $\bar{b}$ is defined ~~as~~ to be a vector such that

a) its magnitude is $|a|.|b| \sin\theta$ where θ is the angle between $\bar{a}$ & $\bar{b}$.

b) its direction is perpendicular ~~to~~ both vectors $\bar{a}$ and $\bar{b}$

c) it forms with a right handed system.



$$\bar{a} \times \bar{b} = |a||b| . \sin\theta \, \hat{n}$$

* Work done as a scalar product.

If a constant force F acting on a particle displaces it from A to B then

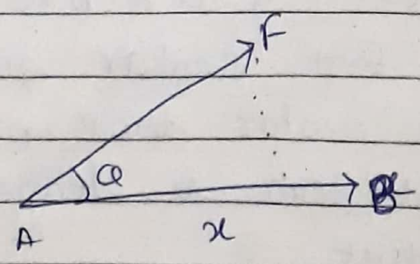
workdone = (component of F along AB). displacement

$= F \cdot \cos\theta . (AB)$

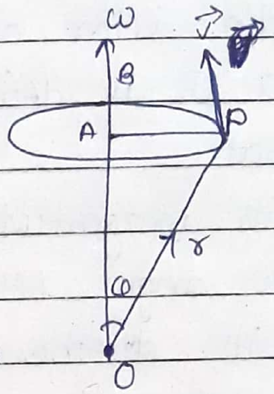
$= F \cdot \vec{AB}$

$= F \cdot x$

= force . displacement.



* ~~Angu~~ Angular velocity as vector product.



Let, a rigid body be rotating about an axis OA with angular velocity ω which is a vector & its magnitude is ω radians per second.

Let P be any point on the body such that $\vec{OP} = \vec{r}$ & $\angle AOP = \theta$ $AP \perp OA$.

Let the velocity of P be v.

$\hat{n}$ be unit vector perpendicular to $\vec{\omega}$ and $\vec{r}$.

$\vec{\omega} \times \vec{r} = (\omega r \sin\theta)\hat{n}$.

velocity of P $\perp^{tor}$ to $\vec{\omega}$ and $\vec{r}$.

hence $\vec{v} = \vec{\omega} \times \vec{r}$

VIMP

* ~~Scalar~~ Triple Product ~~of~~ vectors.

~~A~~, let $\vec{b}$ and $\vec{c}$ are vectors and their vector product $(\vec{b} \times \vec{c})$ is a vector quantity so the product $(\vec{b} \times \vec{c})$ may be multiplied scalarly or vectorially with a third vector $\vec{a}$ to give two triple products as

$\vec{a} \cdot (\vec{b} \times \vec{c})$ & $\vec{a} \times (\vec{b} \times \vec{c})$

where the first scalar quantity is termed as scalar triple product & the second term is termed as vector triple product.

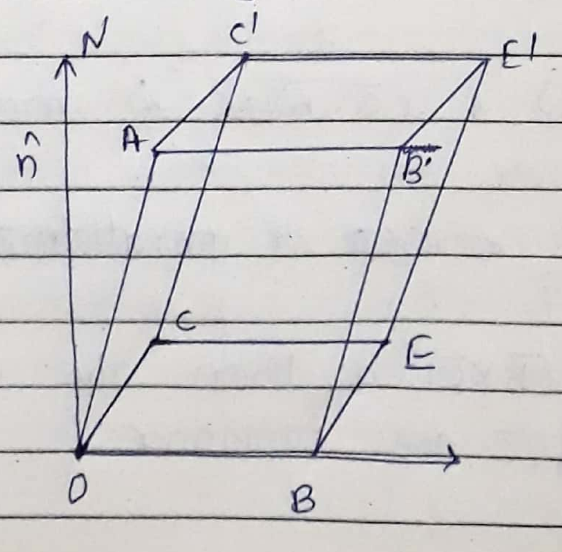
Imp 1) Scalar triple product of vector

let a, b, c are three vectors. Then the scalar product of any of these vectors with the vector product of other two such as $\vec{a}\cdot(\vec{b}\times\vec{c})$ is called scalar triple product of vectors $\bar{a}, \bar{b}, \bar{c}$ and denoted by $[abc]$ or $[a, b, c]$. Obviously this ~~key~~ type of product is a scalar quantity.

Geometric interpretation:

The ~~geometric~~ scalar product of three vectors $\bar{a}, \bar{b}, \bar{c}$ represents the volume of a parallelopiped which ~~has~~ ~~for its~~ has $\bar{a}, \bar{b}$ & $\bar{c}$ as its ~~co-edges~~ co-terminous edges.

Construct a parallelopiped with edges OA, OB and $\vec{OC}$ such that $\vec{OA} = \vec{a}$, $\vec{OB} = \bar{b}$ & $\vec{OC} = \bar{c}$ suppose that $n = b \times c$ and its direction is ON. which is perpendicular to the plane OBEC where adjacent sides are b and c.



or ~~so~~ solve $\bar{b} \times \bar{c}$ and then by $\cdot \bar{a}$

gives $\bar{a} \cdot (\bar{b} \times \bar{c})$

~~The~~ let $\bar{a}, \bar{b}, \bar{c}$ be three vectors then their dot product is given by $\bar{a} \cdot (\bar{b} \times \bar{c})$

If $\bar{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$\bar{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

$\bar{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$.

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \cdot [(b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \cdot (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k})]$$

$$= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) [(b_2 c_3 - b_3 c_2) \hat{i} + (b_3 c_1 - b_1 c_3) \hat{j} + (b_1 c_2 - b_2 c_1) \hat{k}]$$

$$= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Similarly $\bar{b} \cdot (\bar{c} \times \bar{a})$ and $\bar{c} \cdot (\bar{a} \times \bar{b})$ have same values.

$\therefore$ $\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{b} \cdot (\bar{c} \times \bar{a}) = \bar{c} \cdot (\bar{a} \times \bar{b})$.

- $\bar{a} \times (\bar{b} \cdot \bar{c})$ & $(\bar{a} \cdot \bar{b}) \times \bar{c}$ is meaningless.
- $\bar{a} \cdot$ ~~(b×c)~~ ~~area of parallelopiped~~
- If $\bar{a} \cdot (\bar{b} \times \bar{c}) = 0$ then the vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Ex. Find the volume of parallelopiped if $\bar{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $\bar{b} = -3\hat{i} + 7\hat{j} - 3\hat{k}$ and $\bar{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$. are coterminous edges of a parallelopiped.

$$\text{volume} = \bar{a}\cdot(\bar{b}\times\bar{c})$$

$$= \begin{vmatrix} -3 & 7 & 5 \\ -3 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix}$$

$$= 3(-21-15) - 7(9+21) + 5(15-49)$$

$$= 108 - 210 - 170 = -272.$$

Imp

* 2) Vector product of three vectors
(vector triple product) $\Rightarrow$

Let $\bar{a}, \bar{b}, \bar{c}$ are three vectors then their cross product is given by

$$\bar{a}\times(\bar{b}\times\bar{c})$$

$$\bar{a}\times(\bar{b}\times\bar{c}) = (a\cdot c)\, b - (a\cdot b)\, c \quad -①$$

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

To solve eqn ① let us solve LHS.

LHS $= \bar{a}\times(\bar{b}\times\bar{c})$

$$\bar{b}\times\bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\bar b \times \bar c = (b_1\hat i + b_2\hat j + b_3\hat k) \times (c_1\hat i + c_2\hat j + c_3\hat k)$$

$$= b_1c_1(\hat i\times\hat i) + b_1c_2(\hat i\times\hat j) + b_1c_3(\hat i\times\hat k)$$
$$+ b_2c_1(\hat j\times\hat i) + b_2c_2(\hat j\times\hat j) + b_2c_3(\hat j\times\hat k)$$
$$+ b_3c_1(\hat k\times\hat i) + b_3c_2(\hat k\times\hat j) + b_3c_3(\hat k\times\hat k)$$

$$= \cancel{b_1c_2\hat k + b_1c_3\hat j - b_2c_1\hat k + b_2c_3\hat i - b_3c_1\hat j - b_3c_2\hat i}$$

$$\bar b\times\bar c = \begin{vmatrix} \hat i & \hat j & \hat k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat i\,(b_2c_3 - c_2b_3) - \hat j\,(b_3c_1 - b_1c_3) + \hat k\,(b_1c_2 - c_1b_2)$$

$$\bar a \times (\bar b\times\bar c) = \begin{vmatrix} \hat i & \hat j & \hat k \\ a_1 & a_2 & a_3 \\ b_2c_3 - c_2b_3 & b_3c_1 - b_1c_3 & b_1c_2 - c_1b_2 \end{vmatrix}$$

$$= (\vec a\cdot\vec c)\vec b - (\bar a\cdot\bar b)\vec c.$$

Properties (characteristics)

1) $a\times(b\times c) \neq (a\times b)\times c$

$\because a\times(b\times c) = (a\cdot c)b - (a\cdot b)c$

2) $(a\times b)\times c = -c\times(a\times b)$

$= -[(c\cdot b)\cdot a - (c\cdot a)b]$

$= (a\cdot c)b - (b\cdot c)a.$

Imp * 8m

To prove:

vector triple product

$$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a}.\bar{c})\bar{b} - (\bar{a}.\bar{b}).\bar{c}$$

$\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

LHS. $= \bar{a} \times (\bar{b} \times \bar{c})$

let $\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \hat{i}(b_2c_3 - c_2b_3) - \hat{j}(b_1c_3 - c_1b_3) + \hat{k}(b_1c_2 - c_1b_2)$

$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - c_2b_3 & b_3c_1 - b_1c_3 & b_1c_2 - c_1b_2 \end{vmatrix}$

$= [a_2(b_1c_2 - c_1b_2) - a_3(b_3c_1 - b_1c_3)]\hat{i}$
$\quad - \hat{j}[a_1(b_1c_2 - c_1b_2) - a_3(b_2c_3 - c_2b_3)]$
$\quad + \hat{k}[a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - c_2b_3)]$

$= [a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3]\hat{i}$
$\quad + [a_3b_2c_3 - a_3c_2b_3 - a_1b_1c_2 + a_1c_1b_2]\hat{j}$
$\quad + [a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2c_2b_3]\hat{k}$.

adding and subtracting $a_1b_1c_1$ to $\hat{i}$
$a_2b_2c_2$ to $\hat{j}$
& $a_3b_3c_3$ to $\hat{k}$.

$$= [a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3 + a_1b_1c_1 - a_1b_1c_1]\hat{i}$$
$$+ [a_3b_2c_3 - a_3c_2b_3 - a_1b_1c_2 + a_1c_1b_2 + a_2b_2c_2 - a_2b_2c_2]\hat{j}$$
$$+ [a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2c_2b_3 + a_3b_3c_3 - a_3b_3c_3]\hat{k}$$

$$= \cancel{b_1(a_1b_1 + a_2}$$

$$= [b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3)]\hat{i}$$
$$+ [b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3)]\hat{j}$$
$$+ [b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3)]\hat{k}$$

$$= b_1(a_1c_1 + a_2c_2 + a_3c_3)\hat{i} - c_1(a_1b_1 + a_2b_2 + a_3b_3)\hat{i}$$
$$+ b_2(a_1c_1 + a_2c_2 + a_3c_3)\hat{j} - c_2(a_1b_1 + a_2b_2 + a_3b_3)\hat{j}$$
$$+ b_3(a_1c_1 + a_2c_2 + a_3c_3)\hat{k} - c_3(a_1b_1 + a_2b_2 + a_3b_3)\hat{k}$$

$$= (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})(a_1c_1 + a_2c_2 + a_3c_3)$$
$$- (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})(a_1b_1 + a_2b_2 + a_3b_3)$$

$$= \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b})$$

$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a}\cdot\bar{c}) - \bar{c}(\bar{a}\cdot\bar{b})$$

Exam 2016

1) If $\bar{A} = 2\hat{i} - \hat{j} + 4\hat{k}$, $\bar{B} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\bar{c} = 3\hat{i} - 4\hat{j} + 5\hat{k}$.
then prove $\bar{A} \cdot (\bar{B} \times \bar{C}) = 0$.

2) Find vector triple product ~~of~~ for given vectors.
$\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ $\quad \bar{b} = 3\hat{i} + 2\hat{j} - \hat{k}$
$\bar{c} = 2\hat{j} + 4\hat{j} + 3\hat{k}$.

Soln →

1) Given $\bar{A} = 2\hat{i} - \hat{j} + 4\hat{k}$
$\bar{B} = \hat{i} + 2\hat{j} + 3\hat{k}$
$\bar{c} = 3\hat{i} - 4\hat{j} + 5\hat{k}$

$$\bar{B} \times \bar{C} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -4 & 5 \end{vmatrix}$$

$$= \hat{i}(10+12) - \hat{j}(5-9) + \hat{k}(-4-6)$$
$$= 22\hat{i} + 4\hat{j} - 10\hat{k}$$

$$\cancel{\bar{A} \times (\bar{B} \times \bar{C})} = \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 5 \\ 22 & 4 & -10 \end{vmatrix}}$$

$$\bar{a} \cdot (\bar{b} \times \bar{c}) = (2\hat{i} - \hat{j} + 4\hat{k}) \cdot (22\hat{i} + 4\hat{j} - 10\hat{k})$$
$$= 2(22) + (-1)(4) + (4)(-10)$$
$$= 44 - 4 - 40$$
$$= 0.$$

2) →

Given $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\bar{c} = 3\hat{i} - 4\hat{j} + 5\hat{k}$.

$\bar{a} \times (\bar{b} \times \bar{c}) = ?$

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 3 & -4 & 5 \end{vmatrix}$$

$$= \hat{i}(10-4) - \hat{j}(15+3) + \hat{k}(-12-6)$$

$$= 6\hat{i} - 18\hat{j} - 18\hat{k}.$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & -18 & -18 \end{vmatrix}$$

$$= \hat{i}(-36+54) - \hat{j}(-18-18) + \hat{k}(-18-12)$$

$$\boxed{\bar{a} \times (\bar{b} \times \bar{c}) = 18\hat{i} + 36\hat{j} - 30\hat{k}.}$$

3) $\bar{a} = \hat{i} + \hat{j} - \hat{k}$, $\bar{b} = \hat{i} - \hat{j} + \hat{k}$, $c = \hat{i} - \hat{j} - \hat{k}$

$\bar{a} \times (\bar{b} \times \bar{c}) =$

$= 2\hat{i} - 2\hat{j}$

Ex. If $\vec{A} = 4\hat{i} - 5\hat{j} + 3\hat{k}$ $\vec{B} = 2\hat{i} - 10\hat{j} - 7\hat{k}$ and $\vec{C} = 5\hat{i} + 7\hat{j} - 4\hat{k}$ deduce the values of $(A \times B) \cdot C$ and $A \times (B \times C)$

$\longrightarrow$

$\vec{A} = 4\hat{i} - 5\hat{j} + 3\hat{k}$

$\vec{B} = 2\hat{i} - 10\hat{j} - 7\hat{k}$

$\bar{C} = 5\hat{i} + 7\hat{j} - 4\hat{k}$

1) $(A \times B) \cdot C$

Let, $A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -5 & 3 \\ 2 & -10 & -7 \end{vmatrix}$

$= \hat{i}(35 + 30) - \hat{j}(-28 - 6) + \hat{k}(-40 + 10)$

$= 65\hat{i} + 34\hat{j} - 30\hat{k}$

$(A \times B) \cdot C = (65\hat{i} + 34\hat{j} - 30\hat{k}) \cdot (5\hat{i} + 7\hat{j} - 4\hat{k})$

$= 325 + 238 + 120$

$= 683$

2) $A \times (B \times C)$

$B \times C = \begin{vmatrix} i & j & k \\ 2 & -10 & -7 \\ 5 & 7 & -4 \end{vmatrix}$

$= 89\hat{i} - 27\hat{j} + 64\hat{k}$

$A \times (B \times C) = (4\hat{i} - 5\hat{j} + 3\hat{k}) \times (89\hat{i} - 27\hat{j} + 64\hat{k})$

$= -239\hat{i} + 11\hat{j} + 337\hat{k}$.

* Scalar and vector fields -

A physical quantity which is expressible as a continuous function and which can assume one or more definite values at each point of a region of space is said to be a point function in that region and the region satisfying the physical quantity is called as field.

Point functions are of two types

1) scalar point function
2) vector point function.

according to the nature of quantity concerned.

* Vector differential operator (∇)

3) The vector differential operator Del is denoted by ∇ and it is defined as

$$\nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

Scanned by CamScanner

15m
V. Imp

* Gradient of a scalar field:

If $\phi(x,y,z)$ be a scalar function then

$$\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$ is called as

Gradient of a scalar function ϕ.
and ~~the~~ is denoted by grad ϕ or $(\nabla\phi)$.
Thus,

$$\text{grad}\,\phi=\nabla\phi=\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}.$$

$$=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(\phi(x,y,z))$$

$$=\nabla\phi.$$

Explanation:
Suppose, we have a function of three variables say temperature $T(x,y,z)$ in the room.
Start out at one point and set up a system of axes, then at each point (x,y,z) in the room, T gives ~~p~~ temperature at that spot.

$$\therefore \nabla T=\left(\frac{\partial T}{\partial x}\hat{x}+\frac{\partial T}{\partial y}\hat{y}+\frac{\partial T}{\partial z}\hat{k}\right)$$

The ~~derivative~~ gradient of temperature tells us that how temp. T changes when we alter all three variables by infinitesimal amounts dx, dy, dz.

∴ gradient is a vector quantity with three components.

Geometrical interpretation:

- ∇T points in the dir^n of maximum increase of function T.
- also the $|\nabla T|$ gives the slope (rate of increase) along this maximal dir^n.

Ex. Find the gradients of following functions.

1) $f(x,y,z) = x^2 + y^3 + z^4$.

→ $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}\,2x + \hat{j}\,3y^2 + \hat{k}\,4z^3$.

2) $f(x,y,z) = x^2y^3z^4$

→ $\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}\frac{\partial}{\partial x}(x^2y^3z^4) + \hat{j}\frac{\partial}{\partial y}(x^2y^3z^4) + \hat{k}\frac{\partial}{\partial z}(x^2y^3z^4)$

$= \hat{i}(2xy^3z^4) + \hat{j}(x^2 3y^2z^4) + \hat{k}(x^2y^3 4z^3)$

$= (2xy^3z^4)\hat{i} + (3x^2y^2z^4)\hat{j} + (4x^2y^3z^3)\hat{k}$.

3) $f(x,y,z) = e^x y^2 z$.

→ $\nabla f = \frac{\partial}{\partial x}(e^xy^2z)\hat{i} + \frac{\partial}{\partial y}(e^xy^2z)\hat{j} + \frac{\partial}{\partial z}(e^xy^2z)\hat{k}$

$= e^x y^2 z + e^x 2yz\,\hat{j} + e^x y^2\,\hat{k}$

Page No. 57
Date

V. Imp * 15 marks

Divergence of a vector field:

The divergence of a vector function $\bar{F}$ is denoted by div F or $\vec{\nabla}\cdot\vec{F}$ and is defined as,

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

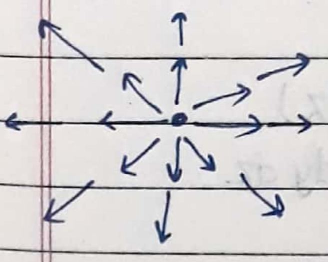
$$\operatorname{div}\bar{F} = \bar{\nabla}\cdot\bar{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(F_1\hat{i}+F_2\hat{j}+F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

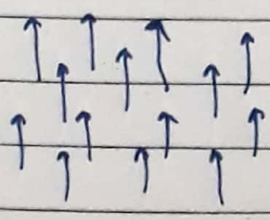
- the divergence of a vector function is a scalar function.
- If divergence is equal to ~~eq.~~ zero ie ($\nabla\cdot\bar{F}=0$) then $\bar{F}$ is known as solenoidal vector function.

Interpretation:

1) The ~~same~~ divergence measures of how much the vector v spreads out (diverges) from the point.



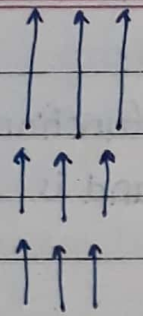
the given figure has large divergence. whereas if the arrows pointed inside, then it would have a negative divergence.



whereas this figure has zero divergence.

Scanned by CamScanner

Solenoidal div $F = 0$



whereas this figure has positive divergence.

Physical interpretation:

consider the case of a fluid flow.

consider small parallelopiped of dimensions dx, dy, dz parallel to x, y, z axes resp.

let $\bar{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$. be the velocity of fluid at point $P(x, y, z)$.

mass of fluid flowing in through the face ABCD in unit time

mass = velocity x area of face.

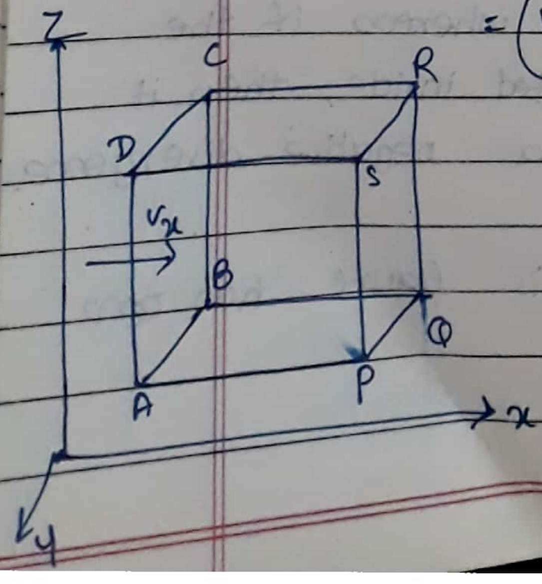
$= V_x \, (dy \, dz)$.

Mass of fluid flowing out across the face PQRS per unit time.

$= V_x(x+dx)\,(dy\,dz)$

$= \left(V_x + \frac{\partial V_x}{\partial x} dx\right) dy\, dz$

$\left(V_x \frac{dx}{dx} + \frac{dV_x}{dx} dx\right) dy\, dz$



net decrease in mass of fluid in the parallelopiped corresponding to the flow of along x - axis per unit time

$$= V_x \, dy \, dz - \left(V_x + \frac{\partial V_x}{\partial x} dx \right) dy \, dz$$

$$= - \frac{\partial V_x}{\partial x} dx \, dy \, dz$$

ie.

the net decrease along x- axis

$$= \frac{\partial V_x}{\partial x} dx \, dy \, dz$$

along y. axis $= \frac{\partial V_y}{\partial y} dx \, dy \, dz$

along z-axis $= \frac{\partial V_z}{\partial z} dx \, dy \, dz$

total decrease of the amount of liquid per unit time $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx \, dy \, dz$

rate of loss of fluid per unit volume

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \right)$$

$$= \vec{\nabla} \cdot \vec{V} \quad = \cancel{\text{div} \, \nabla} \; \text{div} \, \vec{V}$$

Ex: Find divergence of a vector field $\vec{V}$

$\vec{V} = (x-2y)\hat{i} + (3y+4z)\hat{j} + (3x+2y+3z)\hat{k}$

$\rightarrow$ $\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(x-2y) + \frac{\partial}{\partial y}(3y+4z) + \frac{\partial}{\partial z}(3x+2y+3z)$

$= 1+3+3 = 7$ //.

Ex: Find divergence of $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2-y^2z)\hat{k}$. at $(2,-1,1)$

$\rightarrow$ $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2-y^2z)\hat{k}$

$\text{div}.\vec{V} = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2-y^2z)$

$= yz + 3x^2 + 2xz - y^2$

$= -1 + 12 + 4 - 1 = 14$ at point $(2-1\ 1)$

% Imp * 15 marks

Curl of a vector function:

The curl of a vector function F is defined as

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times \left(F_x \hat{i} + F_y \hat{j} + F_z \hat{k} \right)$$

- **Physical meaning of curl:**

we know that $\vec{v} = \vec{\omega} \times \vec{r}$

where $\vec{\omega}$ is angular velocity

$\vec{v}$ linear velocity and $\vec{r}$ is the position vector of a point on the rotating body.

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

~~curl $\vec{v}$ =~~	$\hat{i}$	$\hat{j}$	$\hat{k}$

$$\text{curl } \vec{v} = \vec{\nabla} \times \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

let, $$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$\therefore$ curl $\vec{V} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$

$= \vec{\nabla} \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}]$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}\right]$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \omega_2 z - \omega_3 y & -(\omega_1 z - \omega_3 x) & \omega_1 y - \omega_2 x \end{vmatrix}$$

$= (\omega_1 + \omega_1)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$

$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$

$= 2\omega.$

$\therefore$ curl $\vec{V} = 2\omega$

this shows that the curl of a vector is related to the rotational properties of the vector field and justifies the name rotation used for curl.

If curl $\vec{F} = 0$ then the field is said to be irrotational.

Ex. Find curl of $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$. at $(2, -1, 1)$

$\rightarrow$ curl $\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$

Scanned by CamScanner

$= -2yz\,\hat{i} + (xy - z^2)\,\hat{j} + (6xy - xz)\,\hat{k}.$

curl at $(2,-1,1)$

$= -2(-1)(1)\,\hat{i} + ((2)(-1) - 1)\,\hat{j} + (6(2)(1) - 2(1))\,\hat{k}$

$= 2\hat{i} - 3\hat{j} - 14\hat{k}.$

Ex: Prove that

$(y^2 - z^2 + 3yz - 2x)\,\hat{i} + (3xz + 2xy)\,\hat{j} + (3xy - 2xz + 2z)\,\hat{k}$

is both irrotational and solenoidal.

$\rightarrow$

let $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$

for solenoidal, $\vec{\nabla} \cdot \vec{F} = 0$

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$= (-2) + (2x) + (-2z + 2)$

$= 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right]$$

$$- \hat{j}\left[\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right]$$

$= \hat{i}[3x - 3x] - \hat{j}[3y - 3y] + \hat{k}[3z - 3z]$

$= 0\hat{i} - 0\hat{j} + 0\hat{k}$

$= 0.$

* Laplace Operator :-

Divergence of a gradient is known as laplacian operator and is defined as

$$\vec{\nabla}\cdot(\vec{\nabla}T) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

also the Laplacian ~~oper~~ of a vector is $\nabla^2\vec{V}$ by means of which a vector quantity whose x component is the laplacian of V_x and so on

$$\nabla^2 V = (\nabla^2 V_x)\hat{i} + (\nabla^2 V_y)\hat{j} + (\nabla^2 V_z)\hat{k}.$$

oct-16

• 1) Curl of gradient is always zero.

$$\nabla \times (\nabla T) = 0.$$

2) gradient of divergence $\nabla(\nabla\cdot V)$ is not same as the laplacian operator

$$\nabla^2 V = (\nabla\cdot\nabla)V \neq \nabla(\nabla\cdot V)$$

oct-16

3) divergence of curl is always zero.

$$\nabla\cdot(\nabla \times V) = 0$$

Scanned by CamScanner

1) Curl of gradient is always zero.

$$\vec{\nabla} \times (\vec{\nabla} T) = 0$$

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

$$\vec{\nabla} \times (\vec{\nabla} T) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial T/\partial x & \partial T/\partial y & \partial T/\partial z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial y}\right)\right] - \hat{j}\left[\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial T}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial x}\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)\right]$$

$$= 0.$$

~~2) Gradient of divergence~~ $\cancel{\nabla(\nabla \cdot V)}$

$$\cancel{\vec{\nabla} \cdot \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}.$$

to show $\nabla \times (\nabla f) = 0$

Ex: $f(x,y,z) = x^2 y^3 z^4$.

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= (2xy^3z^4)\hat{i} + (3x^2y^2z^4)\hat{j} + (4x^2y^3z^3)\hat{k}.$$

$$\nabla \times (\nabla T) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(4x^2y^3z^3) - \frac{\partial}{\partial z}(3x^2y^2z^4)\right] - \hat{j}\left[\frac{\partial}{\partial x}(4x^2y^3z^3) - \frac{\partial}{\partial z}(2xy^3z^4)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2y^2z^4) - \frac{\partial}{\partial y}(2xy^3z^4)\right]$$

$$= \hat{i}\left[12x^2y^2z^4 - 12x^2y^2z^3\right] + \hat{j}\left[8y^3z^3x - 8xy^3z^3\right]$$

$$+ \hat{k}\left[6xy^2z^4 - 6xy^2z^4\right]$$

$$= 0.$$

2) divergence of curl is zero.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = 0$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V}) = \frac{\partial}{\partial x}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

$$= 0$$

Ex. $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k}$

$$\nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2-y^2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xz^2-y^2z) - \frac{\partial}{\partial z}(3x^2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xz^2-y^2z) - \frac{\partial}{\partial z}(xyz)\right]$$
$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \hat{i}[-2yz] - \hat{j}[z^2 - xy] + \hat{k}[6xy - xz]$$

$$\nabla\cdot(\nabla\times\vec{V}) = \frac{\partial}{\partial x}(-2yz) - \frac{\partial}{\partial y}(z^2 - xy) + \frac{\partial}{\partial z}(6xy - xz)$$

$$= 0 + x - x$$

$$= 0$$

4

③ Gradient of divergence

$$\text{grad}\,(\vec{\nabla}\cdot\vec{V})$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$= \left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right)\hat{i}$$

$$+ \left(\frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_x}{\partial y \partial x} + \frac{\partial^2 V_z}{\partial y \partial z}\right)\hat{j}$$

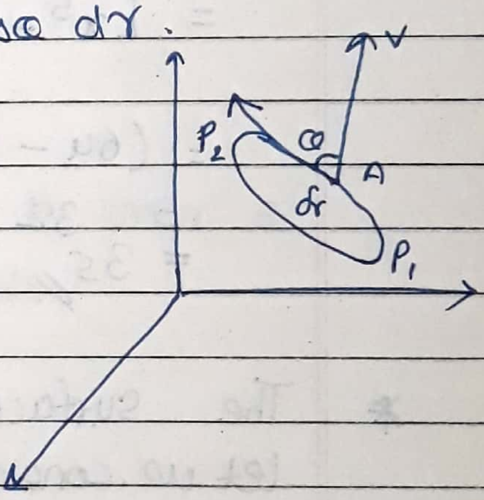
$$+ \left(\frac{\partial^2 V_x}{\partial z \partial x} + \frac{\partial^2 V_y}{\partial z \partial y} + \frac{\partial^2 V_z}{\partial z^2}\right)\hat{k}$$

* The line integral:

The integral of a vector along a curve is called as line integral.
let us consider a vector field defined by $V(x,y,z)$.
consider a small element of length dr at A. The $V \cdot dr$ is the scalar product of the length element and component of V in its dirⁿ i.e. tangentially to the curve.

Thus $V \cdot dr = V \cos\theta \, dr$.

$$\int_{P_1}^{P_2} V \cdot dr$$



Thus the integral $\int V \cdot dr$ is known as line integral

Ex. If $F = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$.
find $\int F \cdot dr$ along the curve C given by $y = x^3$ in xy plane from the point (1,1) to (2,8)

→

$y = x^3$

$\therefore F = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j}$
$= (5x \cdot x^3 - 6x^2)\hat{i} + (2x^3 - 4x)\hat{j}$.

$r = x\hat{i} + y\hat{j}$
$= x\hat{i} + x^3\hat{j}$.

$dr = dx\hat{i} + 3x^2\hat{j}$.

* ~~line integral~~

$$\int F \cdot dr = \int_1^2 \{(5x^4 - 6x^2 y)\hat{i} + (2x^3 - 4x)\hat{j}\} (\hat{i} + 3x^2 \hat{j}) \, dx$$

$$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3) \, dx$$

$$= \left[\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4} \right]_1^2$$

$$= \left[x^5 - 2x^3 + x^6 - 3x^4 \right]_1^2$$

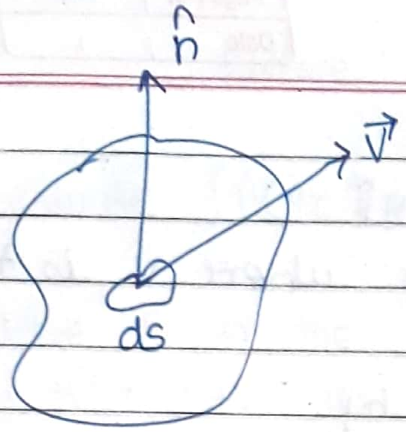
$$= (64 - 16 + 32 - 48] - (1 - 2 + 1 - 3)$$

$$= 32 + 3$$

$$= 35$$

* The surface integral :-
let us consider a vector field $V = V(x, y, z)$ and draw a surface S in this field. let us now take an element of area δS on the surface.

Draw the positive normal n of unit length on this element.
and let θ be the angle between the dirn of n and $\vec{V}$.



component of $\vec{V}$ along the normal
$= \vec{V} \cdot \hat{n}$
where n is the unit normal vector to an element ds.
and $\hat{n} = \frac{\text{grad } \vec{V}}{|\text{grad } \vec{V}|}$

$\therefore$ surface integral of F over s
$$= \iint_S (\vec{F} \cdot \hat{n})\, ds$$

* Volume integral :-
- let $\vec{F}$ be a vector function and volume V enclosed by a closed surface.

The volume integral
$$= \iiint_V F \cdot dv$$

volume

Ex. If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$

Evaluate $\iiint_R F\,dv$ where v is the region bounded by

$x=0 \quad y=0 \quad z=x^2$

$x=2 \quad y=4 \quad z=2$

$$\iiint F\cdot dv = \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$= \int_0^2 dx \int_0^4 dy \int_{x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})\,dz$$

$$= \int_0^2 dx \int_0^4 dy \, . \left(z^2\hat{i} - xz\hat{j} + yz\hat{k}\right)\Big|_{x^2}^{2}$$

$$= \int_0^2 dx \int_0^4 dy \, . \left\{\left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - (x^4\hat{i} + x^3\hat{j} - x^2y\hat{k})\right]\right\}$$

$$= \int_0^2 dx \left(4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\,\hat{i} + x^3y\,\hat{j} - x^2y\hat{k}\right)$$

$$= \int_0^2 \left(16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right)dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4x^5}{5}\hat{i} + x^4\hat{j} - \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{64}{3}\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32\hat{k}}{3} = \frac{32}{15}(3\hat{i} + 5\hat{k})$$

Scanned by CamScanner

surface.

Ex. Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\, d\vec{s}$

where S is the surface of the sphere. $x^2+y^2+z^2=a^2$ in the first octant.

$\phi = x^2+y^2+z^2-a^2$

vector normal to the surface

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-a^2) + \hat{j}\frac{\partial}{\partial \phi}(x^2+y^2+z^2-a^2)$$

$$+ \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-a^2)$$

$$= \hat{i}\,2x + \hat{j}\,2y + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4(x^2+y^2+z^2)}}$$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}.$$

$$\bar{F}\cdot\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)$$

$$= \frac{xyz+zxy+xzy}{a}$$

$$= \frac{3xyz}{a}$$

Stoke's theorem:

Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

mathematically,

$$\oint F \cdot dr = \iint \operatorname{curl} \vec{F} \cdot \hat{n} \, ds.$$

* Gauss Theorem of divergence -

The surface integral of the normal component of a vector function taken around a closed surface S is equal to the integral of divergence of f taken over the volume V enclosed by the surface S.

mathematically,

$$\iint \bar{F} \cdot \hat{n} \, ds = \iiint_V \operatorname{div}(\bar{F}) \, dv$$

$$dv = dx \cdot dx \, dy.$$

— The end —