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Unit (I): Complex Valued Functions Of Complex Variables

Definition (complex valued function of complex variables):

A function $f: \mathbb{C} \to \mathbb{C}$ is called complex valued function of complex variables,
Suppose the $f: \mathbb{C} \to \mathbb{C}$ is defined as $f(z) = w$ where $z = x+iy$ & $w = u+iv$, then f is said to be a mapping from xy-plane to uv-plane

Example: If $f: \mathbb{C} \to \mathbb{C}$ is defined as $f(z) = z^2$ $\forall z \in \mathbb{C}$, then

$f(z) = w \Rightarrow z^2 = w$

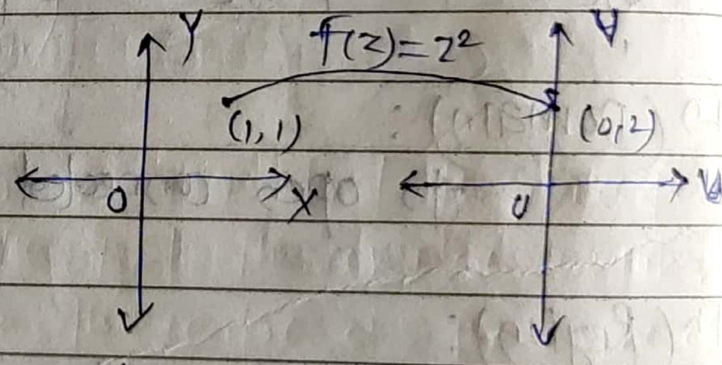
Let $f(z) = z^2$ $\forall z \in \mathbb{C}$ then $f(z) = w$

$z^2 = w$

$(x+iy)^2 = (u+iv)$ $\quad$ $z = (1,1)$ is $w = (0,2)$ under the mapping $w = z^2$

$x^2 - y^2 + 2ixy = u+iv$

$u = x^2 - y^2$ & $2xy = v$



$u = Re(w) = x^2 - y^2$

$v = Im(w) = 2xy$

z-plane $\qquad$ w-plane

suppose $x = 1$, $y = 1$

$\therefore u = x^2 - y^2 = 1^2 - 1^2$

$u = 0$

and

$v = 2xy = 2 \cdot 1 \cdot 1$

$v = 2$

$w = u+iv = (u, v)$

$= (0, 2)$

Thus, image of complex number.

F Linear Transformations :→ A fun^n $W = f(z) = az + b$, 'a' and 'b' are complex const. defined on z-plane is called.

- We know that, the complex valued functions of complex variables maps the points in z-plane onto the points in w-plane.

Here in this article we study some special functions

1) ✱ Translation

- The function $W = z + b$ where b is complex constant is called as translation.
- This function maps the set in z-plane onto the set in w-plane displaced to a vector b.

Note :- Under the translation map the set in w-plane has same size and shape as that of the set in z-plane

Problem :- Show that the map $W = z + (1+2i)$ maps the square having vertices $(\pm 1, \pm i)$ onto a square having vertices i, $2+i$, $3i$, $2+3i$

Solution :- Given, $W = z + (1+2i)$

$$= x + iy + 1 + 2i$$

$$u + iv = (x+1) + i(y+2)$$

$$u = x+1 \quad \& \quad v = y+2$$

Now we will find images of all vertices

i) $z = 1 + i = (1,1)$

$x = 1 \quad y = 1$

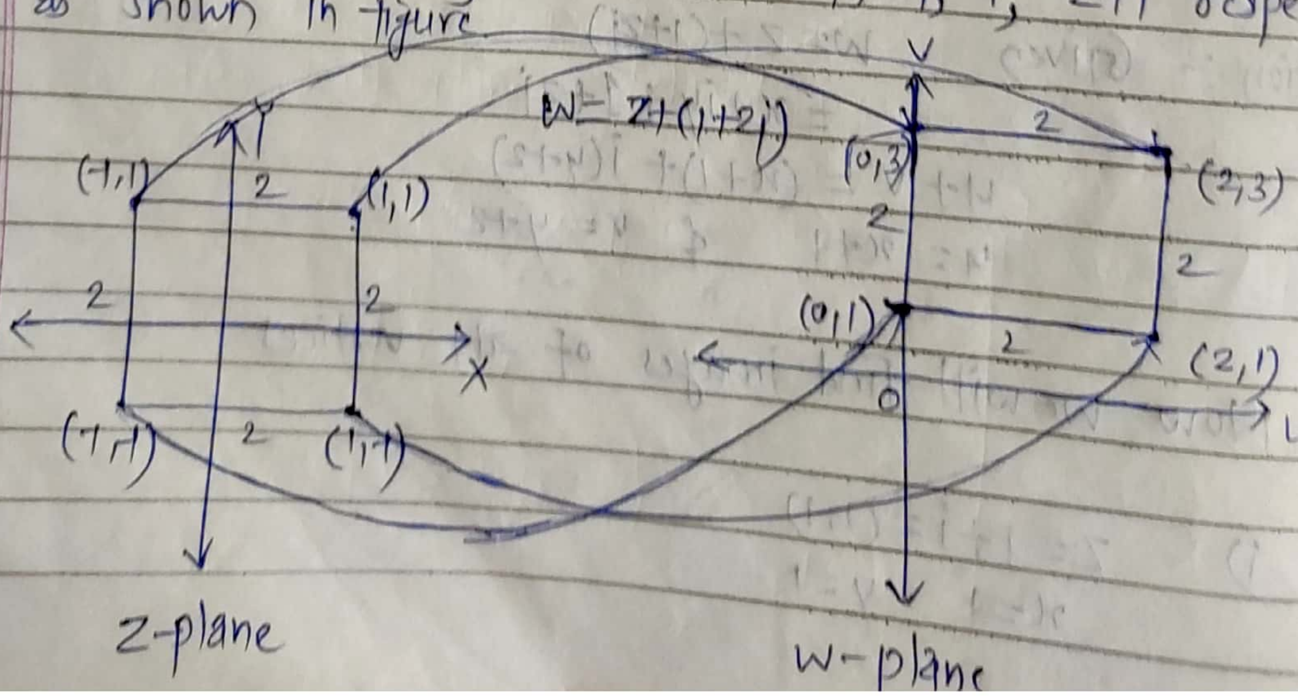
$\Rightarrow (u, v) = (x+1, y+2)$
$= (2, 3)$

ii) $z = (-1, 1)$
$x = -1 \quad y = 1$
$\Rightarrow (u, v) = (x+1, y+2)$
$= (0, 3)$

iii) $z = (-1, -1)$
$x = -1, \quad y = -1$
$\Rightarrow (u, v) = (x+1, y+2)$
$= (0, 1)$

iv) $z = (1, -1)$
$x = 1 \quad y = -1$
$\Rightarrow (u, v) = (x+1, y+2)$
$= (2, 1)$

Thus, the sphere in the (Z-plane) having vertices $1+i$, $-1+i$, $-1-i$, $1-i$ is mapped onto the square in w-plane having vertices $2+3i$, $3i$, i, $2+i$ respectively as shown in figure



Here, both the square in z-plane and w-plane has same size. Only the square in w-plane is displaced through a vector (1, 2)

2) * Rotation Mapping:

The mapping $w = az$ where $a = \cos\alpha + i\sin\alpha$ is called as the rotation mapping which maps the point in z-plane onto the point in w-plane whose modulus is same but argument is changed by 'α' where $\alpha = \arg a$.

If $z = re^{i\theta}$ & $a = \cos\alpha + i\sin\alpha = e^{i\alpha}$, then

$w = az$

$\Rightarrow w = re^{i\theta} \cdot e^{i\alpha}$

$\Rightarrow w = re^{i(\theta+\alpha)}$

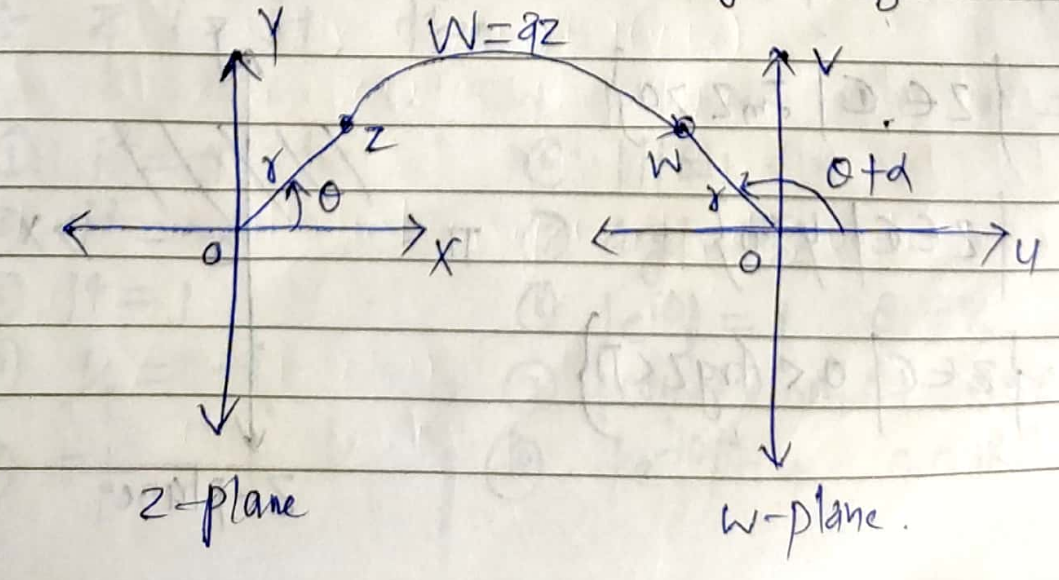
$\therefore |w| = r \quad \arg w = \theta + \alpha$

but $|z| = r$ & $\arg z = \theta$

$\therefore |w| = |z|$ & $\arg w = \arg(az)$

$= \theta + \alpha$

$= \arg a + \arg z$.



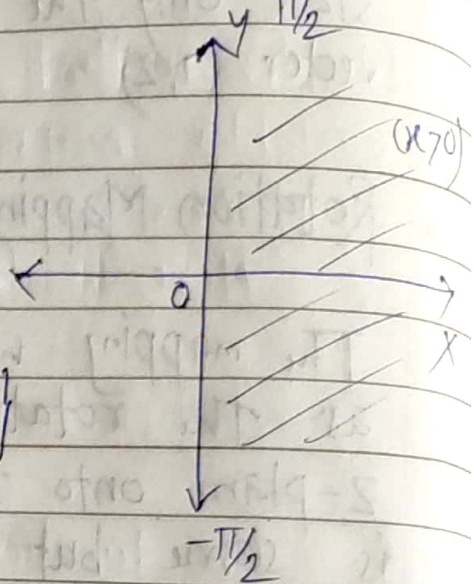
Notation :-

① **Right half plane :**

The right half plane is defined as

$$= \{ z \in \mathbb{C} \mid \operatorname{Re} z > 0 \}$$

$$= \{ z \in \mathbb{C} \mid -\pi/2 < \operatorname{Arg} z < \pi/2 \}$$



Z-plane

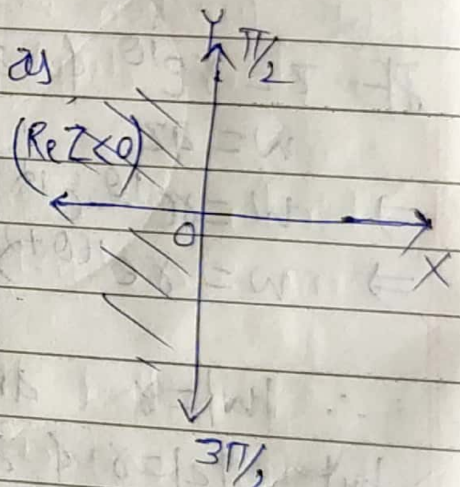
② **Left half plane :**

The left half plane is defined as

$$= \{ z \in \mathbb{C} \mid \operatorname{Re} z < 0 \}$$

$$= \{ z \in \mathbb{C} \mid 3\pi/2 > \operatorname{Arg} z > \pi/2 \}$$

$$= \{ z \in \mathbb{C} \mid \pi/2 < \operatorname{Arg} z < 3\pi/2 \}$$



Z-plane

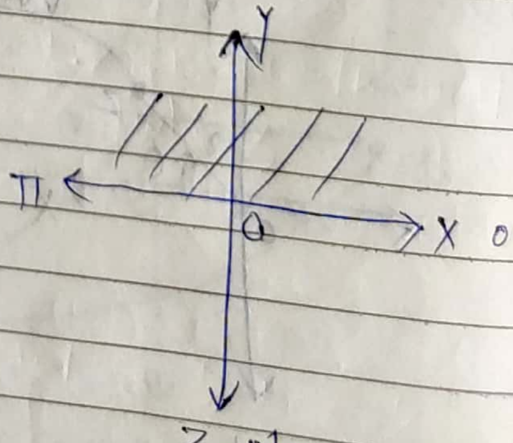
③ **Upper half plane :**

The upper half plane is defined as

$$= \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

$$= \{ z \in \mathbb{C} \mid y > 0 \}$$

$$= \{ z \in \mathbb{C} \mid 0 < \operatorname{Arg} z < \pi \}$$



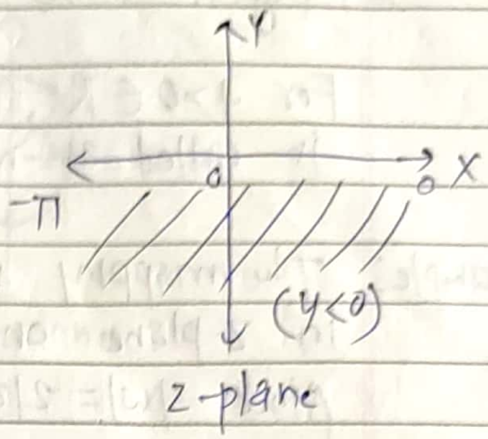
Z-plane

④ Lower half plane :-

The lower half plane in defined as

$$= \{ z \in \mathbb{C} \mid y < 0 \}$$

$$= \{ z \in \mathbb{C} \mid -\pi < \operatorname{Arg} z < 0 \}$$

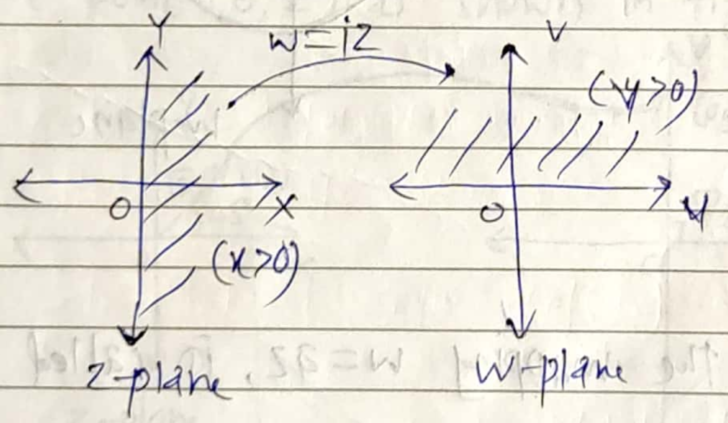


* Example of rotation map :-

The rotation map $w = iz$ maps the right half plane onto upper half plane

$w = iz$

$|i| = 1 \quad \arg i = \pi/2$



Note:- If $i = \sqrt{-1}$, then

① $i^2 = -1$
② $i^3 = -i$
③ $i^4 = 1$
④ $\frac{1}{i} = -i$
⑤ $\frac{-1}{i} = i$
⑥ $|i| = 1$
⑦ $\arg i = \pi/2$
⑧ $|e^{i\theta}| = 1 \quad \theta \in \mathbb{R}$
⑨ $|\cos\theta + i\sin\theta| = 1 \quad \theta \in \mathbb{R}$
⑩ $|e^{-i\theta}| = 1 \quad \theta \in \mathbb{R}$

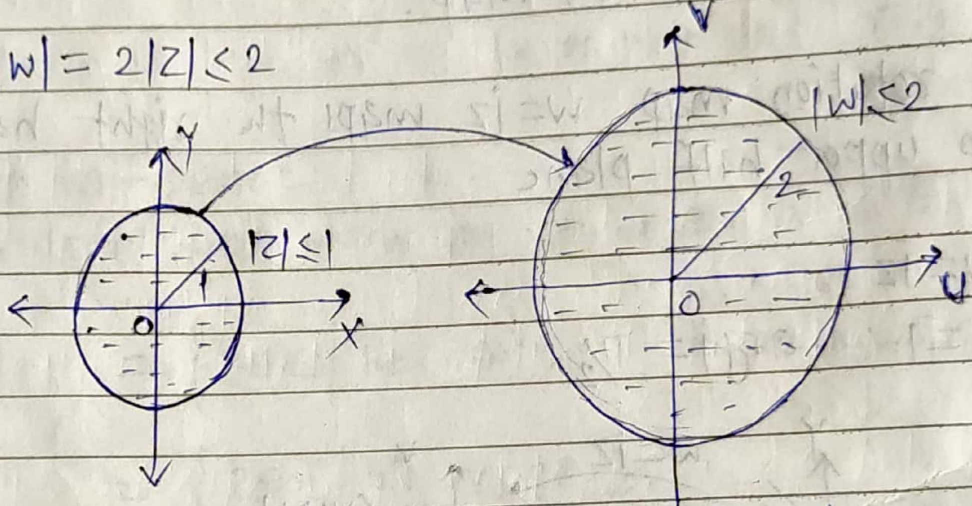
* Magnification :-

For $a > 0 \in R$, $a \neq 1$ and $a > 1$, the mapping $w = az$ is called as magnification.

Example: The mapping $w = 2z$ maps the unit disc $|z| \leq 1$ in z-plane onto the disc $|w| \leq 2$.

since $|w| = 2|z| \leq 2$ $\quad \because |z| \leq 1$

$$|w| = 2|z| \leq 2$$



z-plane

w-plane

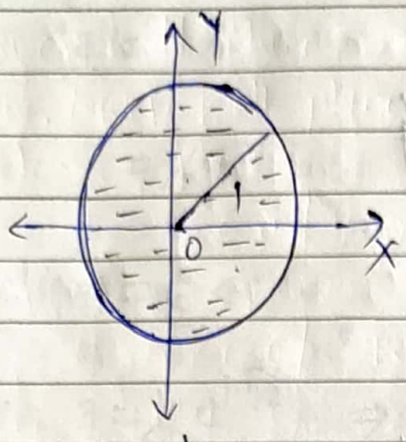
* Contraction :-

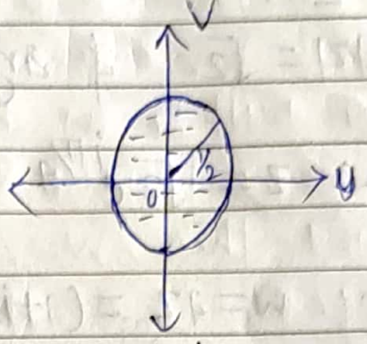
For $0 < a < 1$ the mapping $w = az$, is called as contraction.

Example:- The contraction mapping $w = \frac{1}{2} \cdot z$ maps the unit disc $|z| \leq 1$ onto the disc $|w| \leq \frac{1}{2}$ as shown in figure.

$$w = \frac{1}{2} \cdot z$$

$|z| \leq 1$ $\quad \Rightarrow |w| = \frac{1}{2}|z| \leq \frac{1}{2} \cdot 1 \leq \frac{1}{2}$





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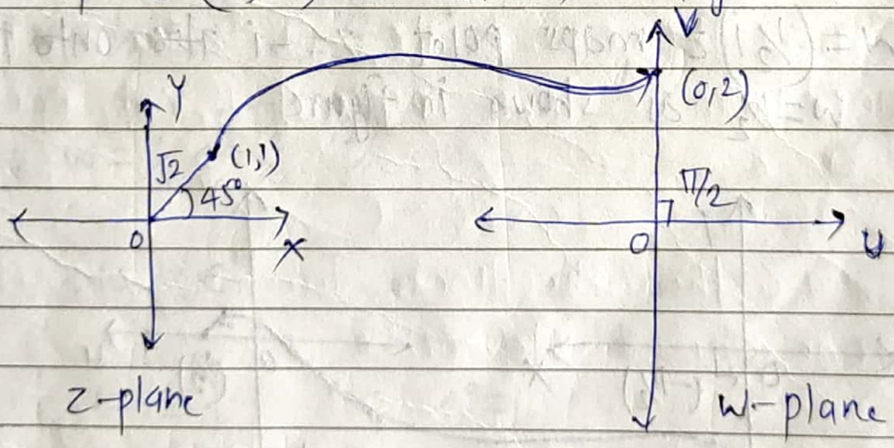
* Rotation and Magnification :

The mapping $w = az$ where $a \in \mathbb{C}$ is such that $|a|>1$ then is called as rotation and magnification.

Example: The mapping $w = (1+i)z$ maps the point $z = 1+i$ onto the point $(0, 2)$ as shown in figure.



$\therefore$ Given $w = (1+i)z = az$ say

$\therefore a = 1+i \Rightarrow |a| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore |a|>1$

& $\arg a = \tan^{-1}(1/1) = \pi/4$

$\therefore a = 1+i = \sqrt{2}\cdot e^{i\pi/4}$

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Given $z = 1+i$

$\therefore |z| = \sqrt{2}$ & $\arg z = \pi/4$

$\therefore z = \sqrt{2}\, e^{i\pi/4}$

Hence, $w = az = (1+i)z$

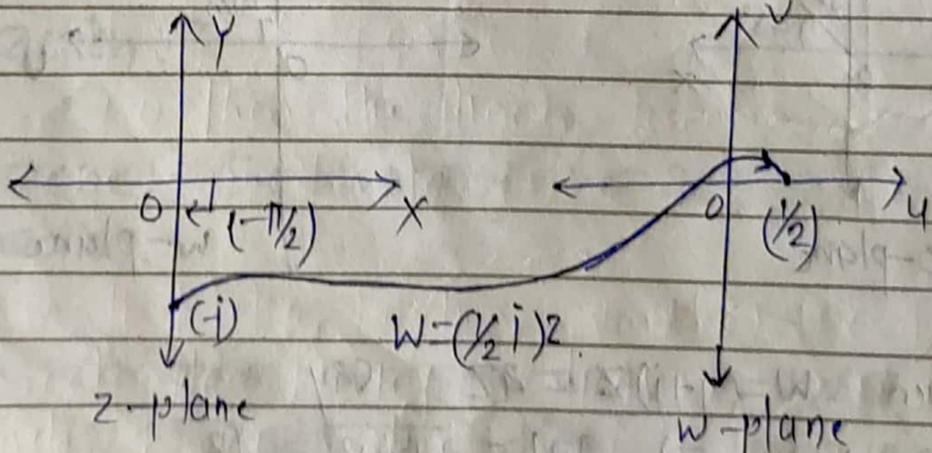
$\Rightarrow w = \left[\sqrt{2}\, e^{i\pi/4}\right]\left[\sqrt{2}\, e^{i\pi/4}\right]$

$= 2e^{i\pi/2}$

* Rotation and Contraction :-

The mapping ~~a =~~ $w = az$ where $a \in \mathbb{C}$ is such that $|a| < 1$ is called as rotation & contraction mapping

Example :- $w = (\frac{1}{2} i) z$ maps point $z = -i$ also onto the point $w = \frac{1}{2}$ as shown in figure



$\therefore w = (\frac{1}{2} i) z = az$

$\Rightarrow a = \frac{1}{2} i$ $\therefore \arg a = \pi/2$

$|a| = |\frac{1}{2} i|$

$= \frac{1}{2} |i|$

$\therefore |a| = \frac{1}{2} < 1$

$\therefore a = \frac{1}{2} e^{i\pi/2} \Rightarrow (\frac{1}{2} i) = \frac{1}{2} e^{i\pi/2}$

$z = -i \Rightarrow |z| = |-i| = 1$

$\arg z = -\pi/2$

$\therefore z = e^{-i\pi/2}$

Hence, $w = az = (+\frac{1}{2} i) z$

$\Rightarrow w = \frac{1}{2} e^{i\pi/2} \cdot e^{-i\pi/2}$

$\therefore w = \frac{1}{2} e^{i(\pi/2 - \pi/2)}$

$\therefore w = \frac{1}{2} e^{0}$

$\therefore \boxed{w = \frac{1}{2}} \quad \because e^{0} = 1$

* Linear function:

- The function $w = az + b$ is called as a linear function where a and b are complex constants & z is complex variable.

Note:- ① If $a=1$ and $b=0$ then we have $w=z$ i.e, $f(z)=z \quad \forall z \in \mathbb{C}$
$\therefore$ w will be an identity mapping.

② If $a=1$ then $w=z+b$ i.e, w is a translational mapping.

③ If $b=0$ then $w=az$ i.e, w is rotation map.

④ A linear function is the combination of rotation and translation.

⑤ Linear function maps circles & straight lines in z-plane onto circles & straight lines in w-plane respectively.

⑥ If $a=0$ then $w=b$ i.e, w is the constant mapping from $\mathbb{C}$ to $\mathbb{C}$ hence it is not one-one and onto on $\mathbb{C}$.

⑦ If $a \neq 0$ then the linear function $w=az+b$ is both one-one & onto on $\mathbb{C}$

Example:- Show that the function $w=(1-i)z+(2+i)$ maps the rectangle having vertices $0, 2, 2+i, i$ onto a rectangle having vertices $2+i, 3+2i, 4-i, 5$ having twice the area than rectangle in z-plane.

Solution:- Given $w=(1-i)z+(2+i)$ i.e, $u+iv=(1-i)(x+iy)+(2+i)$

$$u+iv=(x+iy-ix-i^2y)+(2+i)$$

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$$u+iv = (x+iy-iy+y)+2+i \quad \because i^2=-1$$

$$\therefore \quad u+iv = (x+y+2)+i(y-x+1)$$

$$u = x+y+2 \quad \& \quad v = y-x+1$$

Given a rectangle say $\square PQRS$ in z-plane having vertices $P(0,0)$ $Q(2,0)$ $R(2,1)$ & $S(0,1)$

i) The image of point $P(0,0)$ is (u,v) where $u=x+2+y$ & $v=y-x+1$

here $x=0$ $y=0$

$$u=2 \quad \& \quad v=1$$

$\therefore$ Image of point $P(0,0)$ is $P'(2,1)$

ii) The image of point $Q(2,0)$ is (u,v)

where $u=x+y+2$ & $v=y-x+1$

here $x=2$ & $y=0$

$$u=4 \quad \& \quad v=-1$$

$\therefore$ Image of point $Q(2,0)$ is $Q'(4,-1)$

iii) The image of point $R(2,1)$ is (u,v)

where $u=x+y+2$ & $v=y-x+1$

here $x=2$ & $y=1$

$$\therefore \quad u=2+1+2 \quad \& \quad v=1-2+1$$

$$u=5 \quad \& \quad v=0$$

$\therefore$ Image of point $R(2,1)$ is $R'(5,0)$

iv) The image of point $S(0,1)$ is (u,v).

where $u=x+y+2$ & $v=y-x+1$

here $x=0$ & $y=1$

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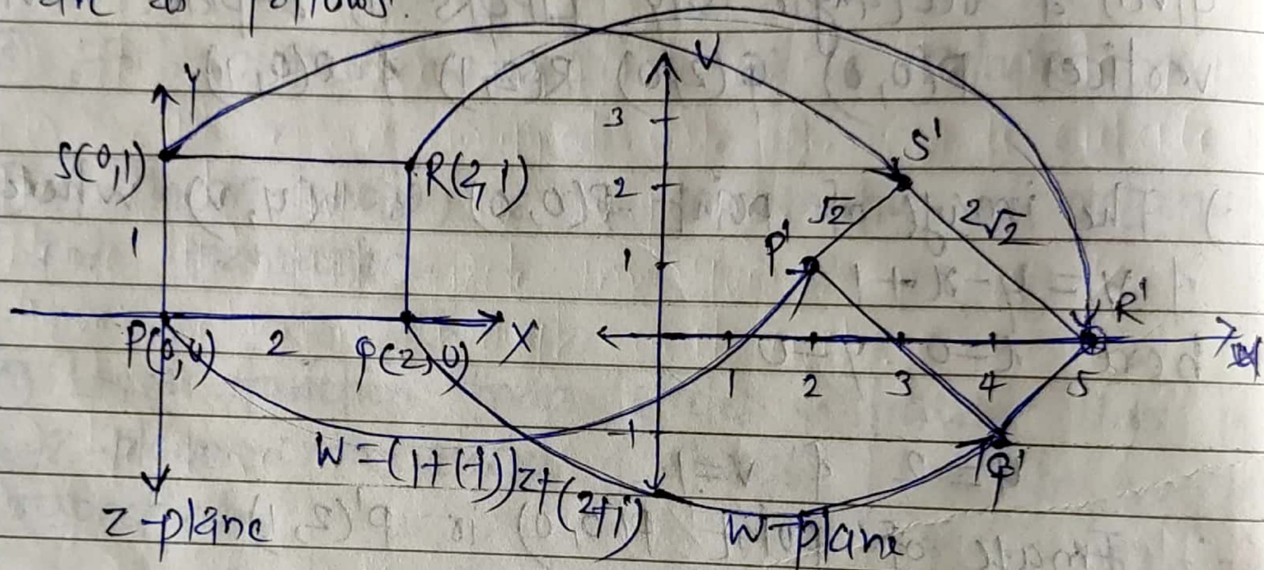
$u = 0+1+2$ & $v = 1-0+1$

$u = 3$ & $v = 2$

$\therefore$ Image of point $S(0,1)$ is $S'(3,2)$

Thus, $\square PQRS$ in z-plane & $\square P'Q'R'S'$ in w-plane are as follows.



Here $P(0,0)$ & $Q(2,0)$ $\therefore PQ = \sqrt{(2-0)^2+(0-0)^2} = 2$

& $S(0,1)$ $\therefore PS = \sqrt{(0-0)^2+(0-1)^2} = 1$

$\therefore A[\square PQRS] = PQ \times PS$

$= 2 \times 1$

$= 2$ sq.units

Also in w-plane,

$P'(2,1)$ $Q'(4,-1)$ & $S'(3,2)$

$\therefore P'Q' = \sqrt{(4-2)^2+(1-(-1))^2} = \sqrt{2^2+2^2}$

$= \sqrt{8} = 2\sqrt{2}$

$P'S' = \sqrt{(3-2)^2+(2-1)^2} = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore A[\square P'Q'R'S'] = P'Q' \times R'S'$

$= 2\sqrt{2} \times \sqrt{2}$

$= 4$ sq. units

Note:- Here the rectangle in w-plane is rotated through an angle $-\pi/4$ because here $a = 1-i$

$\therefore \arg a = \tan^{-1}\left(\frac{-1}{1}\right) = -\pi/4$

② The rectangle in w-plane is displaced through a vector $2+i$ or $(2, 1)$ because here we have $b = 2+i$

③ The rectangle in w-plane ~~got~~ is magnified. i.e, side lengths of rectangle are increased by $\sqrt{2}$ unit because we have

$|a| = |1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2} > 1$

* Inversion :-

The mapping $w = \frac{1}{z}$ is called as an inversion mapping where $z \neq 0$

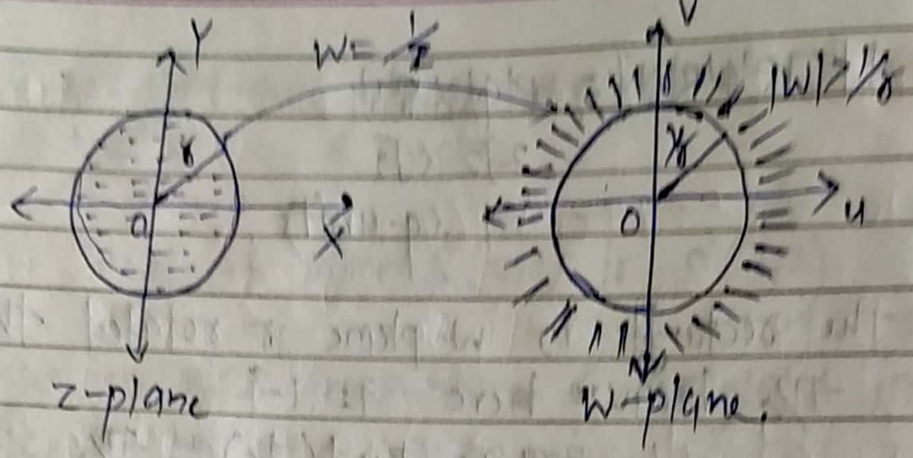
This maps takes the points close to the origin in z-plane onto the points far away from the origin in w-plane, and vice versa.

Because as $z \to 0$, $w \to \infty$

as $z \to \infty$, $w \to 0$, under the origin.

Note:- ① $w = \frac{1}{z}$ maps the points inside the circle $|z| = r$ onto the points outside the circle $|w| = \frac{1}{r}$

i.e, if $|z| < r \Rightarrow |w| > \frac{1}{r}$ as shown in fig.



(e) If $r=1$ then the points inside the circle $|z|=1$ in z-plane are mapped onto the points outside the circle $|w|=1$ in w-plane.
Hence the unit circle inversion map $w=\frac{1}{z}$ is ~~inve~~ reflection through unit circle.

(3) We have $w=\frac{1}{z}$

$$\therefore u+iv=\frac{1}{x+iy}$$

$$=\frac{x-iy}{(x+iy)(x-iy)}=\frac{x-iy}{x^2+y^2}$$

$$u+iv=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$$

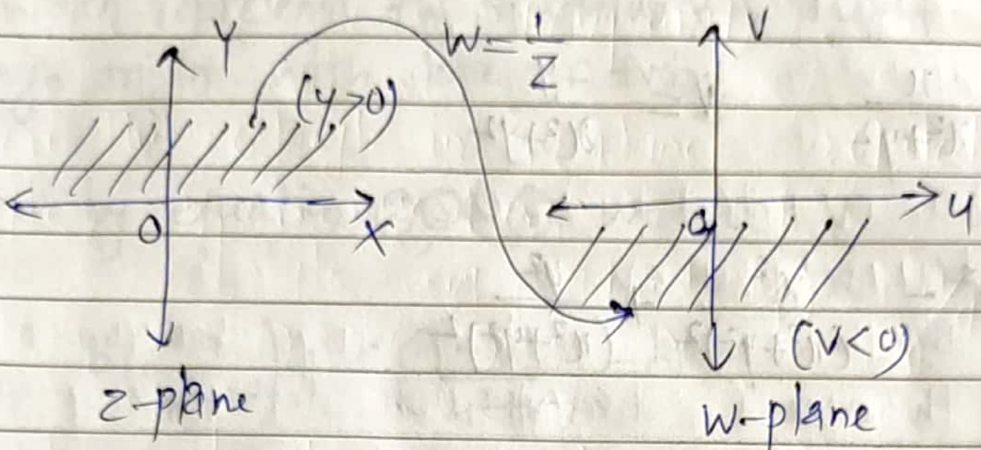
Hence

$$u=\frac{x}{x^2+y^2} \quad \& \quad v=\frac{-y}{x^2+y^2}$$

We know that, $x^2+y^2 \geq 0 \quad \forall\, x,y \in R$
Now if $y>0 \Rightarrow v=\frac{-y}{x^2+y^2}<0$

i.e, $y>0 \Rightarrow v<0$
i.e, upper half plane $(y>0)$ in z-plane is mapped

onto lower half plane $(v<0)$ in w-plane and vice versa as shown in figure.



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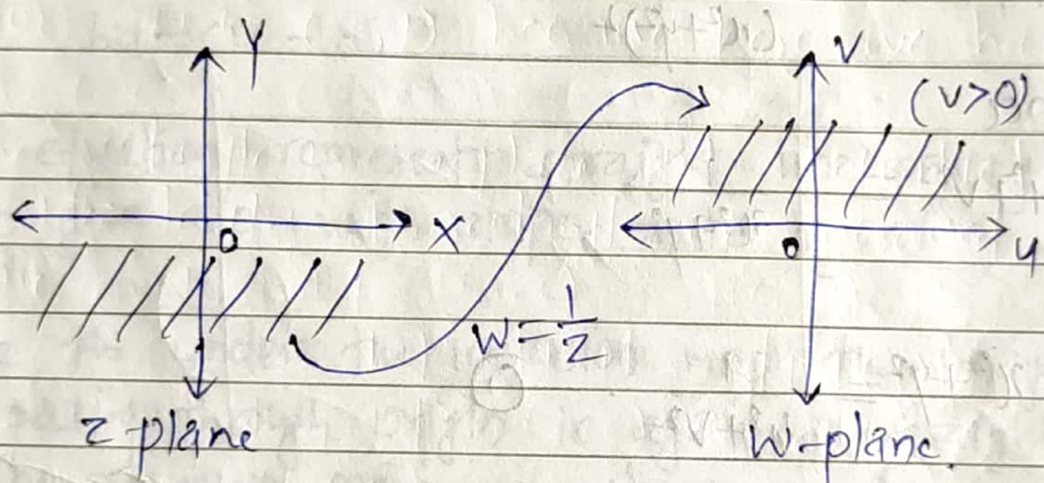


Hence inversion map $w = 1/z$ is reflection with respect to x-axis.

Problem: Find the images of straight lines & circles under the inversion map $w = 1/z$.

Solution:- Given $w = 1/z$

$$u + iv = \frac{1}{x+iy}$$

$$u + iv = \frac{x - iy}{x^2 + y^2}$$

$$u+iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$u = \frac{x}{x^2+y^2} \qquad v = \frac{-y}{x^2+y^2}$$

$$u^2+v^2 = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2}$$

$$u^2+v^2 = \frac{x^2+y^2}{(x^2+y^2)^2}$$

hence

$$u^2+v^2 = \frac{1}{x^2+y^2}$$

$$x^2+y^2 = \frac{1}{u^2+v^2} \quad \cdots (1)$$

$$x = u(x^2+y^2)$$

$$x = \frac{u}{u^2+v^2} \quad \cdots (2)$$

$$y = -v(x^2+y^2)$$

$$y = \frac{-v}{u^2+v^2} \quad \cdots (3)$$

Now consider the general equation,

$$a(x^2+y^2) + bx + cy + d = 0 \quad \cdots (4)$$

Case I: If $a=0$ and $d=0$ then eqn ④ becomes,

$$bx+cy=0 \quad \cdots ⑤$$

which represents the equation of straight line passing through origin with slope $m = -b/c$

Now by equation ②,③ & ⑤ we get,

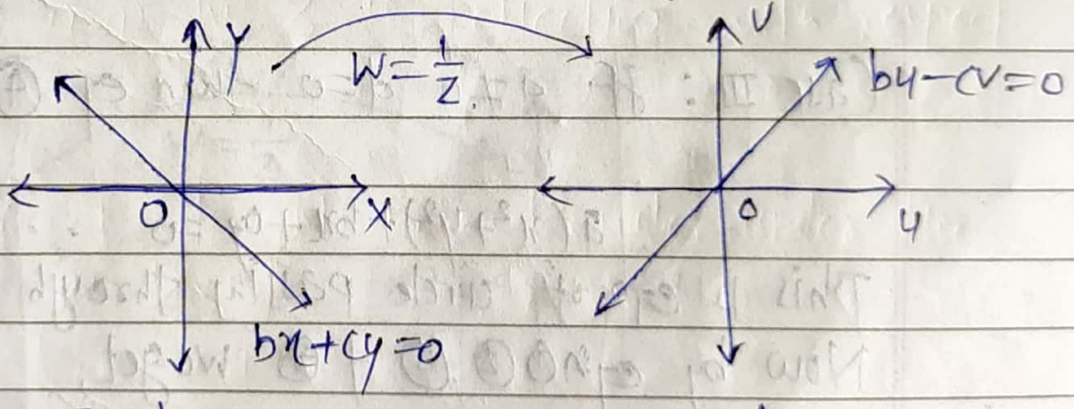
$$b\left(\frac{u}{u^2+v^2}\right)+c\left(\frac{-v}{u^2+v^2}\right)=0$$

i.e, $bu+c(-v)=0$

i.e, $bu-cv=0$

This equation represents straight line passing through origin ~~give sto~~ with slope $m' = b/c$

Hence ~~the~~ under the inversion map the image of straight line passing through origin is again a straight line passing through origin as shown in fig.



Case II: If $a=0$ & $d\neq0$ then ④ becomes

$$bx+cy+d=0 \quad \cdots ⑥$$

Which represents a straight line not passing through origin.

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$a(x^2+y^2)+bx+cy+d=0$ It is eqⁿ of circle
(u^2+v^2) or (x^2+y^2) का term आला तर eqⁿ of circle असते

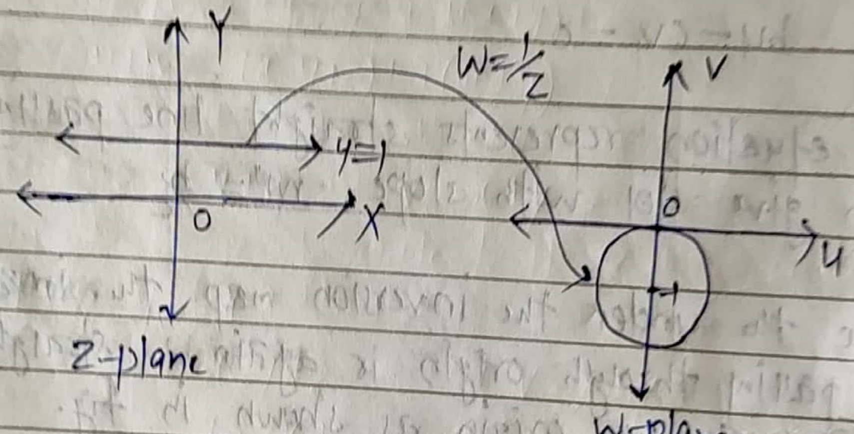
Now by eqⁿ ②, ③ & ⑥ we get

$$\frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0$$

$$bu - cv + d(u^2+v^2) = 0$$

$$\therefore d(u^2+v^2) + bu - cv = 0$$

This is the eqⁿ of circle passing through origin.
Hence the image of straight line not passing through origin is circle passing through origin.



Case III : If $a \neq 0$ $d=0$ then eqⁿ ④ becomes

$$a(x^2+y^2)+bx+cy=0 \quad \ldots ⑦$$

This is eqⁿ of circle passing through origin.
Now by eqⁿ ②, ③ & ⑦ we get,

$$\frac{a}{u^2+v^2} + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} = 0$$

i.e, $\quad a + bu - cv = 0$

i.e, $\quad bu - cv + a = 0$

This is eqⁿ of straight line not passing through origin

x, y, u, v चार term आल्या तर eqⁿ of line असते
for ex $y = mx + c$, $bu - cv + a = 0$.

Thus the image of circle passing through origin is ~~the~~ a straight line not passing through origin.

Case IV: If $a \neq 0$, $d \neq 0$ then eqⁿ (4) becomes

$$a(x^2+y^2) + bx + cy + d = 0 \quad \cdots (8)$$

This is eqⁿ of circle not passing through ~~origin~~. origin.
Now by eqⁿ (1), (2), (3) & (8) we get

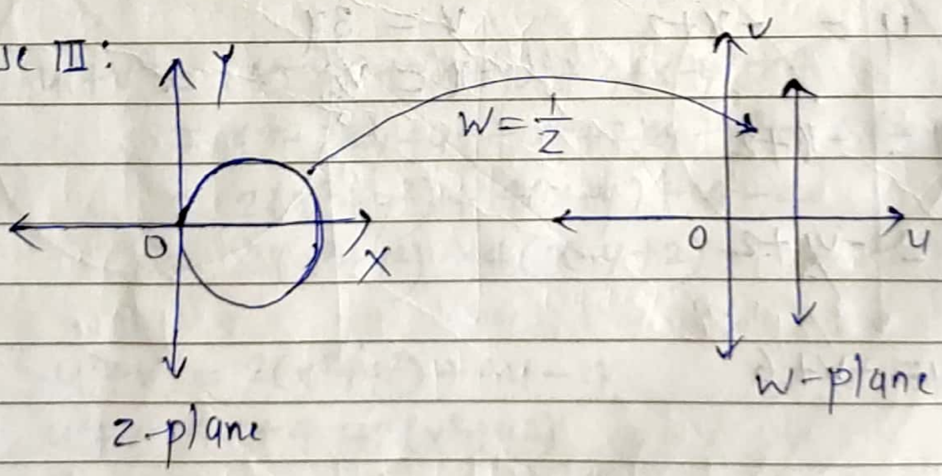
$$\frac{a}{u^2+v^2} + \frac{bu}{u^2+v^2} - \frac{cv}{u^2+v^2} + d = 0$$

$$a + bu - cv + d(u^2+v^2) = 0$$

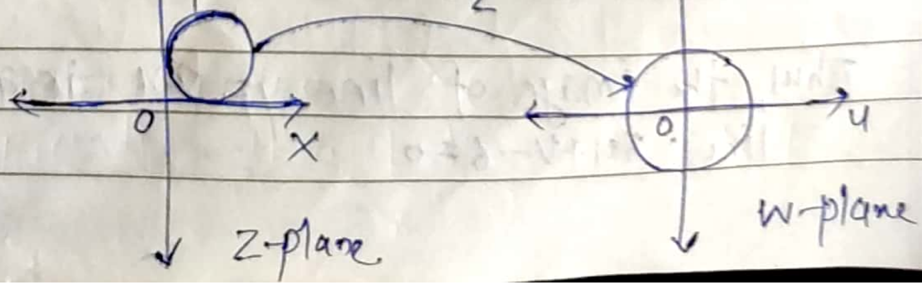
i.e, $d(u^2+v^2) + bu - cv + a = 0$

This is the eqⁿ of circle not passing through origin.
Thus image of circle not passing through origin is a circle not passing through origin as shown in fig.

Case III:



Case IV:



Problem:- Under the map $w=(1+i)\cdot z+2$. Find the image of

Assignment{
i] line $y=2x$
ii] circle $|z|=3$
iii] line $y=3x+2$
iv] circle $|z-1|=2$.

Solution:- Given $w=(1+i)z+2$

$$u+iv=(1+i)(x+iy)+2$$
$$u+iv = x+iy+xi+i^2y+2$$
$$u+iv = x-y+i(x+y)+2$$
$$u+iv=(x-y+2)+i(x+y)$$

$$u=x-y+2 \qquad v=x+y$$

i] Given $y=2x$

$\therefore u=x-y+2$ $\qquad$ $v=x+y$
$= x-2x+2$ $\qquad$ $= x+2x$
$u = -x+2$ $\qquad$ $v = 3x$
$\qquad$ $x = v/3$

$\therefore u=-x+2$

$\Rightarrow u=-v/3+2$

$\Rightarrow 3u=-v+6$

$\Rightarrow 3u+v-6=0$

Thus, the image of line $y=2x$ is a straight line. $3u+v-6=0$

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iv] Given $|z-1|=2$

i.e, $|x+iy-1|=2$

$|(x-1)+iy|=2$

$\therefore \sqrt{(x-1)^2+y^2}=2$

$\therefore (x-1)^2+y^2=4$

$\therefore x^2-2x+1+y^2=4$

$\therefore x^2+y^2-2x+1=4$

$\therefore x^2+y^2-2x=3 \quad \cdots (1)$

We have,

$u=x-y+2$ and $v=x+y$

$u^2=[(x-y)+2]^2$

$u^2=(x-y)^2+4(x-y)+4$

$v^2=(x+y)^2$

$u^2+v^2=(x-y)^2+(x+y)^2+4(x-y)+4$

$=x^2-2xy+y^2+x^2+2xy+y^2+4(x-y)+4$

$=2(x^2+y^2)+4(x-y)+\cancel{8}-4$

$=2(x^2+y^2)+4(x-y+2)-4$

$u^2+v^2=2(x^2+y^2)+4u-4$

$u^2+v^2-4u+4=2(x^2+y^2)$

$\therefore x^2+y^2=\frac{1}{2}(u^2+v^2)-2u+2 \quad \cdots (2)$

Now, $u+v=x-y+2+x+y$

$=2x+2$

$2x=u+v-2 \quad \cdots (3)$

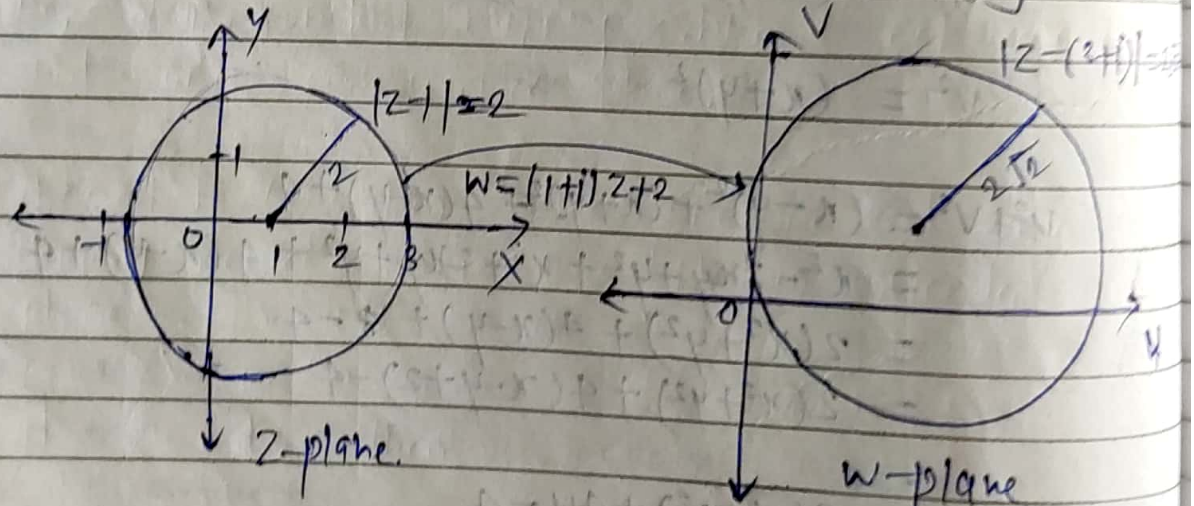
by ①, ② & ③ we get,

$$\frac{1}{2}(u^2+v^2) - 2u + 2 - u - v + 2 = 3$$

$$u^2+v^2-4u+4-2u-2v+4=6$$
$$u^2+v^2-6u-2v+8-6=0$$
$$u^2+v^2-6u-2v+2=0$$
$$u^2+v^2-6u+9-2v+1+2=9+1$$
$$\cancel{(u^2+}$$
$$(u^2-6u+9)+(v^2-2v+1)+2=10$$
$$(u-3)^2+(v-1)^2=10-2$$
$$(u-3)^2+(v-1)^2=8$$

$$(u-3)^2+(v-1)^2=(2\sqrt{2})^2$$

Thus the image of circle $|z-1|=2$ is a circle centred at $(3,1)$ & radius $2\sqrt{2}$ as shown in fig.



Assignment:

Problem :- Find the image of ~~the~~ right half plane under the map $w=(2i)z-i$ & $\cancel{w=\frac{i}{2}}$ $w=\frac{1}{z}-1$

* Linear fractional transformation or Bilinear Transformation or Mobius transformation.

Definition: The quotient of two linear transformations is called as linear Fractional transformation i.e, the mapping

$$w = T(z) = \frac{az+b}{cz+d}$$

where a, b, c and d are complex number constants. such that $ad-bc \neq 0$ is called as linear fractional transformation

* Observations OR Notes :-

i] If $c=0$ then $w = \frac{az+b}{d} = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$

$$w = a'z + b'$$

i.e, w is linear fract function.

ii] If $a=d=0$ & $b=c=1$ then $w = 1/z$ i.e, w is an inversion mapping.

iii] If $b=c=0$ & $d=1$ then $w = az$
i.e, w is rotation and magnification (contraction)

iv] $w = \frac{az+b}{cz+d}$ can be re-written as

$$w = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d} \qquad (c \neq 0)$$

v] Linear fractional transformation is the sequence of transformations i.e, rotation followed by translation followed by inversion again followed by rotation which is followed by translation as follows.

$$z \xrightarrow{\text{Rotation}} cz \xrightarrow{\text{translation}} cz+d \xrightarrow{\text{inversion}} \frac{1}{cz+d}$$

$$\xrightarrow{\text{Rotation}} \frac{b-ad/c}{cz+d} \xrightarrow{\text{translation}} \frac{a}{c} + \frac{b-ad/c}{cz+d} = w.$$

vi] If $ad-bc=0$ then $w = \frac{a}{c} + \frac{b-ad/c}{cz+d}$

$$\Rightarrow \quad w = \frac{a}{c} + \frac{bc-ad/c}{cz+d}$$

$$\Rightarrow \quad w = \frac{a}{c} \qquad \because ad-bc=0.$$

i.e, w is constant mapping which is not one-one and onto on complee plane.

vii] If $ad-bc \neq 0$ then w is one-one and onto ~~on~~ mapping hence w is invertible in extended complex plane

viii] If $z = -d/c$, then $w = \frac{az+b}{cz+d}$

$$\Rightarrow \quad w = \frac{a \times \left(\frac{-d}{c}\right) + b}{c \times \left(\frac{-d}{c}\right) + d}$$

$$w = \frac{-ad+bc/c}{-cd+dc/c}$$

$$W = \frac{-ad+bc/c}{-d+d}$$

$\Rightarrow \quad W = \infty$

ix] If $W = \frac{a}{c}$ then $W = \frac{a}{c} + \frac{b - ad/c}{cz+d}$

$\Rightarrow \quad \frac{a}{c} = \frac{a}{c} + \frac{b-ad/c}{cz+d}$

$\Rightarrow \quad \frac{b - ad/c}{cz+d} = 0$

$\Rightarrow \quad \frac{cz+d}{b-ad/c} = \infty$

$\Rightarrow \quad cz+d = \infty$

$\Rightarrow \quad cz = \infty$

$\Rightarrow \quad z = \infty$

Thus image of ∞ under linear fractional transformation is a/c.

Hence the mapping $T: C \cup \{\infty\} \longrightarrow C \cup \{\infty\}$

i.e, $\quad T: C_\infty \longrightarrow C_\infty$.

is one-one and onto.
Where T is defined as $T(z) = \frac{az+b}{cz+d}$.

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x] $T: C_\infty \longrightarrow C_\infty$ is invertible: where the inverse of $w = T(z)$ is given by

$$T(z) = w = \frac{az+b}{cz+d}$$

$$\Rightarrow \quad w(cz+d) = az+b$$

$$\Rightarrow \quad wcz + dw = az+b$$

$$\Rightarrow \quad (wc-a)z = b - wd$$

$$\Rightarrow \quad z = \frac{b-wd}{wc-a}$$

$$\Rightarrow \quad z = \frac{(-d)w+b}{cw+(-a)}$$

$$T^{-1}(w) = z = \frac{Aw+B}{Cw+D}$$

where $A = -d$, $B = b$, $C = c$, $D = -a$
Hence $AD - BC = ad - bc \neq 0$.
This is the linear fractional transformation in w.
Thus the ~~invertio~~ inverse of linear fractional transformation is again a linear fractional transformation.

xi] $w = \frac{az+b}{cz+d}$

$$\Rightarrow \quad wcz + dw = az+b$$

$$\Rightarrow \quad cwz + dw - az - b = 0$$

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$$\Rightarrow \quad AWZ + BW + CZ + D = 0 \quad \cdots (1)$$

eqn (1) is linear eqn in both variables w and z.

∴ eqn (1) is called Bilinear eq.

∴ Linear fractional transformation is called as Bilinear transformation.

Theorem:- Prove the linear fractional transformation maps circle and straight line onto the circles & straight lines.

Proof:- We have,

$$T(z) = w = \frac{az+b}{cz+d}$$

$$\therefore \quad T(z) = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d} \quad \cdots (1)$$

Now,

$$T_1(z) = cz + d$$

and $T_2(z) = \frac{1}{z}$

Also define

$$T_3(z) = \left(b - \frac{ad}{c}\right)z + \frac{a}{c} \qquad \forall\ z \text{ in } C_\infty$$

Consider,

$$(T_3 \circ T_2 \circ T_1)(z) = T_3[T_2[T_1]]$$

$$= T_3[T_2[cz+d]]$$

$$= T_3\left[\frac{1}{cz+d}\right]$$

$$= \left(b - \frac{ad}{c}\right)\left(\frac{1}{cz+d}\right) + \frac{a}{c}$$

$$= \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

$$\therefore (T_3 \circ T_2 \circ T_1)(z) = T(z) \qquad \forall\, z \in \mathbb{C}_\infty$$

$$\therefore T_3 \circ T_2 \circ T_1 = T$$

$\therefore$ Linear fractional transformation is composition of T_3, T_2 & T_1.
Where T_1 & T_3 are linear functions & T_2 is an inversion.
We know that linear transformation & inversion maps circles and straight lines onto circles and straight lines.
Hence their composition maps circles & straight line onto circles & straight lines.

Hence the bilinear transformation maps circles & straight lines onto circles & straight lines.

Note:- Def^n (Fixed point):

- If $f: A \to A$ then the point $c \in A$ is said to be fixed point of f if $f(c) = c$.

Note:- If The bilinear transformation $T: \mathbb{C}_\infty \to \mathbb{C}_\infty$ has more than two fixed points, then T is an identity transformation identity transformation.

2015 10M / 2014 8M

Theⁿ: For given three distinct points z_1, z_2, z_3 in extended z plane and three distinct points w_1, w_2, w_3 in extended. w plane. prove that there exist the unique bilinear transformation $w = T(z)$ such that $w_k = T(z_k)$ where $k = 1, 2, 3$.

Proof: Given three distinct point z_1, z_2, z_3 in extended z-plane & Three distinct points w_1, w_2, w_3 in extended w-plane i.e, given that $z_1 \neq z_2 \neq z_3$ and $w_1 \neq w_2 \neq w_3$

Claim: There exist unique $T: C_\infty \to C_\infty$ such that

$$T(z_k) = w_k \qquad k = 1, 2, 3.$$

& T is bilinear transformation

We firstly we will prove the existance of bilinear transformation.

Case I: Suppose none of z_1, z_2, z_3, w_1, w_2 & w_3 is infinity (∞)

$$\text{Let } w = T(z) = \frac{az+b}{cz+d} \quad \dots (1)$$

is a bilinear transformation which maps z_k onto w_k

Here we have to find the values of a, b, c and d.

put $z = z_1$ in (1) then we get,

$$w_1 = T(z_1) = \frac{az_1+b}{cz_1+d} \quad \dots (2)$$

similarly,

$$w_2 = T(z_2) = \frac{az_2+b}{cz_2+d} \quad \dots (3)$$

$$w_3 = T(z_3) = \frac{az_3+b}{cz_3+d} \quad \cdots (4)$$

Consider for $k=1,2,3$

$$w - w_k = \frac{az+b}{cz+d} - \frac{az_k+b}{cz_k+d}$$

$$w - w_k = \frac{(az+b)(cz_k+d) - (az_k+b)(cz+d)}{(cz+d)(cz_k+d)}$$

$$= \frac{aczz_k + adz + bcz_k + bd - (aczz_k + adz_k + bcz + bd)}{(cz+d)(cz_k+d)}$$

$$= \frac{aczz_k + adz + bcz_k + bd - aczz_k - adz_k - bcz - bd}{(cz+d)(cz_k+d)}$$

$$= \frac{ad(z-z_k) - bc(z-z_k)}{(cz+d)(cz_k+d)}$$

$$\therefore w - w_k = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)} \qquad k=1,2,3.$$

$$\therefore w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)} \quad \cdots (5)$$

$$\therefore w - w_2 = \frac{(ad-bc)(z-z_2)}{(cz+d)(cz_2+d)} \quad \cdots (6)$$

$$\therefore w - w_3 = \frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)} \quad \cdots (7)$$

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Consider

$$\frac{w-w_1}{w-w_3} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)} \times \frac{(cz+d)(cz_3+d)}{(ad-bc)(z-z_3)} \quad \text{-- by ⑤ \& ⑦}$$

$$\therefore \frac{w-w_1}{w-w_3} = \frac{(z-z_1)(cz_3+d)}{(cz_1+d)(z-z_3)} \quad \text{--- ⑧}$$

Replace z by z_2 & w by w_2 in ⑧ then we get,

$$\frac{w_2-w_1}{w_2-w_3} = \frac{(z_2-z_1)(cz_3+d)}{(cz_1+d)(z_2-z_3)} \quad \text{--- ⑨}$$

$$\frac{w_2-w_3}{w_2-w_1} = \frac{(cz_1+d)(z_2-z_3)}{(z_2-z_1)(cz_3+d)} \quad \text{--- ⑩}$$

Now consider,

$$\frac{w-w_1}{w-w_3} \times \frac{w_2-w_3}{w_2-w_1} = \frac{(cz_3+d)(z-z_1)}{(cz_1+d)(z-z_3)} \times \frac{(cz_1+d)(z_2-z_3)}{(cz_3+d)(z_2-z_1)}$$

$$\frac{w-w_1}{w-w_3} \times \frac{w_2-w_3}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\therefore \frac{w-w_1}{w-w_3} \times \frac{w_2-w_3}{w_2-w_1} = \frac{(z-z_1)}{(z-z_3)} \times \frac{(z_2-z_3)}{(z_2-z_1)} \quad \text{--- ⑪}$$

If we solve eqn ⑪ for w then we will get a bilinear transformation. such that, that bilinear transformation

maps z_k onto w_k for $k=1,2,3$.
To check this we put $z=z_2$ in eqn (11) then we get

$$\frac{(w-w_1)}{(w-w_3)} \times \frac{(w_2-w_3)}{(w_2-w_1)} = \frac{(z_2-z_1)}{(z_2-z_3)} \times \frac{(z_2-z_3)}{(z_2-z_1)}$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = 1$$

$$(w-w_1)(w_2-w_3) = (w-w_3)(w_2-w_1)$$
$$ww_2 - ww_3 - w_1w_2 + w_1w_3 = ww_2 - ww_1 - w_2w_3 + w_1w_3$$
$$-ww_3 - w_1w_2 = -ww_1 - w_2w_3$$
$$ww_3 + w_1w_2 = ww_1 + w_2w_3$$
$$ww_3 - ww_1 = w_2w_3 - w_1w_2$$
$$w(w_3-w_1) = w_2(w_3-w_1)$$
$$w = w_2$$

$$\therefore T(z_2) = w_2$$

Thus $z=z_2 \Rightarrow w=w_2=T(z_2)$

Similarly, we can show that,
if $z=z_1$ then $w=w_1$
& if $z=z_3$ then $w=w_3$

Hence eqn (11) represents the required bilinear transformation.

Case (II): Suppose any one of z_1, z_2, z_3 & w_1, w_2, w_3 is infinity

In perticular suppose $z_3 = \infty$, then the ratio,

$$\frac{z_2 - z_3}{z - z_3} \rightarrow 1$$

Hence the bilinear transformation in eqn (II) becomes,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)}{(z_2-z_1)}$$

Thus in both cases (I) & (II) there exist a bilinear transformation which maps z_k onto w_k, $k=1,2,3$.

Now we have to prove uniqueness bilinear transformation

<u>uniqueness</u>

Suppose there exists two bilinear transformation say S and T such that,

$$S(z_k) = w_k \text{ \& } T(z_k) = w_k, \quad k=1,2,3.$$

$\therefore \quad z_k = S^{-1}(w_k) \quad \forall \; k=1,2,3.$

clearly S^{-1} is a bilinear transformation.
More over composition of two bilinear transformations is again a bilinear transformations.
Hence

$S^{-1} \circ T : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ is a bilinear transformation

such that,

$$(S^{-1} \circ T)(z_k) = z_k \qquad k=1,2,3.$$

$$(S^{-1} \circ T)(z_k) = S^{-1}[T(z_k)] = S^{-1}(w_k)$$
$$= z_k$$

Thus, z_k $(k=1,2,3)$ are fixed points for the bilinear transformation $S^{-1}oT$.
Thus $S^{-1}oT$ has 3 fixed points z_1, z_2, z_3.
Thus $S^{-1}oT$ has more than 2 fixed points
by theorem

$$S^{-1}oT = I$$

$$\Rightarrow \quad \boxed{T=S}$$

$\therefore$ there exist a unique bilinear transformation given in (11) which maps z_k onto w_k $(k=1,2,3)$
Hence proved.

* Cross Ratio :-

The cross ratio of point z_1, z_2, z_3 & z is denoted by (z_1, z_2, z_3, z) and it is defined as

$$(z_1, z_2, z_3, z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

imp 6M 2016 10M & Define.

Theorem: For given three distinct points z_1, z_2, z_3 in extended z plane & three distinct points w_1, w_2, w_3 in extended w plane. prove that the cross ratio (z_1, z_2, z_3, z) of the points z_1, z_2, z_3 & z. is invarient under the bilinear transformation which maps z_k onto w_k, $k=1,2,3$.

Proof :- Given three distinct points z_1, z_2, z_3 in extended z z plane of w_1, w_2, w_3 in extended w-plane.

$\therefore$ By theorem there exist a unique bilinear transformation $T: C_\infty \longrightarrow C_\infty$ which maps z_k onto w_k. $k=1,2,3$.

i.e, $T(z_k) = w_k \quad k=1,2,3.$

Where T is define as

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \quad \ldots \text{(1)}$$

Now the cross ratio of z_1, z_2, z_3 & z is defined as

$$(z_1, z_2, z_3, z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \quad \ldots \text{(2)}$$

similarly, the cross ratio of w_1, w_2, w_3 & w is defined as

$$(w_1, w_2, w_3, w) = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \quad \ldots \text{(3)}$$

$\therefore$ By (1), (2) & (3) we get

$$(z_1, z_2, z_3, z) = (w_1, w_2, w_3, w)$$

Hence the value of both cross ratios (z_1, z_2, z_3, z) & (w_1, w_2, w_3, w) are same

$\therefore$ Cross ratios (z_1, z_2, z_3, z) is invarient under bilinear transformation T.

Hence proved.

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imp 2014 4M

Example: Find bilinear transformation which maps $z_1 = i$, $z_2 = 2$ & $z_3 = -2$ onto the point $w_1 = i$, $w_2 = 1$ & $w_3 = -1$ respectively

Solution:- Given $z_1 = i$, $z_2 = 2$, $z_3 = -2$ & $w_1 = i$, $w_2 = 1$, $w_3 = -1$

The bilinear transformation which maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Put $z_1 = i$, $z_2 = 2$, $z_3 = -2$, $w_1 = i$, $w_2 = 1$, $w_3 = -1$ then we get

$$\frac{(w-i)(1-(-1))}{(w-(-1))(1-i)} = \frac{(z-i)(2-(-2))}{(z-(-2))(2-i)}$$

$$\frac{(w-i)\times 2}{(w+1)(1-i)} = \frac{(z-i)(4)}{(z+2)(2-i)}$$

$$(w-i)(z+2)(2-i) = 2(z-i)(w+1)(1-i)$$

$$(wz+2w-iz-2i)(2-i) = 2[(zw+z-iw-i)(1-i)]$$

$$2wz+4w-2iz-4i-iwz-2iw+i^2z+2i^2 =$$
$$2zw+2z-2iw-2i-2izw-2iz+2i^2w+2i^2$$

$$-iwz+4w+i^2z-4i = -2iwz+2z+2i^2w-2i$$

$$-iwz+4w-z-4i = -2iwz+2z-2w-2i$$

$$-iwz+4w-z-4i+2iwz-2z+2w+2i = 0$$

$$iwz+6w-3z+2i = 0$$

$$w(iz+6)-(3z-2i) = 0$$

$$w(iz+6) = 3z+2i$$

$$w = \frac{3z+2i}{(iz+6)}$$

i.e, $$w = \frac{3z+2i}{iz+6}$$

This is the required bilinear transformation.

2016 4M

Problem: F. B.T which maps $z = 1-i, 1+i, -1+i$ and $w = 0, 1, \infty$

Solution: Given $z_1 = 1-i$, $z_2 = 1+i$, $z_3 = -1+i$ and $w_1 = 0$, $w_2 = 1$ $w_3 = \infty$

The bilinear transformation mapping the points z_1, z_2, z_3 onto w_1, w_2, w_3 respectively is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(1-\infty)}{(w-\infty)(1-0)} = \frac{(z-(1-i))((1+i)-(-1+i))}{(z-(-1+i))((1+i)-(1-i))} \times$$

Here $w_3 = \infty$ ∴ the fraction $w_2 - w_3 / w - w_3$ tends to 1

∴ We have

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-0}{1-0} = \frac{(z-(1-i))(1+i-(-1+i))}{(z-(-1+i))(1+i-(1-i))}$$

$$w = \frac{(z-1+i)(1+i+1-i)}{(z+1-i)(1+i-1+i)}$$

$$w = \frac{(z-1+i)(2)}{(z+(1-i))(2i)}$$

$$w = \frac{z-(-1+i)}{iz+i(1-i)} \qquad \frac{z-(1-i)}{iz+i(1-i)}$$

$$w = \frac{z+(-1+i)}{iz+(i-i^2)}$$

$$w = \frac{z+(-1+i)}{iz+(1+i)}$$

$$w = \frac{z-(1-i)}{iz+(1+i)}$$

H.W.

Problem: Find Bilinear transformation which maps

Imp i] $z = 0, i, -i$ & $w = i, -i, 0$ Ans: $w = (iz-1)/(-3z+1)$

ii] $z = 1, i, \infty$ & $w = i, -1, 0$

iii] $z = 2, -1, -i$ & $w = \infty, -1, -i$

iv] $z = 2, \infty, i$ & $w = \infty, -1, 1$ Ans: $w = (-z+2(1-i))/(z-2)$

(OR) $w = (z-2(1-i))/2-z$

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Example: Under the bilinear transformation $w = iz/(z-1)$ find the image of closed unit disc $|z| \leq 1$

Solution: Given $w = \frac{iz}{(z-1)}$

i.e.,

$$u+iv = \frac{i(x+iy)}{(x+iy-1)}$$

$$u+iv = \frac{i(x+iy)}{(x-1)+iy}$$

$$= \frac{(i(x+iy))((x-1)-iy)}{(x-1+iy)(x-1-iy)}$$

$$= \frac{(ix+i^2y)((x-1)-iy)}{(x-1+iy)(x-1-iy)}$$

$$= \frac{(ix-y)(x-1-iy)}{(x-1+iy)(x-1-iy)}$$

$$= \frac{ix^2-ix-i^2xy-xy+y+iy^2}{(x-1)^2+y^2}$$

$$= \frac{ix^2-ix+xy-xy+y+iy^2}{(x-1)^2+y^2}$$

$$= \frac{i(x^2+y^2-x)+y}{(x-1)^2+y^2}$$

$\therefore u+iv = \dfrac{y}{(x-1)^2+y^2} + \dfrac{i(x^2+y^2-x)}{(x-1)^2+y^2}$

$\therefore u = \dfrac{y}{(x-1)^2+y^2}$ & $v = \dfrac{x^2+y^2-x}{(x-1)^2+y^2}$

$\therefore u = \dfrac{y}{x^2-2x+1+y^2}$, $v = \dfrac{x^2+y^2-x}{x^2-2x+1+y^2}$

$\therefore u = \dfrac{y}{x^2+y^2-2x+1}$, $v = \dfrac{x^2+y^2-x}{x^2+y^2-2x+1}$

Now given that $|z| \leq 1$

$\Rightarrow |z|^2 \leq 1$

$\Rightarrow x^2+y^2 \leq 1$

$\Rightarrow v = \dfrac{x^2+y^2-x}{x^2+y^2-2x+1} \leq \dfrac{1-x}{1-2x+1}$

$\Rightarrow v \leq \dfrac{1-x}{2-2x} \leq \dfrac{(1-x)}{2(1-x)} \leq \dfrac{1}{2}$

$v \leq 1/2$

Here the boundary $|z|=1$ is mapped onto the line $v = 1/2$

Now for given origin we have $z=0$

i.e, $x+iy=0$ $\therefore x=0, y=0$

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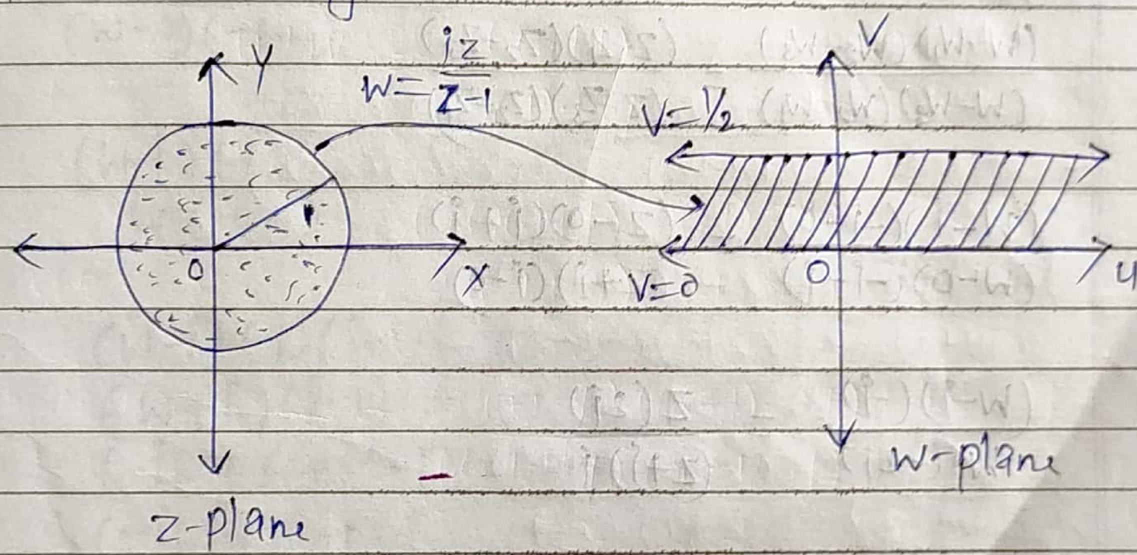
$v = \frac{1}{2}$ is the eqn of straight line.

then $v = \frac{x^2+y^2-x}{x^2+y^2-2x+1} = 0$

$\Rightarrow \quad v = 0$

The origin is mapped onto the line $v=0$
$\therefore$ the closed unit disc $|z| \leq 1$ is mapped onto the closed infinite strip $0 \leq v \leq \frac{1}{2}$
As shown in fig.



* ~~Other Mappings~~

H.W.

Problem: Find Bilinear transformation which maps if $z = 0, i, -i$ and $w = i, -i, 0$

Solution:- Given $z_1 = 0$ $z_2 = i$, $z_3 = -i$ & $w_1 = i$, $w_2 = -i$, $w_3 = 0$

The bilinear transformation mappings the point z_1, z_2, z_3 onto w_1, w_2, w_3 respectively given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(-i-0)}{(w-0)(-i-i)} = \frac{(z-0)(i+i)}{(z+i)(i-0)}$$

$$\frac{(w-i)(-i)}{w(-2i)} = \frac{z(2i)}{(z+i)i}$$

$$\frac{(w-i)}{2w} = \frac{2z}{(z+i)}$$

$$(w-i)(z+i) = 2z \cdot 2w$$

$$wz + wi - iz - i^2 = 4zw$$

$$wz + wi - iz + 1 = 4zw$$

$$wz + wi - 4zw = iz - 1$$

$$w(z + i - 4z) = iz - 1$$

$$w(i - 3z) = iz - 1$$

$$\Rightarrow \quad w = \frac{(iz-1)}{(i-3z)}$$

ii] $z_1 = 1$, $z_2 = i$, $z_3 = \infty$ & $w_1 = i$, $w_2 = -1$, $w_3 = 0$

The bilinear transformation mapping the point z_1, z_2, z_3 onto w_1, w_2, w_3 respectively given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(z-z_1)}{(z_2-z_1)} \quad \because z_3 = \infty$$

$$\frac{(w-i)(-1-0)}{(w-0)(-1-i)} = \frac{(z-1)}{(i-1)}$$

$$\frac{(w-i)(-1)}{w(-1-i)} = \frac{(z-1)}{(i-1)}$$

$$(w-i)(-1)(i-1) = (z-1)(w)(-1-i)$$
$$(-w+i)(i-1) = (z-1)(-w-iw)$$
$$(-iw+w+i^2-i) = (-zw-izw+w+iw)$$
$$(-iw+w-1-i) = (-zw-izw+w+iw)$$
$$-iw+w+zw+izw-w-iw = 1+i$$
$$-2iw+zw+izw = 1+i$$

$$w(z-2i+iz) = (1+i)$$

$$w = \frac{(1+i)}{(z-2i+iz)}$$

$$\Rightarrow w = \frac{(1+i)}{z(1+i)-2i}$$

iii] $z_1 = 2$, $z_2 = -1$, $z_3 = -i$ and $w_1 = \infty$, $w_2 = -1$, $w_3 = -i$

The bilinear transformation map which maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively given by,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Here $w_1 = \infty$ ∴ the fraction $(w-w_1)/(w_2-w_1)$ tends to 1

∴ We have,

$$\frac{(w_2-w_3)}{(w-w_3)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(-1+i)}{(w+i)} = \frac{(z-2)(-1+i)}{(z+i)(-1-2)}$$

$$\frac{(-1+i)}{(w+i)} = \frac{(z-2)(-1+i)}{(z+i)(-3)}$$

$$(-1+i)(z+i)(-3) = (z-2)(-1+i)(w+i)$$

$$(3-3i)(z+i) = (-z+zi+2-2i)(w+i)$$

$$3z+3i-3iz-3i^2 = -zw-zi+izw+zi^2+2w+2i-2iw-2i^2$$

$$3z+3i-3iz+3 = -zw-zi+izw-z+2w+2i-2iw+2$$

$$-zw-zi+izw-z+2w+2i-2iw+2 = 3z+3i-3iz+3$$

$$-zw+izw+2w-2iw = 3z+3i-3iz+3+zi+z-2i-2$$

$$w(iz+2-z-2i) = 4z+i-2iz+1$$

$$w(iz-2i-z+2) = 4z+i-2iz+1$$

$$w = \frac{4z+i-2iz+1}{(iz-2i-z+2)} = \frac{z(4-2i)+(i+1)}{z(i-1)+2(-i+1)}$$

(OR) →

OR) $\frac{1}{w+i} = \frac{z-2}{(z+i)(-3)}$ $= -3z - 3i = (w+i)(z-2)$

$-3z - 3i = wz - 2w + iz - 2i$

$\cancel{-3z - 3i = wz}$ $w(z-2) = -3z - 3i - iz + 2i$

$w = \frac{z(-3-i)-i}{z-2}$

iv] $z_1 = 2, z_2 = \infty, z_3 = i$ and $w_1 = \infty, w_2 = -1, w_3 = 1$

The bilinear transformation which maps z_1, z_2, z_3 onto w_1, w_2, w_3 respectively given by,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Here $z_2 = \infty$ & $w_1 = \infty$ $\therefore$ $(w-w_1)/(w_2-w_1)$ & $(z_2-z_3)/(z_2-z_1)$ tends to 1

$$\therefore \frac{(w_2-w_3)}{(w-w_3)} = \frac{(z-z_1)}{(z-z_3)}$$

$$\frac{(-1-1)}{(w-1)} = \frac{(z-2)}{(z-i)}$$

$$(-1-1)(z-i) = (z-2)(w-1)$$

$$-2z + 2i = zw - z - 2w + 2$$

$$zw - z - 2w + 2 = -2z + 2i$$

$$zw - 2w = -2z + 2i + z - 2$$

$$zw - 2w = -z + 2i - 2$$

$$w(z-2) = -z + 2i - 2$$

$$w = \frac{-z + 2i - 2}{z-2}$$

$$\Rightarrow w = \frac{-z + 2(i-1)}{z-2}$$

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