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Unit I

* Riemann Integration Defination and exeistance of Riemann integral.

Def^n - Let,

[a,b] be a given interval by the partition P of [a,b] we ~~hav~~ mean a finite set of point $x_0, x_1, \ldots\ldots, x_n$.

where,

$$x_0 = a \leq x_1, \leq x_2, \leq \ldots\ldots, \leq x_{i-1} \leq x_i \ldots\ldots, \leq x_n = b$$

we write

$$\Delta x_i = x_i - x_{i-1}$$

where $i = 1, 2, \ldots\ldots, n$.

Now,

Suppose f is bounded real function on [a,b] corresponding to each partition 'P' of [a,b] we put,

$$M_i = \sup f(x) \qquad x_{i-1} \leq x \leq x_i$$

$$m_i = \inf f(x) \qquad x_{i-1} \leq x \leq x_i$$

Here,

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

and finally,

$$\int_a^b f\,dx = \sup L(P,f) \qquad \text{—— (1)}$$

$$\overline{\int_a^b} f.\,dx = \inf U(P,f) \quad \text{——} \ ②$$

where the infrimum and superimum are taken over all partition 'P' of [a,b]. left member of eqn ① is called lower reimann integral and left member of eqn ② is called upper reimann integral of f. over [a,b].

If the lower and upper integral are equal we say that f is reimann integrable on [a,b].

$f \in R$ where R is set of Reimann integrible function denote the common value of 1, 2 by

$$\int_a^b f\,dx \quad \text{or} \quad \int_a^b f(x)\,dx.$$

Riemann Stieltgen Integration

Def^n - let,

α be monotonically increasing function on [a,b] α is bounded ~~an~~ on [a,b] corresponding to each partition P on [a,b] we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

It is clear $\Delta\alpha_i \to 0$ for any real function f which is bounded on $[a,b]$

we ~~get~~ put,

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where, $M_i = \sup f(x) \qquad x_{i-1} \le x \le x_i$.

$m_i = \inf f(x) \qquad x_{i-1} \le x \le x_i$

and we define

$$\underline{\int_a^b} f\, d\alpha = \sup L(P,f,\alpha) \quad \text{———} \quad (1)$$

$$\overline{\int_a^b} f\, d\alpha = \inf U(P,f,\alpha) \quad \text{———} \quad (2)$$

This is Riemann stieltjes integral or simply stieltjes integral of f with respect to α over $[a,b]$.

when ever

$$\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha.$$

$R(\alpha)$ this is set of all Riemann Stielties integrible function with respect to α and $f \in R(\alpha)$.

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Refinement of partition P

We say that partition P^* is Refinement of P if $P < P^*$.

i.e. every point of P is a point of P^*

Given two partition P_1 and P_2, we say P^* is there comman refinement if $P^* = P_1 \cup P_2$.

Theorem

If P^* is refinement of P then

1) $L(P, f, \alpha) \leq L(P^*, f, \alpha)$

2) $U(P, f, \alpha) \geq U(P^*, f, \alpha)$

→ Proof -

1) Given that P^* is refinement of P

i.e. $P^* \supset P$ to show

$$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

i.e. $L(P^*, f, \alpha) - L(P, f, \alpha) \geq 0$

Let us assume that.

P^* contains just one element more than P say x^* such that

$$x_{i-1} \leq x^* \leq x_i$$

Let,

$m_i = \inf f(x)$ $\quad x_{i-1} \leq x \leq x_i$

$w_1 = \inf f(x)$ and $\quad x_{i-1} \leq x \leq x_i$

$w_2 = \inf f(x)$ $\quad x^* \leq x \leq x_i$

$w_1 \geq m_i$ and $w_2 \geq m_i$

$L(P^*, f, \alpha) - L(P, f, \alpha) = w_1[\alpha(x^*) - \alpha(x_{i-1})]$
$+ w_2 [\alpha(x_i) - \alpha(x^*)] - m_i [\alpha(x_i) - \alpha(x_{i-1})]$

$= w_1 [\alpha(x^*) - \alpha(x_{i-1})] + w_2 [\alpha(x_i) - \alpha(x^*)] - m_i$
$[\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})]$

$= (w_1 - m_i) [\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i) [\alpha(x_i) -$
$\alpha(x^*)]$.

$= (w_1 - m_i) [\alpha(x^*) - \alpha(x_{i-1})] \geq 0$
$+ (w_2 - m_i) [\alpha(x_i) - \alpha(x^*)] \geq 0$

Hence we get,

$L(P^*, f, \alpha) - L(P, f, \alpha) \geq 0$.

~~Now~~, i.e. $L(P^*, f, \alpha) \geq L(P, f, \alpha)$

Now, If P^* contains K point more than P we repeat this resing K times to rich the result.

Given that $P^* \supseteq P$ to show that,

2) $U(P, f, \alpha) \geq U(P^*, f, \alpha)$.

$U(P, f, \alpha) - U(P^*, f, \alpha) \geq 0$

Let,

$M_i = \sup f(x) \quad x_{i-1} \leq x \leq x_i$

$V_1 = \sup f(x) \quad x_{i-1} \leq x \leq x^*$

$V_2 = \sup f(x) \quad x^* \leq x \leq x_i$

$M_i \geq V_1, \quad M_i \geq V_2$

$U(P, f, \alpha) - U(P^*, f, \alpha) \geq 0$

$= M_i [\alpha(x_i) - \alpha(x_{i-1})] -$

$\{ V_1 [\alpha(x^*) - \alpha(x_{i-1})] + V_2 [\alpha(x_i) - \alpha(x^*)] \} \geq 0$

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$$= m_i[\alpha(x_i)-\alpha(x^*)+\alpha(x^*)-\alpha(x_{i-1})] - \{v_1[\alpha(x^*)-\alpha(x_{i-1})]+v_2[\alpha(x_i)-\alpha(x^*)]\} \geq 0$$

$$=(m_i-v_1)[\alpha(x^*)-\alpha(x_{i-1})]+(m_i-v_2)[\alpha(x_i)-\alpha(x^*)] \geq 0$$

$U(P,f,\alpha)-U(P^*,f,\alpha) \geq 0.$

Since α is monotonically increasing

$\Rightarrow \alpha(x_i)-\alpha(x^*) \geq 0$

$\alpha(x^*)-\alpha(x_{i-1}) \geq 0$

$\therefore U(P,f,\alpha)-U(P^*,f,\alpha) \geq 0.$

i.e. $U(P,f,\alpha) \geq U(P^*,f,\alpha).$

Similarly, P contains K points more than P^* we repeat these rezoning K times to reach the result.

Theorem

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

→ Proof - Let,

P^* be common refinement of partition P_1 and P_2

$L(P_1,f,\alpha) \leq L(P^*,f,\alpha)\} \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha)]$

$\because$ by previous theorem.

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$$\Rightarrow L(P_1, f, \alpha) \le U(P_2, f, \alpha)$$

By taking suprimum over P_1.
we get,

$$\sup L(P_1, f, \alpha) \le \sup U(P_2, f, \alpha)$$

$$\int_a^b f \, d\alpha \le U(P_2, f, \alpha)$$

by taking infimum over P_2.

$$\inf \int_a^b f \, d\alpha \le \inf U(P_2, f, \alpha)$$

$$\underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha$$

Hence proved.

Theorem

$f \in R(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ $\exists$. partition P such a point
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

→ Proof -

1) Assume that.

for given $\varepsilon > 0$ $\exists$ partition P such point
$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

To show

$f \in R(\alpha)$ on $[a, b]$

We have for any partition.

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$$L(P,f,\alpha) \leq \underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha \leq U(P,f,\alpha)$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha < \varepsilon$$ $\because \varepsilon$ is arbitrary

$$\Rightarrow \overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$.
Hence proved.

2) Assume that,
$f \in R(\alpha)$ on $[a,b]$. To show for every $\varepsilon > 0$ $\exists$ partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

for given $f \in R(\alpha)$ on $[a,b]$ for given $\varepsilon > 0$ $\exists$ partition P_1 and P_2 such that

$$U(P_2,f,\alpha) - \overline{\int_a^b} f\,d\alpha < \frac{\varepsilon}{2}$$

$$\underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) < \frac{\varepsilon}{2}$$

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let, $P = P_1 \cup P_2$

$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \frac{\varepsilon}{2} + \int_a^b f\,d\alpha < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + L(P_1, f, \alpha) \leq \varepsilon + L(P_1, f, \alpha)$

$\Rightarrow U(P, f, \alpha) \leq \varepsilon + L(P, f, \alpha)$

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$

Hence ε is arbitary.
$\exists$ partition P which is refinement of P_1 and P_2
Hence proved.

Theorem

a) If $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ holds for some P and some ε then $U(P, f, \alpha) - L(P, f, \alpha)$ holds (for some ε) and for every refinement of P.

b) If $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ hold for $P = \{x_0, x_1, \ldots, x_n\}$ if s_i and t_i are arbitary points in $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i < \varepsilon$$

c) If $f \in R(\alpha)$ and the hypothesis of (b) holds

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then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta \alpha_i - \int_a^b f \, d\alpha \right| < \varepsilon$$

→ Proof -

a) Given that for any $\varepsilon > 0$ we have,

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

Let,

P^0 is refinement of P such that $P^0 \supseteq P$.

$$L(P, f, \alpha) \leq L(P^0, f, \alpha) \leq U(P^0, f, \alpha) \leq U(P, f, \alpha)$$

Given that,

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

$$\Rightarrow U(P^0, f, \alpha) - L(P^0, f, \alpha) < \varepsilon.$$

b) Given that,

$$x_{i-1} \leq s_i, t_i \leq x_i$$

let,

$$M_i = \sup f(x) \qquad x \in [x_{i-1}, x_i]$$
$$m_i = \inf f(x) \qquad x \in [x_{i-1}, x_i]$$

$$m_i \leq f(s_i), f(t_i) \leq M_i$$
for $i = 1, 2, \ldots, n$.

$$\Rightarrow |f(s_i) - f(t_i)| \leq M_i - m_i$$
$i = 1, 2, \ldots, n$.

multiply both side in $\Delta\alpha_i$

$$\Rightarrow |f(s_i)-f(t_i)|\,\Delta\alpha_i \le (M_i-m_i)\,\Delta\alpha_i \qquad \because \alpha_i \ge 0.$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i)-f(t_i)|\,\Delta\alpha_i \le \sum_{i=1}^{n} (M_i-m_i)\,\Delta\alpha_i$$

$$\sum_{i=1}^{n} (M_i-m_i)\,\Delta\alpha_i = U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon.$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i)-f(t_i)|\,\Delta\alpha_i < \varepsilon.$$

Hence proved.

c) for given $f \in R(\alpha)$

i.e. $U(P,f,\alpha)-L(P,f,\alpha) < \varepsilon$

also, $t_i \in [x_{i-1}, x_i]$

$$L(P,f,\alpha) \le \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \le U(P,f,\alpha) \quad \text{——} (1)$$

$\because m_i \le f(t_i) \le M_i$

we know,

$$L(P,f,\alpha) \le \int_a^b f\,d\alpha \le U(P,f,\alpha) \quad \text{——} (2)$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| \le U(P,f,\alpha)-L(P,f,\alpha)$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \varepsilon$$

Hence proved.

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Theorem

If f is continous on $[a,b]$ the $f \in R(\alpha)$ on $[a,b]$.

→ Proof

Given f is continous on $[a,b]$.

α is monotonically increasing on $[a,b]$.

$\therefore \alpha(b) - \alpha(a) > 0$

$\therefore$ for given $\varepsilon > 0$ $\exists$ $\eta > 0$ such that,

$[\alpha(b) - \alpha(a)]\eta < \varepsilon$.

Now,

f is continuous on $[a,b]$ which is compact.

$\therefore f$ is uniformaly continuous.

[$\because$ We know, if f is continuous function form compact set to metric space x then f is uniformaly continuous.]

$\therefore$ for $\eta > 0$ $\exists$ $\delta > 0$ such that.

$|f(x) - f(t)| < \eta$ whenever

$x.t \in [a,b]$ and $|x-t| < \delta$.

Let,

partition P of $[a,b]$ such that $\Delta x_i < \delta$

$\Rightarrow M_i - m_i < \eta$

Now,

$U(P,f,\alpha) - L(P,f,\alpha)$

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$$= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$U(P,f,\alpha) - L(P,f,\alpha) < \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$< \eta[\alpha(b) - \alpha(a)]$$

$$< \varepsilon$$

$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

$\Rightarrow f \in R(\alpha)$ on $[a,b]$.

Case I: If f is monotonic on $[a,b]$ and if α is continuous on $[a,b]$ then $f \in R(\alpha)$ on $[a,b]$.

→ Proof

Given that,

f is monotonic on $[a,b]$

and α is continuous. we know,

α is monotonically increasing.

$\alpha(b) \geq \alpha(a)$

Let,

partition P such that for any +ve integer n.

$$\alpha(x_i) - \alpha(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n}$$

for $i = 1, 2, \ldots, n$.

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$$

Let us consider f is monotonically increasing.

$$M_i = f(x_i) \text{ and } m_i = f(x_{i-1})$$

for $x_{i-1} \leq x < x_i$.

~~$U(P,f,x)$~~

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} [M_i - m_i] \Delta\alpha_i$$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$$

$$= \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)].$$

as $n \to \infty$

$$\left[\frac{\alpha(b) - \alpha(a)}{n} \left\{f(b) - f(a)\right\}\right] \to 0$$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$.

i.e. $f \in R(\alpha)$.

case II

If f is monotonically decresing.

$m_i = f(x_i)$ $\quad M_i = f(x_{i-1})$ $\quad x_{i-1} \leq x < x_i$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} f(x_{i-1}) - f(x_i)$$

$$\left(\frac{\alpha(b) - \alpha(a)}{n}\right)$$

$$= \left(\frac{\alpha(b)-\alpha(a)}{n}\right) \sum_{i=1}^{n} (f(x_{i-1}) - f(x_i))$$

$$= \frac{(\alpha(b)-\alpha(a))}{n} (f(a) - f(b)) \to 0 \text{ as } n \to \infty$$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$\Rightarrow f \in R(\alpha)$ if f is mono decreasing.

from case ① and ② $f \in R(\alpha)$ is f is monotonic.

Properties of Integral

Theorem

ⓐ If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in R(\alpha)$ on $[a,b]$

If $f \in R(\alpha)$ then $cf \in R(\alpha)$ for every constant c and

$$\int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \text{ and}$$

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$$

ⓑ If $f_1 \leq f_2$ on $[a,b]$ then

$$\int_a^b f_1\,d\alpha \leq \int_a^b f_2\,d\alpha$$

ⓒ If $f \in R(\alpha)$ and if $a<c<b$ then $f \in R(\alpha)$ on $[a,c]$ and on $[c,b]$.

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

(d) If $f \in R(\alpha)$ on $[a,b]$ and $|f(x)| < M$ then

$$\left| \int_a^b f d\alpha \right| \leq M(\alpha(b) - \alpha(a))$$

(e) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ then $f \in R(\alpha_1+\alpha_2)$

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

and if c is possitive constant then

$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha$$

→ Proof -

(a) $\exists$ partition P_1 such that $U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\varepsilon}{2}$ $\because f_1 \in R(\alpha)$ and

$\exists$ partition P_2 such that $U(P_2, f_1, \alpha) - L(P_2, f_2, \alpha) < \frac{\varepsilon}{2}$ $\because f_2 \in R(\alpha)$

Let,

$P = P_1 \cup P_2$ be comman refinement of P_1 and P_2.

$\therefore U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \frac{\varepsilon}{2}$ ----(1)

and

$U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \frac{\varepsilon}{2}$ ----(2)

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let,

$$\left.\begin{array}{l} \inf f_1(x) = m_{1i} \\ \inf f_2(x) = m_{2i} \\ \sup f_1(x) = M_{1i} \\ \sup f_2(x) = M_{2i} \end{array}\right\} \text{ on } x_{i-1} \le x \le x_i .$$

$$\sup(f_1+f_2) = M_{0i}$$
$$\inf(f_1+f_2) = m_{0i}$$

$$m_{1i} + m_{2i} \le m_{0i} \le M_{0i} \le M_{1i} + M_{2i}$$

multiply by $\Delta\alpha_i$

$$(m_{1i}+m_{2i})\Delta\alpha_i \le (m_{0i})\Delta\alpha_i \le (M_{0i})\Delta\alpha_i \le (M_{1i}+M_{2i})\Delta\alpha_i$$

applying Σ

$$\Sigma(m_{1i}+m_{2i})\Delta\alpha_i < \Sigma(m_{0i})\Delta\alpha_i \le \Sigma(M_{0i})\Delta\alpha_i \le \Sigma(M_{1i}+M_{2i})\Delta\alpha_i$$

$$L(P,f_1,\alpha) + L(P,f_2,\alpha) \le L(P,f_1+f_2,\alpha) \le U(P,f_1+f_2,\alpha) \le U(P,f_1,\alpha) + U(P,f_2,\alpha) \quad (3)$$

from (1) and (2)

$$U(P,f_1+f_2,\alpha) - L(P,f_1+f_2,\alpha) < \varepsilon$$

from (3)

$$U(P,f_1+f_2,\alpha) - L(P,f_1+f_2,\alpha) < \varepsilon$$

$$\Rightarrow f_1+f_2 \in R(\alpha)$$

If $U(P,f_1,\alpha) \le \int_a^b f_1\,d\alpha + \frac{\varepsilon}{2}$ and

$$U(P,f_2,\alpha) \le \int_a^b f_2\,d\alpha + \frac{\varepsilon}{2}$$

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$$\Rightarrow U(P,f_1,\alpha) + U(P,f_2,\alpha) \le \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon$$

$$\Rightarrow \int_a^b (f_1+f_2)\, d\alpha \le U(P, f_1+f_2, \alpha) \le U(P,f_1,\alpha) + U(P,f_2,\alpha) \le \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon$$

$$\Rightarrow \int_a^b (f_1+f_2)\, d\alpha \le \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + \varepsilon$$

$\because \varepsilon$ is arbitrary

$$\Rightarrow \int_a^b (f_1+f_2)\, d\alpha \le \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \quad \text{-----} (4)$$

If we replace f_1 and f_2 by $-f_1$ and $-f_2$ we get,

$$\int_a^b (-f_1-f_2)\, d\alpha \le \int_a^b -f_1 d\alpha + \int_a^b -f_2 d\alpha \quad \text{----} (5)$$

(mul by (-1) on both)

from (4) and (5) we get,

$$\int_a^b (f_1+f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha.$$

(c) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ then $f \in R(\alpha_1+\alpha_2)$ on $[a,b]$.

let,

$$\alpha = \alpha_1 + \alpha_2$$

Given that $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$

$$\therefore U(P,f,\alpha_1) - L(P,f,\alpha_1) < \frac{\varepsilon}{2} \quad \text{----} (1)$$

$$U(P,f,\alpha_2) - L(P,f,\alpha_2) < \frac{\varepsilon}{2} \quad \text{----} (2)$$

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$$\Delta\alpha_{1i} = \alpha_1(x_i) - \alpha_1(x_{i-1})$$
$$\Delta\alpha_{2i} = \alpha_2(x_i) - \alpha_2(x_{i-1})$$
$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$
$$\Delta\alpha_i = \alpha_1(x_i) + \alpha_2(x_i) - \alpha_1(x_{i-1}) - \alpha_2(x_{i-1})$$
$$\Delta\alpha_i = \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

Now,

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$
$$= \sum_{i=1}^{n} M_i (\Delta\alpha_{1i} + \Delta\alpha_{2i})$$
$$= \sum_{i=1}^{n} M_i \Delta\alpha_{1i} + \sum_{i=1}^{n} M_i \Delta\alpha_{2i}$$
$$= U(P,f,\alpha_1) + U(P,f,\alpha_2)$$

Similarly,

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$
$$= \sum_{i=1}^{n} m_i (\Delta\alpha_{1i} + \Delta\alpha_{2i})$$
$$= \sum_{i=1}^{n} m_i \Delta\alpha_{1i} + \sum_{i=1}^{n} m_i \Delta\alpha_{2i}$$
$$= L(P,f,\alpha_1) + L(P,f,\alpha_2)$$

Now,

$$U(P,f,\alpha) - L(P,f,\alpha)$$
$$= U(P,f,\alpha_1) + U(P,f,\alpha_2) - L(P,f,\alpha_1) - L(P,f,\alpha_2)$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{from (1) and (2)}$$

i.e. $U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$

$\Rightarrow f \in R(\alpha)$

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i.e. $f \in R(\alpha_1+\alpha_2)$ where $\alpha = \alpha_1+\alpha_2$

Now to prove

$$\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$$

$$\begin{aligned}\int_a^b f\,d\alpha &= \inf U(P,f,\alpha)\\ &= \inf \{U(P,f,\alpha_1) + U(P,f,\alpha_2)\}\\ &\geq \inf U(P,f,\alpha_1) + \inf U(P,f,\alpha_2)\\ &= \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2\end{aligned}$$

$$\Rightarrow \int_a^b f\,d\alpha \geq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \quad \text{----} (3)$$

Now,

$$\begin{aligned}\int_a^b f\,d\alpha &= \sup L(P,f,\alpha)\\ &= \sup \{L(P,f,\alpha_1) + L(P,f,\alpha_2)\}\\ &\leq \sup L(P,f,\alpha_1) + \sup L(P,f,\alpha_2)\\ &= \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2\end{aligned}$$

$$\Rightarrow \int_a^b f\,d\alpha \leq \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 \quad \text{----} (4)$$

from (3) and (4)

$$\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2.$$

Theorem

Suppose f is bounded on $[a,b]$ & f be has only finitely many points of discontinuity on $[a,b]$ and α is continuous at every point at which f is discontinuous, then $f \in R(\alpha)$.

→ Given that,

f is bounded.

$|f(x)| \leq M$ for some M.

let E be the set where f is discontinuous and given that α is continuous at each point of E.

let,

$[u_j, v_j]$ be the interval containing each point of E.

$[u_j, v_j] \subseteq [a,b]$ such that,

sum of $\alpha(v_j) - \alpha(u_j) < \varepsilon$

and each point of $E \cap (a,b)$ lies in the interior of $[u_j, v_j]$.

Remove (u_j, v_j) from $[a,b]$

Remaining set is compact say K.

Function f is continuous on compact set K.

As we know, every continuous function on compact set to matric space is uniformly continuous.

$\therefore$ Here f is uniformly continuous on K.

$\Rightarrow$ for any $\varepsilon > 0$ $\exists \delta > 0$ such that,

$|f(s) - f(t)| < \varepsilon$ if ~~such that~~ $s, t \in K$ & $|s-t| < \delta$

let,

take partition P such that,

$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ such that each $u_j \in P$ and each $v_j \in P$. If x_{i-1} is not one of the u_j then take $\Delta x_i < \delta$.

since $|f(x)| < M$

$\Rightarrow M_i - m_i < 2M \quad \forall i$

and if x_{i-1} is not of the u_j then $M_i - m_i < \varepsilon$

Now,

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{K} (M_i - m_i)\Delta\alpha_i + \sum_{[a,b]-K} (M_i - m_i)\Delta\alpha_i$$

$$< \varepsilon(\alpha(b) - \alpha(a)) + 2M\varepsilon$$

$\because \varepsilon$ is arbitrary number

$\Rightarrow f \in R(\alpha)$.

Theorem

Let $f \in R(\alpha)$ on $[a,b]$, $m \le f \le M$. Let ϕ be continuous function on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$ then $h \in R(\alpha)$ on $[a,b]$.

→ ϕ is continuous function on compact set $[m, M]$.

$\Rightarrow \phi$ is uniformly continuous on $[m, M]$

$\Rightarrow$ for given $\varepsilon > 0$ $\exists\ \delta > 0$ ----- $(\delta < \varepsilon)$

such that,

$|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$

Given that,

$f \in R(\alpha)$

$\Rightarrow \exists$ partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a,b]$.

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such that,

$$U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

$M_i = \sup f(x)$ $\quad x_{i-1} \leq x \leq x_i$

$m_i = \inf f(x)$ $\quad x_{i-1} \leq x \leq x_i$

Let, $M_i^* = \sup h(x)$ $\quad x_{i-1} \leq x \leq x_i$

and $m_i^* = \inf h(x)$ $\quad x_{i-1} \leq x \leq x_i$

divide $i = 0, 1, 2, \ldots, n$ in two sets A, B.

if $M_i - m_i < \delta$ then $i \in A$

if $M_i - m_i \geq \delta$ then $i \in B$

$$\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta\alpha_i < \delta^2$$

$$\Rightarrow \sum_{i \in B} \Delta\alpha_i < \delta$$

Let, $i \in A \Rightarrow M_i - m_i < \delta$

$\Rightarrow M_i^* - m_i^* < \varepsilon$

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i=0}^{n} (M_i^* - m_i^*) \Delta\alpha_i$$

$$= \sum_{i \in A} (M_i^* - m_i^*) \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta\alpha$$

$$\leq \varepsilon(\alpha(b) - \alpha(a)) + 2k\delta$$

where k is such that,

$|\phi(x)| \leq k \Rightarrow M_i^* - m_i^* \leq 2k$

$$\Rightarrow < \varepsilon\,[(\alpha(b) - \alpha(a)) + 2k]$$

$\because \varepsilon$ is arbitrary.

$\Rightarrow h \in R(\alpha)$ on $[a,b]$.

Theorem

If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a,b]$ then

ⓐ $f \cdot g \in R(\alpha)$ on $[a,b]$.

ⓑ $|f| \in R(\alpha)$ and $\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$.

→ ⓐ Given that,

$f \in R(\alpha)$ and $g \in R(\alpha)$.

In theorem $f \in R(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ be continuous function on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$ then $h \in R(\alpha)$ on $[a,b]$.

Let, $\phi(t) = t^2$ which is continuous.

$\therefore h(x) = \phi(f(x)) = f^2(x) \quad \forall x \in [a,b]$.

$\therefore f^2 \in R(\alpha)$.

We know that if $f, g \in R(\alpha)$ then $f+g$ and $f-g \in R(\alpha)$.

We can write,

$$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

$\therefore fg \in R(\alpha)$.

ⓑ In the theorem $f \in R(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ be continuous function on $[m, M]$ and $h(x) = \phi(f(x))$ on $[a,b]$ then $h \in R(\alpha)$ on $[a,b]$.

If we take,

$\phi(t) = |t|$ on $[m, M]$ where $m \leq f \leq M$.

Here, ϕ is continuous.

$h(x) = \phi(f(x)) = |f(x)| \quad \forall x \in [a,b]$

$\therefore h = |f| \in R(\alpha)$.

Now,

choose $c = \pm 1$.

so that, $c\int_a^b f\, d\alpha \geq 0$

i.e. if $\int_a^b f\, d\alpha \geq 0$ then take $c = +1$.

and if $\int_a^b f\, d\alpha < 0$ then take $c = -1$

$$\left|\int_a^b f\, d\alpha\right| = c\int_a^b f\, d\alpha = \int_a^b c f\, d\alpha \leq \int_a^b |f|\, d\alpha$$

$(\because cf \leq |f|)$

$$\Rightarrow \left|\int_a^b f\, d\alpha\right| \leq \int_a^b |f|\, d\alpha.$$

Unit step function

The ~~fun.~~ unit step function I is defined by

$I(x) = 0$ if $x \leq 0$

$= 1$ if $x > 0$.

Theorem

If $a < s < b$, f is bounded on $[a,b]$. Let $\alpha(x) = I(x-s)$. f is continuous at s then

$$\int_a^b f\, d\alpha = f(s)$$

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→ let partition P of $[a,b]$
$\{a = x_0, x_1, x_2, x_3 = b\}$

such that,
$x_1 = s < x_2$

$$U(P,f,\alpha) = \sum_{i=1}^{3} M_i \Delta\alpha_i = M_2$$

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$\Delta\alpha_1 = I(x_1 - s) - I(x_0 - s)$

$= 0 - 0$

$\Delta\alpha_1 = 0$ for $i = 1$

$\Delta\alpha_2 = 1 - 0 = 1$

$\Delta\alpha_3 = \alpha(x_3 - s) - \alpha(x_2 - s) = 1 - 1 = 0$

Similary,
$L(P,f,\alpha) = m_2$

as f is continuous at s

$\Rightarrow M_2$ and $m_2 \to f(s)$

$$\Rightarrow \int_a^b f\,d\alpha = f(s)$$

Hence the proof.

Theorem

Let $c_n \geq 0$ for $n = 1, 2, 3 \ldots\ldots$, $\sum_{n=1}^{\infty} c_n$ converges. Let $\{s_n\}$ be a sequences of distinct points of (a,b). Let $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-s_n)$

and f is continuous on each s_n then $[a,b]$

$$\int_a^b f\,d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

Given that,

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-s_n) \qquad a<s_n<b$$

$$\leq \sum_{n=1}^{\infty} c_n \cdot 1 \qquad \because I(x-s_n) \leq 1 \ \forall x$$

It is given that $\sum_{n=1}^{\infty} c_n$ converges. $\therefore$ by comparision test.

$\sum_{n=1}^{\infty} c_n I(x-s_n)$ also converges.

$\Rightarrow \alpha$ is well defined.

$\alpha(a) = \sum_{n=1}^{\infty} c_n I(a-s_n) = 0$ and $\alpha(b) = \sum_{n=1}^{\infty} c_n$

$\Rightarrow \alpha$ is monotonically incresing and bdd.

Now,

$\sum_{n=1}^{\infty} c_n$ is convergent, $\therefore$ its seq^n of partial

sum conveges and we know any convergent seq^n in $(a,b) \subseteq R$ is the cauchy sequence. for any $\varepsilon>0$ $\exists$ +ve int N such that

$|t_n - t_N| < \varepsilon$ if $n \geq N$ where $t_n = c_1+c_2+\cdots\cdots+c_n$

$$|c_1+c_2+\cdots+c_N+c_{N+1}+\cdots+c_n-(c_1+c_2+\cdots+c_N)| < \varepsilon$$

$$\Rightarrow |c_{N+1}+\cdots\cdots+c_n| < \varepsilon$$

$$\left|\sum_{N+1}^{n} c_n\right| < \varepsilon$$

as $n\to\infty$ $\cdot$ $\sum_{N+1}^{\infty} c_n < \varepsilon$ —— (1)

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Now,
$$\alpha(x) = \sum_{n=1}^{N} C_n I(x-S_n) + \sum_{n=N+1}^{\infty} C_n I(x-S_n)$$

$$= \alpha_1(x) + \alpha_2(x)$$

$\because$ f is continuous on $[a,b]$

$\Rightarrow f \in R(\alpha)$ on $[a,b]$

$$\int_a^b f\, d\alpha = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2$$

consider
$$\int_a^b f\, d\alpha_1(x) = \int_a^b f\, d\left(\sum_{n=1}^{N} C_n I(x-S_n)\right)$$

$$= \int_a^b f\, d(C_1 I(x-S_1)) + \int_a^b f\, d(C_2 I(x-S_2))$$

$$+ \cdots + \int_a^b f\, d(C_N I(x-S_N)) \quad \left(\because \int_a^b f\, d(\alpha_1 + \alpha_2) = \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2\right)$$

$$= C_1 \int_a^b f\, d(I(x-S_1)) + C_2 \int_a^b f\, d(I(\alpha-S_2)) + \cdots +$$

$$C_N \int_a^b f\, d(I(x-S_N))$$

$$= C_1 f(S_1) + C_2 f(S_2) + \cdots + C_N f(S_N)$$

$$\int_a^b f\, d\alpha_1(x) = \sum_{n=1}^{N} C_n f(S_n)$$

Now,

f is constinous fun on $[a,b]$

$\Rightarrow f$ is bounded

$\therefore \exists\, M > 0$ such that $M = \sup |f(x)|$

$\Rightarrow |f(x)| \leq M$

$$\left| \int_a^b f\, d\alpha_2(x) \right| = \int_a^b |f|\, d\alpha_2(x) \leq \int_a^b M\, d\alpha_2(x) =$$

$$M \int_a^b d\alpha_2(x) = M[\alpha_2(b) - \alpha_2(a)] = M \sum_{n=N+1}^{\infty} c_n < M\varepsilon$$

$$\left| \int_a^b f(x)\, d\alpha - \sum_{n=1}^{n} c_n f(s_n) \right| = \left| \int_a^b f\, d\alpha_1 + \int_a^b f\, d\alpha_2 - \int_a^b f\, d\alpha_1 \right|$$

$$= \left| \int_a^b f\, d\alpha_2 \right| < M\varepsilon$$

$\because M$ is finite and ε is arbitrary.

$$\Rightarrow \int_a^b f(x)\, d\alpha = \sum_{n=1}^{N} c_n f(s_n)$$

As $N \to \infty$ we get, $\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$

and Hence proved.

Theorem

Assume α is monotonically increasing, $\alpha' \in R$ on $[a,b]$, Let f be bdd real function $[a,b]$ then $f \in R(\alpha)$ if and only if $f\alpha' \in R$.

In that,

$$\int_a^b f\, d\alpha = \int_a^b f\alpha'\, dx$$

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→ Given that,

$\alpha' \in R$ (set of Riemann integrable function)

$\Rightarrow$ for given $\varepsilon > 0$ $\exists$ partition

$P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ of $[a,b]$

such that,

$$U(P,\alpha) - L(P,\alpha') < \varepsilon \quad \ldots\ldots\ldots (1)$$

Given that, α' exists i.e. function α is differentiable.

$\Rightarrow$ α is continuous on $[a,b]$

$\Rightarrow$ α is continuous on $[x_{i-1}, x_i]$, $\forall\ i = 1,2,\ldots,n$

By mean value thm there exists,

$t_i \in [x_{i-1}, x_i]$ $\forall\ i = 1,2,\ldots\ldots,n$.

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i)$$

i.e. $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$

from (1) $\sum |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon$

where $s_i, t_i \in [x_{i-1}, x_i]$.

$\because$ f is bdd function on $[a,b]$

$\Rightarrow$ $\exists M > 0$ such that $M = \sup |f(x)|$.

Now, consider

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i$$

consider,

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right| =$$

$$\left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

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$$\leq \left| \sum_{i=1}^{n} M\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} M\alpha(s_i)\Delta x_i \right| \quad \because f(s_i) \leq M.$$

$$\leq M \sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)| \Delta x_i$$

$$\leq M\varepsilon$$

Thus,

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i \right| \leq M\varepsilon$$

$$\Rightarrow -M\varepsilon \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i \leq M\varepsilon$$

Now, let

$$\sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i + M\varepsilon$$

taking supremum on L.H.S. we get,

$$U(P,f,\alpha) \leq \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i + M\varepsilon$$

$$U(P,f,\alpha) \leq U(P,f,\alpha') + M\varepsilon.$$

Now, $U(P,f,\alpha) - U(P,f\alpha') \leq M\varepsilon$ ----- (2)

$$-M\varepsilon \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\Delta\alpha_i \leq M\varepsilon$$

$$\Rightarrow \sum_{i=1}^{n} f\alpha'(s_i)\Delta x_i \leq \sum_{i=1}^{n} f(s_i)\Delta\alpha_i + M\varepsilon$$

$$\Rightarrow U(P,f\alpha') \leq U(P,f,\alpha) + M\varepsilon$$

$-M\varepsilon \leq U(P,f,\alpha) - U(P,f\alpha')$ ----- (3)

from (2) and (3)

$$|U(P,f,\alpha) - U(P,f\alpha')| \leq M\varepsilon$$

$$\Rightarrow \left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\alpha'\,dx \right| \leq M\varepsilon \quad \text{------} (4)$$

$\because \varepsilon$ is arbitrary

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\alpha'\,dx \quad \text{------} (5)$$

similarly we get,

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\alpha'\,dx \quad \text{------} (6)$$

Now, If $f \in R(\alpha)$ then L.H.S. of (5) & (6) will be equal

$$\Rightarrow \overline{\int_a^b} f\alpha'\,dx = \underline{\int_a^b} f\alpha'\,dx$$

$\Rightarrow f\alpha' \in R$

and conversily if $f\alpha' \in R$ then R.H.S. of (5) and (6) will be equal

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$\Rightarrow f \in R(\alpha)$.

Theorem (change of variable)

If ϕ is strictly incresing continuous function that maps $[A, B]$ onto $[a, b]$. suppose α is monotonically incresing on $[a, b]$ $f \in R(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by $\beta(y) = \alpha(\phi(y))$ and $g(y) = f(\phi(y))$ then $g \in R(\beta)$ and

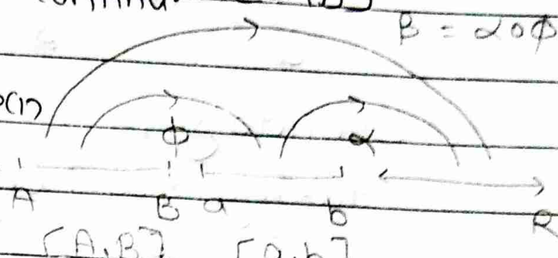
$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha.$$

Given that

$$\phi : [A,B] \xrightarrow[\text{continu.}]{\text{onto}} [a,b]$$

Let partition

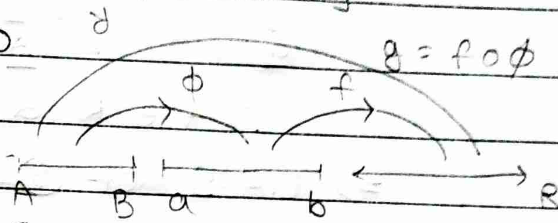
$P = \{a = x_0, x_1, \dots, x_n = b\}$



of $[a,b]$.

This partition corrsponds to partition

$Q = \{A = y_0, y_1, \dots, y_n = B\}$ of $[A,B]$



Such that,

$$\phi(y_i) = x_i$$

Given that,

$$g(y) = f(\phi(y))$$

$$\sup g(y) \quad y_{i-1} \le y \le y_i$$

$$= \sup f(\phi(y)) \quad y_{i-1} \le y \le y_i$$

$$= \sup f(x) \quad x_{i-1} \le x \le x_i$$

$$= M_i$$

$\sup f(x) = M_i = \sup g(y) \qquad y_{i-1} \le y \le y_i$

$\inf f(x) = m_i = \inf g(y) \qquad y_{i-1} \le y \le y_i$

$$\Delta\beta_i = \beta(y_i) - \beta(y_{i-1})$$

$$= \alpha(\phi(y_i)) - \alpha(\phi(y_{i-1}))$$

$$= \alpha(x_i) - \alpha(x_{i-1})$$

$$= \Delta\alpha_i$$

$U(P, f, \alpha) = U(Q, g, \beta) \quad \text{------} (1)$

$L(P, f, \alpha) = L(Q, g, \beta) \quad \text{------} (2)$

taking inf. on both sides of eq^n (1)

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_A^B} g\,d\beta \quad \text{------- (3)}$$

Similarly,

taking sup. on both sides of eq^n (2)

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_A^B} g\,d\beta \quad \text{------ (4)}$$

$\because f \in R(\alpha)$

$\Rightarrow$ L.H.S. of (3) and (4) are equal

$\therefore$ R.H.S. of (3) and (4) will be equal

$$\text{i.e. } \overline{\int_A^B} g \cdot d\beta = \underline{\int_A^B} g \cdot d\beta$$

$\Rightarrow g \in R(\beta)$ and

$$\int_a^b f\,d\alpha = \int_A^B g\,d\beta$$

and hence proved.

* Integration and Differentiation Theorem

let $f \in R$ on $[a,b]$, $a \le x \le b$, put $F(x) = \int_a^x f(t).dt$ then F is continuous on $[a,b]$, further more, if f is continous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

→ Given that,

$f \in R$ on $[a,b]$

$\Rightarrow f$ is bounded

$\Rightarrow \exists M>0$ such that $\sup |f(x)| \le M$

i.e. $|f(x)| \le M$. for $a \le t \le s < b$

$$|F(s) - F(t)| = \left| \int f(x)\,dx - \int_a^t f(x)\,dx \right|$$

$$= \left| \int_a^t f(x)\,dx + \int_t^s f(x)\,dx - \int_a^t f(x)\,dx \right|$$

$$= \left| \int_t^s f(x)\,dx \right|$$

$$\le \int_t^s |f(x)\,dx$$

$$\le \int_t^s M\,dx$$

$$= M[s-t]$$

$\Rightarrow |F(s) - F(t)| \le M(s-t)$

for given $\varepsilon > 0$ we can choose $\delta = \varepsilon/M > 0$ such that,

$|F(s) - F(t)| < \varepsilon$ if $|s-t| < \delta = \varepsilon/M$

$\Rightarrow F$ is uniformly continuous on $[a,b]$.

$\Rightarrow F$ is continuous on $[a,b]$.

Given that f is continuous at x_0

$x_0 \in [a,b]$

for given $\varepsilon > 0$ $\exists \delta > 0$ such that

$|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$.

Let, ~~$s < x_0 < t$~~ $x_0 - \delta < t < x_0 < s < x_0 + \delta$.

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consider $\left| \dfrac{F(s)-F(t)}{s-t} - f(x_0) \right| =$

$$\left| \frac{\int_a^s f(x)\,dx - \int_a^t f(x)\,dx}{s-t} - \frac{(s-t)\,f(x_0)}{s-t} \right|$$

$$= \left| \frac{\int_t^s f(x)\,dx}{s-t} - \frac{\int_t^s f(x_0)\,dx}{s-t} \right|$$

$$= \frac{1}{s-t} \left| \int_t^s (f(x) - f(x_0))\,dx \right|$$

$$\leq \frac{1}{s-t} \int_t^s |f(x) - f(x_0)|\,dx$$

$$< \frac{1}{s-t} (s-t)\,\varepsilon = \varepsilon$$

$$\Rightarrow \left| \frac{F(s)-F(t)}{s-t} - f(x_0) \right| < \varepsilon$$

$\because$ ε is arbitrary $\Rightarrow F'(x_0) = f(x_0)$

Fundamental theorem of calculus

If $f \in R$ on $[a,b]$, If there exist a differentiable function F on $[a,b]$ such that $F' = f$ on $[a,b]$ then

$$\int_a^b f\,dx = F(b) - F(a)$$

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Given that, $f \in R$

$\Rightarrow$ for every $\varepsilon > 0$ $\exists$ partition

$P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$ of $[a,b]$

such that, $U(P,f) - L(P,f) < \varepsilon$ ...... ①

Now,

F is differentiable on (a,b)

$\therefore$ F is differentiable on (x_{i-1}, x_i) for ~~for~~ $i = 1, 2, \ldots\ldots, n$

and F is continuous on $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$

by mean value theorem, $\exists$ $t_i \in [x_{i-1}, x_i]$

such that,

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i)$$

②

$\Rightarrow$ $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$ $(\because F' = f)$

from ① we can write

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f\,dx \right| < \varepsilon \quad \ldots\ldots \text{ where } t_i \in [x_{i-1}, x_i]$$

from ②

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))$$

$$= F(b) - F(a)$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f\,dx \right| < \varepsilon$$

$\because$ ε is arbitary

$$\Rightarrow \int_a^b f\,dx = F(b) - F(a)$$

Hence proved.

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Theorem

Suppose F and G are differenciable funs on (a, b). Such that $F' = f$ and $G' = g$.

$f, g \in R$ then

$$\int_a^b F(x) g(x)\, dx = F(b) G(b) - F(a) G(a) - \int_a^b f(x) G(x) . dx$$

→ Let,

$$H(x) = F(x) G(x)$$

$$H'(x) = F(x) g(x) + f(x) G(x)$$

$$\exists H \quad H' = Fg + fG$$

by previous theorem

$$\int_a^b Fg + fG\, dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b Fg\, dx + \int_a^b fG\, dx = F(b) G(b) - F(a) G(a)$$

Thus,

$$\int_a^b Fg\, dx = F(b) G(b) - F(a) G(a) - \int_a^b fG\, dx$$

Hence proved

Integration of vector valued function

Let $f_1, f_2, \text{------}, f_k$ be real fun^{ns} on $[a, b]$.

Let $\bar{f} = (f_1, f_2, \text{-----}, f_k)$ be a corresponding mapping of $[a, b]$ into R^k

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Unit - III

sequences and series of functions

If α is increses monotonicaly on $[a,b]$ to say $\bar{f} \in R(\alpha)$ means that each $f_j \in R(\alpha)$ for $j = 1, 2, \ldots, k$.

If this is the case we define

$$\int_a^b \bar{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \ldots, \int_a^b f_k\,d\alpha\right)$$

in other words

$\int_a^b \bar{f}\,d\alpha$ is the point in R^k whose j^{th} componant is $\int_a^b f_j\,d\alpha$

Properties

a. If $\bar{f}$ and $\bar{g}$ are vector valued function on $[a,b]$ such that $\bar{f}$ and $\bar{g} \in R(\alpha)$ over $[a,b]$ then $\int_a^b (\bar{f}+\bar{g})\,d\alpha = \int_a^b \bar{f}\,d\alpha + \int_a^b \bar{g}\,d\alpha$

→ Let,

i) $\bar{f} = (f_1, f_2, \ldots, f_k)$ and $\bar{g} = (g_1, g_2, \ldots g_k)$

then

$(\bar{f}+\bar{g}) = (f_1+g_1, f_2+g_2, \ldots, f_k+g_k)$

$\because \bar{f}, \bar{g} \in R(\alpha)$

$$\Rightarrow \int_a^b \bar{f}\,d\alpha = \left(\int_a^b f_1\,d\alpha, \int_a^b f_2\,d\alpha, \ldots \int_a^b f_k\,d\alpha\right)$$

and

$$\int_a^b \bar{g}\,d\alpha = \left(\int_a^b g_1\,d\alpha, \int_a^b g_2\,d\alpha, \ldots, \int_a^b g_k\,d\alpha\right)$$

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Now,

$$\int_a^b \bar{f}\, d\alpha + \int_a^b \bar{g}\, d\alpha = \left(\int_a^b f_1 d\alpha_1, \ldots\ldots, \int_a^b f_k d\alpha\right)$$

$$+ \left(\int_a^b g_1 d\alpha, \ldots\ldots, \int_a^b g_k d\alpha\right)$$

$$= \left(\int_a^b f_1 d\alpha + \int_a^b g_1 d\alpha, \ldots\ldots, \int_a^b f_k d\alpha + \int_a^b g_k d\alpha\right)$$

$$= \left(\int_a^b (f_1+g_1) d\alpha, \ldots\ldots, \int_a^b (f_k+g_k) d\alpha\right)$$

$$= (j = 1, 2, \ldots\ldots k)$$

$\because$ f_j and $g_j \in R(\alpha)$

$$\Rightarrow \int_a^b (f_j+g_j)\, d\alpha = \int_a^b f_j\, d\alpha + \int_a^b g_j\, d\alpha)$$

$$= \int_a^b (\overline{f+g})\, d\alpha$$

$$= \int_a^b (\bar{f}+\bar{g})\, d\alpha$$

$$\Rightarrow \int_a^b (\bar{f}+\bar{g})\, d\alpha = \int_a^b \bar{f}\, d\alpha + \int_a^b \bar{g}\, d\alpha$$

ii) If $\bar{f} \in R(\alpha)$ then for any $c \in R$
$c\bar{f} \in R(\alpha)$

let,

$$\bar{f} = (f_1, f_2, \ldots\ldots, f_k)$$

where $f_j \in R(\alpha)$ over $[a, b]$

$$c\bar{f} = (cf_1, cf_2, \ldots\ldots, cf_k)$$

$\because f_j \in R(\alpha)$, $\forall$ $j = 1, 2, \ldots, k$

$\Rightarrow$ for all $c \in R$, $cf_j \in R(\alpha)$ over $[a, b]$

$\therefore \int_a^b cf_j \, d\alpha$ exists for all $j = 1, 2, \ldots, k$.

$$\Rightarrow \int_a^b c\bar{f} \, d\alpha = \left(\int_a^b cf_1 \, d\alpha, \ldots, \int_a^b cf_k \, d\alpha \right)$$

$$= \left(c\int_a^b f_1 \, d\alpha, \ldots, c\int_a^b f_k \, d\alpha \right)$$

$$\left(\because c\int_a^b f_j \, d\alpha = \int_a^b c \right.$$

$$= c\left(\int_a^b f_1 \, d\alpha, \ldots \int_a^b f_k \, d\alpha \right)$$

$$= c\int_a^b \bar{f} \, d\alpha.$$

b. If $\bar{f} \in R(\alpha)$ over $[a, b]$, $a < c < b$ then $\bar{f} \in R(\alpha)$ over $[a, c]$ and over $[c, b]$ and

$$\int_a^b \bar{f} \, d\alpha = \int_a^b \bar{f} \, d\alpha + \int_a^b \bar{f} \, d\alpha$$

Theorem

If $\bar{f}$ and $\bar{F}$ maps $[a, b]$ into R^k. $\bar{f} \in R(\alpha)$ on $[a, b]$ if $\bar{F}' = \bar{f}$ then $\int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$

$\rightarrow$ Let,

$\bar{f} = (f_1, f_2, \ldots, f_k)$ ~~where f_j are~~ where f_j maps $[a, b]$ into R & $f_j \in R(\alpha)$ on $[a, b]$.

~~So~~ given that,

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$\bar{F}' = \bar{f}$ for $j = 1, 2, \ldots, k$

Let, $\bar{F} = (F_1, F_2, \ldots, F_k)$

where, F_j are differentiable real valued function on $[a,b]$ for $j = 1, 2, \ldots, k$

$$\int_a^b \bar{f}(t)\,dt = \left(\int_a^b f_1\,dt, \int_a^b f_2\,dt, \ldots, \int_a^b f_k\,dt\right)$$

$\ldots$ ($\because$ by def^n) $\ldots$ ($\because F_j' = f_j \quad j = 1, 2, \ldots, k$)

$$= F_1(b) - F_1(a), F_2(b) - F_2(a), \ldots, F_k(b) - F_k(a)$$

($\because F_j' = f_j \quad j = 1, 2, \ldots, k$)

$$= (F_1(b), F_2(b), \ldots, F_k(b)) - (F_1(a), F_2(a), \ldots, F_k(a))$$

$$= \bar{F}(b) - \bar{F}(a).$$

$$\therefore \int_a^b \bar{f}(t)\,dt = \bar{F}(b) - \bar{F}(a)$$

and hence proved.

Theorem

If $\bar{f}$ maps $[a,b]$ into R^k if $f \in R(\alpha)$ for some monotonically increasing fun^n α $[a,b]$ then, $|\bar{f}| \in R(\alpha)$ and

$$\left| \int_a^b \bar{f}\,d\alpha \right| \leq \int_a^b |\bar{f}|\,d\alpha$$

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Given,

$\bar{f} \in R(\alpha)$

$\bar{f} : [a,b] \to R^K$ s.t. $\bar{f} = (f_1, f_2, \text{-----}, f_K)$

to prove $|\bar{f}| \in R(\alpha)$

where $|\bar{f}| = (f_1^2 + f_2^2 + \text{------} + f_K^2)^{1/2}$

Now, $\because$ each $f_j \in R(\alpha)$ for $j = 1, 2, \text{----}, K$

$\therefore f_1^2 + f_2^2 + \text{------} + f_K^2 \in R(\alpha)$ over $[a,b]$

~~Let us take~~

$\exists\, M \geq 0$ such that $f_1^2 + f_2^2 + \text{-----} + f_K^2 \leq M$

$\because$ each f_j is bounded.

Let us define $\phi : [0, M] \to R$ by $\phi(t) = \sqrt{t}$.

which is continuous.

$\therefore h = \phi(f_1^2 + f_2^2 + \text{----} + f_K^2) \in R(\alpha)$ on $[a,b]$

i.e. $h = (f_1^2 + f_2^2 + \text{-----} + f_K^2)^{1/2} \in R(\alpha)$ on $[a,b]$

i.e. $|\bar{f}| \in R(\alpha)$.

$$\bar{f} = (f_1, f_2, \text{------}, f_K)$$

$$\int \bar{f}\, d\alpha = \left(\int f_1 d\alpha, \int f_2 d\alpha, \text{------}, \int f_K d\alpha\right)$$

$$\bar{y} = (y_1, y_2, \text{------}, y_K)$$

where,

$$y_i = \int f_i\, d\alpha \quad \text{and}$$

$$\bar{y} = \int \bar{f}\, d\alpha$$

$$|\bar{y}| = \left(\sum_{i=1}^{K} y_i^2\right)^{1/2}$$

if $|\bar{y}| = 0$ then ① holds.

if $|\bar{y}| \neq 0$

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$$|\bar{y}|^2 = \sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k} y_i \int f_i \, d\alpha = \int \sum_{i=1}^{k} y_i f_i \, d\alpha$$

by Schwarz inequality

$$\sum_{i=1}^{k} y_i f_i \le |\bar{y}| \, |\bar{f}|$$

$$\Rightarrow \int \sum_{i=1}^{k} y_i f_i \le \int |\bar{y}| \, |\bar{f}| \, d\alpha$$

$$\Rightarrow |\bar{y}|^2 \le \int |\bar{y}| \, |\bar{f}| \, d\alpha$$

$$|\bar{y}|^2 \le |\bar{y}| \int |\bar{f}| \, d\alpha$$

$$\because |\bar{y}| \ne 0$$

$$\Rightarrow |\bar{y}| \le \int |\bar{f}| \, d\alpha$$

i.e. $\left| \int \bar{f} \, d\alpha \right| \le \int |\bar{f}| \, d\alpha$

and hence proved.

Rectifiable curves

A continous mapping γ of $[a,b]$ into R^k is called curve in R^k.

i) If γ is one to one then γ is called arc

ii) If $\gamma(a) = \gamma(b)$ then the curve γ is called closed curve.

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Def^n of Rectifiable curves

let $P = \{a = x_0, x_1, \text{------}, x_n = b\}$ be a partition of $[a,b]$ and let $\Lambda(P,\gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$ i^{th} term in the sum

is the distance in R^k bet^n ~~per~~ two points $\gamma(x_{i-1})$ and $\gamma(x_i)$ hence $\Lambda(P,\gamma)$ length of polygonal path with vertices at $\gamma(x_0)$, $\gamma(x_1)$, $\gamma(x_2)$, ------, $\gamma(x_n)$ in this order.

As the partition becomes finer and finer this polygonal approches the range of γ more and more closely. This makes its seem regionable to define length of γ as $\gamma = \sup_P \Lambda(P,\gamma)$ where sup is taken more all partition (a,b) if $\Lambda(\gamma) < \infty$ then γ is rectifible.

- "continuously differential function f means f is differentiable (i.e. f' exists) and f' is continuous".

Theorem

most IMP

If γ' is continous then, γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$.

→ Proof -

Given that,

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ϑ' is continuous.

let P be a partition of $[a,b]$.

$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$

by fundamental theorem of calculous

$$|\vartheta(x_i) - \vartheta(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \vartheta'(t)\,dt\right| \quad \text{for } i=1,2,\ldots,n$$

$$\Rightarrow |\vartheta(x_i) - \vartheta(x_{i-1})| \leq \int_{x_{i-1}}^{x_i} |\vartheta'(t)|\,dt \quad \text{for } i=1,2,\ldots,n$$

$$\sum_{i=1}^{n} |\vartheta(x_i) - \vartheta(x_{i-1})| \leq \sum \int_{x_{i-1}}^{x_i} |\vartheta'(t)|\,dt$$

$$\Rightarrow \Lambda(P,\vartheta) \leq \int_a^b |\vartheta'(t)|\,dt$$

taking supremum over P, we get,

$$\Lambda(\vartheta) \leq \int_a^b |\vartheta'(t)|\,dt$$

claim

$$\int_a^b |\vartheta'(t)|\,dt \leq \Lambda(\vartheta)$$

Given that ϑ' is continuous on $[a,b]$

$\Rightarrow \vartheta'$ is uniformely continous on $[a,b]$

let

$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$

such that $\Delta x_i < \delta$

ϑ' is uniformaly continuous on

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$[x_{i-1}, x_i]$ for each $i = 1, 2, \dots, n$.

$\Rightarrow$ for any $\varepsilon > 0$ $\exists\ \delta > 0$ such that

$|v'(s) - v'(t)| < \varepsilon$ whenever $|s-t| < \delta$

let, $t \in [a,b]$ such that $x_{i-1} \le t \le x_i$

by using Δ inequality$(||a| - |b|| \le |a-b|)$ we get,

$\big||v'(t)| - |v'(x_i)|\big| \le |v'(t) - v'(x_i)| < \varepsilon$

$\because |t - x_i| < \delta$

$\Rightarrow |v'(t)| - |v(x_i)| < \varepsilon$

$\Rightarrow |v'(t)| < |v'(x_i)| + \varepsilon$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |v'(t)|dt < |v(x_i)|\Delta x_i + \varepsilon \Delta x_i$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |v'(t)|\,dt < \int_{x_{i-1}}^{x_i} |v'(x_i)|\,dt + \varepsilon \Delta x_i$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |v'(t)\,dt < \left|\int_{x_{i-1}}^{x_i} v'(x_i)\,dt\right| + \varepsilon \Delta x_i$$

$$= \left|\int_{x_{i-1}}^{x_i} [v'(t) + v'(x_i) - v'(t)]dt\right| + \varepsilon \Delta x_i$$

$$= \left|\int_{x_{i-1}}^{x_i} v'(t)\,dt + \int_{x_{i-1}}^{x_i} [v'(x_i) - v'(t)]dt\right| + \varepsilon \Delta x_i$$

$$\le \left|\int_{x_{i-1}}^{x_i} v'(t)dt\right| + \left|\int_{x_{i-1}}^{x_i} [v'(x_i) - v'(t)]\,dt\right| + \varepsilon \Delta x_i$$

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$$\leq |v'(x_i) - v'(x_{i-1})| + \int_{x_{i-1}}^{x_i} |v'(x_i) - v'(t)|\, dt + \epsilon \Delta x_i$$

$$< |v(x_i) - v(x_{i-1})| + 2\epsilon \Delta x_i$$

$$\Rightarrow \int_{x_{i-1}}^{x_i} |v'(t)|\, dt < |v(x_i) - v(x_{i-1})| + 2\epsilon \Delta x_i$$

adding up all inequality

$$\int_a^b |v'(t)|\, dt < \sum_{i=1}^{n} |v(x_i) - v(x_{i-1})| + 2\epsilon [b-a]$$

$$\int_a^b |v'(t)|\, dt < \Lambda(P, v) + 2\epsilon\,(b-a)$$

taking sup. over P

$$\int_a^b |v'(t)|\, dt < \Lambda(v) + 2\epsilon\,(b-a)$$

$$\Rightarrow \int_a^b |v'(t)|\, dt \leq \Lambda(v) \quad \text{---} (2)$$

from (1) and (2)

$$\int_a^b |v'(t)|\, dt = \Lambda(v).$$

and hence proved.