Page No.
Date 19 07 18

Definition: The complex Number:

The number of the form $z=x+iy$ is called as the complex number. Where x & y are real number and $i=\sqrt{-1}$ is called an imaginary number or unit

Note: ① $\mathbb{C}=\{x+iy \mid x,y\in\mathbb{R},\ i=\sqrt{-1}\}$ is the set of all complex number.

② We denote the complex number $a+ib$ as an ordered pair (a,b) where $a,b\in\mathbb{R}$.

③ Thus, set of all complex numbers is redefined as

$$\mathbb{C}=\{(x,y) \mid x,y\in\mathbb{R}\}$$

- Operations on $\mathbb{C}$ (Algebra of Complex Numbers):

① Equality of Complex Numbers:

Let $z_1=a_1+ib_1=(a_1,b_1)$

& $z_2=a_2+ib_2=(a_2,b_2)$

be any two complex numbers, then $z_1=z_2 \Leftrightarrow a_1+ib_1=a_2+ib_2$

$\Leftrightarrow (a_1,b_1)=(a_2,b_2)$

$\Leftrightarrow a_1=a_2$ & $b_1=b_2$

Thus z_1 & z_2 are equal if & only if their corresponding real & imaginary parts are equal.

② Addition of Complex Numbers:

Consider two complex numbers $z_1=a_1+ib_1$ & $z_2=a_2+ib_2$, Then their addition is defined as

$$z_1+z_2=(a_1+ib_1)+(a_2+ib_2)$$

$$= (a_1+a_2) + i(b_1+b_2)$$

$$(a_1, b_1) + (a_2, b_2) = (a_1+a_2,\ b_1+b_2)$$

(3) Multiplication of complex numbers:

Consider any two complex numbers $z_1 = a_1+ib_1$ and $z_2 = a_2+ib_2$, then the multiplication z_1 and z_2 is denoted by $z_1 \cdot z_2$ and it is defined as

$$z_1 \cdot z_2 = (a_1+ib_1)(a_2+ib_2)$$
$$= a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2$$
$$= a_1a_2 + i(a_1b_2+a_2b_1) - b_1b_2) \qquad \because i^2=-1$$
$$= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1)$$
$$= ((a_1a_2 - b_1b_2), (a_1b_2 + a_2b_1))$$

Thus we have,

$$(a_1, b_1) \cdot (a_2, b_2) = ((a_1a_2 - b_1b_2), (a_1b_2 + a_2b_1))$$

- Complex field :-

The set $\mathbb{C}$ of complex numbers together with the operations addition and multiplication of complex numbers forms a field i.e, addition & multiplication operations satisfy the following property.

(1) Addition is closed :

The sum of two complex numbers is a complex number i.e, if $z_1, z_2 \in \mathbb{C}$ then, $z_1+z_2 \in \mathbb{C}$. Thus, the addition operation is closed.

(2) Addition is commutative :

For any two complex numbers $z_1, z_2 \in \mathbb{C}$

We have $z_1+z_2=z_2+z_1$.
Thus, addition commutative.

③ Addition is associative:
For any three complex numbers $z_1, z_2, z_3 \in \mathbb{C}$.
We have $z_1+(z_2+z_3) = (z_1+z_2)+z_3$
ie, addition is associative.

④ Existance of zero element or additive identity:
There exist an element $0 = 0+0i = (0,0)$ in $\mathbb{C}$
so that $z+0 = 0+z = z \ \forall \ z \in \mathbb{C}$.
Where 0 is called as additive identity.

⑤ Existance of Additive inverse:
For ~~any~~ every $z \in \mathbb{C}$, $\exists$ $(-z) \in \mathbb{C}$ such that,
$z_1+(-z_1) = (-z)+z = 0$
Where $z=(a,b) \Rightarrow (-z) = (-a,-b)$ and $(-z)$ is called as additive inverse of z.
Thus, $(\mathbb{C},+)$ is abelian group.

⑥ Multiplication is closed:
$\forall \ z_1, z_2 \in \mathbb{C}$ we have $z_1 \cdot z_2 \in \mathbb{C}$ i.e, the multiplicat of two complex numbers is again a complex numbers.

⑦ Multiplication is commutative:
$\forall \ z_1, z_2 \in \mathbb{C}$ we have $z_1 \cdot z_2 = z_2 \cdot z_1$ i.e, multiplication of complex number is commutative.

⑧ Multiplication is associative:

$\forall z_1, z_2, z_3 \in \mathbb{C}$ we have $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

Thus multiplication of complex number is associative

⑨ Existance of Multiplicative identity:

There exist an element $1 = 1 + 0i = (1, 0) \in \mathbb{C}$ in $\mathbb{C}$ so that we have $1 \cdot z = z \cdot 1 = z \;\; \forall z \in \mathbb{C}$.

Here '1' is called as multiplicative identity.

⑩ Existance of Multiplication inverse:

$\forall z \neq 0 \in \mathbb{C}$ there exist $z^{-1} \in \mathbb{C}$ such that $z \cdot z^{-1} = z^{-1} \cdot z = 1$ where $z^{-1} = 1/z$ is called as multiplicative inverse of z.

Note that $z = x + iy = (x, y)$

$$\Rightarrow z^{-1} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$$

Thus, $(\mathbb{C} - \{0\}, \cdot)$ is multiplicative abelian group.

⑪ Distributive law:

Both the distributive laws holds in $\mathbb{C}$ i.e, $z_1, z_2, z_3 \in \mathbb{C}$.

We have

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$$
$$(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$$

Thus the set $\mathbb{C}$ of complex numbers is a field with respect to the operations addition & multiplication of complex numbers.

① If $z = x + iy$ is any complex numbers then, x is called as real part of z and y is called as imaginary part of z.
i.e, $x = \text{Re}\, z$ and $y = \text{Im}\, z$
Thus $z = \text{Re}\, z + i\, \text{Im}\, z$.

② If $y = 0$ then, $z = x$ is called as purely real number. Thus, every real number is a complex number whose imaginary part is zero.

③ If $x = 0$ then, $z = iy$ is called as purely imaginary num

④ We denote the complex number $z = x + iy$ as an ordered pair (x, y) hence a purely real number can be denot by $(x, 0)$ and a purely imaginary number can be denoted by $(0, y)$

⑤ We know that $\mathbb{C} = \{(a+ib) \mid a, b \in \mathbb{R}\}$. Also we have

$$\mathbb{R}\times\mathbb{R} = \{(a, b) \mid a, b \in \mathbb{R}\}$$

Thus, there exist a one to one correspondence betn the set $\mathbb{R}\times\mathbb{R}$ & $\mathbb{C}$ i.e, For every complex number $z = x + iy$ in $\mathbb{C}$ there exist an order pair (x, y) in $\mathbb{R}\times\mathbb{R}$ and conversely

⑥ We know that purely real number is denoted by orde pair $(x, 0)$, $\forall\, x \in \mathbb{R}$.

$$\mathbb{R}\times\{0\} = \{(x, 0) \mid x \in \mathbb{R}\}$$

Thus, there exist a one-to one correspondence betn the sets $\mathbb{R}$ & $\mathbb{R}\times\{0\}$ which is defined as
$x \to (x,0)$.

⑦ The field of complex numbers $\mathbb{C}$ has an advantage over the field of real numbers that the every polynomial of any degree has all roots in $\mathbb{C}$

⑧ But $\mathbb{C}$ has a disadvantage over $\mathbb{R}$ i.e, ordered relation does not follow in $\mathbb{C}$ while it holds in $\mathbb{R}$ i.e, for any two real numbers a and b we have ~~a+i~~ either $a<b$ or $a>b$ or $a=b$.
But these does not hold for z_1 & z_2.

Rectangular representation of Complex Numbers:

We know that every real number can be represented by a point on the real line similarly every complex number $z = x+iy$ can be represented by point in the plane $P(x,y)$.
Thus, every complex number correspondence to one & only one point in cartesian plane & conversely.

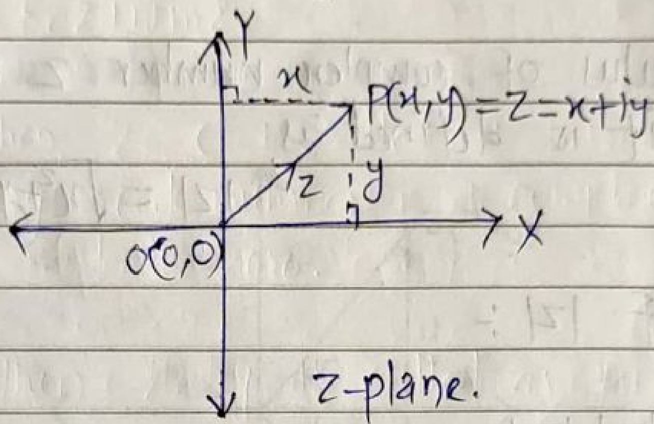
Purely real numbers $(x,0)$ are represented by point on the x-axis hence x-axis is called as real axis. Similarly y-axis is called as imaginary axis. And the cartesian plane is called as complex plane or z-plane.

$O=(0,0)$ is the 0 element of $\mathbb{C}$ which represents

| All axes written in capital letters. |

Page No.
Date

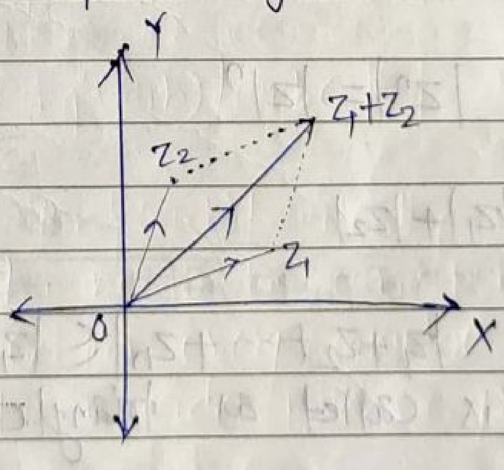
origin in z-plane.
There is one more interpretation for complex numbers. i.e, every complex number can be represented by a two dimensional vectors from origin (0,0) to the point (x,y) as shown in fig.



z-plane.

• Note:

① As every complex number can be denoted by a two dimensional vectors therefore addition of two complex number follows a parallelogram law as shown in fig.



② The magnitude of the vector (x,y) = z is the distance of the point (x,y) from origin. which is called as modulus of complex number of z.

Modulus, Arguments and Conjugate of Complex Number:

Definition: Modulus of Complex Numbers:

If $z = x+iy$, then the magnitude of vector z is called as modulus of complex number z.

The modulus of complex number z is denoted by $|z|$ and it is defined as,

$$|z| = \sqrt{x^2+y^2}$$

- Properties of $|z|$:

① $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

in general $|z_1 \cdot z_2 \cdots z_n| = |z_1| |z_2| \cdots |z_n|$

② $|z^2| = |z|^2$

in general $|z^n| = |z|^n$

③ $|z_1+z_2| \leq |z_1|+|z_2|$

in general $|z_1+z_2+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|$

This is called as triangle inequality.

④ $|z_1-z_2| \geq ||z_1|-|z_2||$

i.e, $||z_1|-|z_2|| \leq |z_1-z_2|$

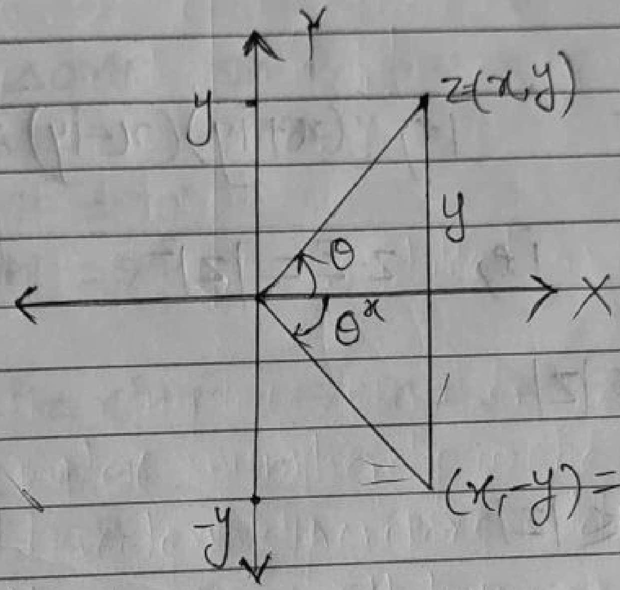
⑤ $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

Conjugate of z (complex Number) :-

If $z = x+iy = (x, y)$ then the conjugate of z is denoted $\bar{z}$ and it is defined as,

$$\bar{z} = x - iy = (x, -y)$$

- Thus, $\bar{z}$ is reflection of point z about x-axis.



• Properties of $\bar{z}$:

① $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

In general $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$

② $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

In general $\overline{z_1 \cdot z_2 \cdot z_3 \cdots z_n} = \bar{z}_1 \cdot \bar{z}_2 \cdot \bar{z}_3 \cdots \bar{z}_n$

③ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

④ $z+\bar{z} = 2x$ i.e, $x = \mathrm{Re}\, z = \frac{z+\bar{z}}{2}$

⑤ $z-\bar{z} = 2iy$ i.e, $y = \mathrm{Im}\, z = \frac{z-\bar{z}}{2}$

⑥ $|\bar{z}| = |z| = \sqrt{x^2+y^2}$

i.e, $\bar{z} = x-iy$

⑦ $z \cdot \bar{z} = x^2+y^2$ i.e, $(x+iy)(x-iy) = x^2+y^2$

i.e, $z \cdot \bar{z} = |z|^2$

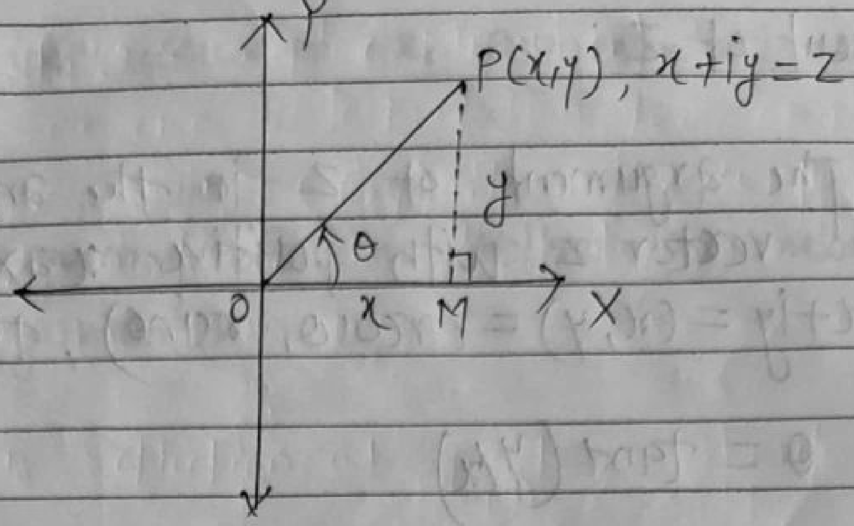
Note: (I) $|x| = |\mathrm{Re}\, z| \leq |z|$

(II) $|y| = |\mathrm{Im}\, z| \leq |z|$

- Polar Representation of Complex Numbers:

We know that, every complex number z can be represented by a two dimensional vector. Suppose, the magnitude of complex number $z = x+iy$ is $|z| = r = \sqrt{x^2+y^2}$.

Suppose θ is the angle made by the vector z with positive real axis, then the polar co-ordinates of point $P(x,y)$ are (r,θ) as shown in figure



Here, in ΔOMP of
$OM = x$, $OP = r$ $PM = y$
and $\angle OMP = 90^\circ$
$\angle POM = \theta$

$\therefore OP^2 = r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$

and $\cos\theta = \dfrac{OM}{OP} = \dfrac{x}{r}$

$\Rightarrow x = r\cos\theta$

Also, $\sin\theta = \dfrac{PM}{OP} = \dfrac{y}{r}$

$\therefore y = r\sin\theta$

$\therefore z = x+iy$
$= r\cos\theta + ir\sin\theta$
$= r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$

& here $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x}$

$\therefore \theta = \tan^{-1}(y/x)$

- Argument of z:

The argument of z is the angle of made by the vector z with positive x-axis. i.e, if $z = x+iy = (x, y) = (r\cos\theta, r\sin\theta)$, then

$$\theta = \tan^{-1}(y/x)$$

is called as argument of z.

- The argument of z is denoted by arg z

$$\arg z = \theta = \tan^{-1}(y/x)$$

We know that

$$\cos(2\pi+\theta) = \cos\theta$$
$$\sin(2\pi+\theta) = \sin\theta$$

If $z = r(\cos\theta + i\sin\theta) \Rightarrow \theta = \arg z$
then

$$z = r(\cos(2\pi+\theta) + i\sin(2\pi+\theta)) \Rightarrow \arg z = 2\pi+\theta$$

In general, if $z = x+iy$ & $\arg z = \theta$, then

$$\arg z = \theta + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

hence for every complex number z, arg z has infinite number of values.

Note:- The value of arg z is unique for $-\pi < \arg z \leq \pi$ (or $-\pi \leq \arg z < \pi$)
This unique value of arg z is called as principle arguments

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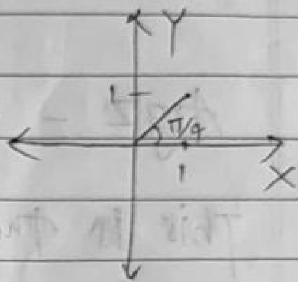
If $x=y$ then angle is $\pi/4$

Page No.
Date

- The principle value of the argument is denoted by 'Arg z'

Q. Find the argument of z for following numbers. Also find principle arguments.

① $z = 1+i$



Solution: Given $z = 1+i = x+iy$

$\therefore x=1 \quad y=1$

$\arg z = \theta = \tan^{-1}(y/x) = \tan^{-1}(1/1) = \tan^{-1}(1) = \pi/4$

$\arg z = \pi/4,\ 2\pi+\pi/4,\ 4\pi+\pi/4, \ldots$

$\arg z = \pi/4 + 2k\pi \quad k = 0, \pm1, \pm2, \ldots$

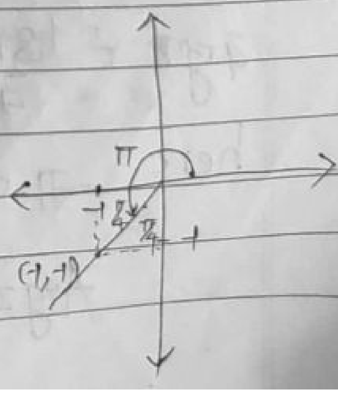
$\operatorname{Arg} z = \pi/4.$

② $z = -1-i$

Solution: Given $z = -1-i = x+iy = (-1,-1)$

$\therefore$ Clearly the complex number z lies in 3rd quadrant

$\therefore \arg z = \pi + \arg(x+iy)$



$\therefore \arg(-1-i) = \pi + \arg(1+i)$

$= \pi + \tan^{-1}(1/1)$

$= \pi + \pi/4 = 5\pi/4$

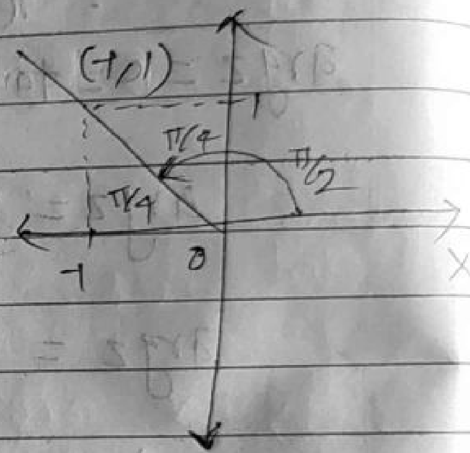
$\arg z = \frac{5\pi}{4} + 2k\pi \quad k = 0, \pm 1, \pm 2, \ldots$

$\therefore \arg z = \frac{5\pi}{4}, \frac{5\pi}{4} - 2\pi, \frac{5\pi}{4} + 2\pi \ldots$

$= \frac{5\pi}{4}, \frac{-3\pi}{4}, \frac{13\pi}{4}$

$\text{Arg } z = \frac{-3\pi}{4} \quad \because -\pi < \frac{-3\pi}{4} \leq \pi$

This is the principle arguments



③ $z = -1 + i$

Solution: Given $z = -1 + i = (-1, 1)$

Clearly z lie in 2nd quadrant.

$\arg z = \frac{\pi}{2} + \arg(|x| + i|y|)$

$= \frac{\pi}{2} + \arg(1 + i)$

$= \frac{\pi}{2} + \frac{\pi}{4}$

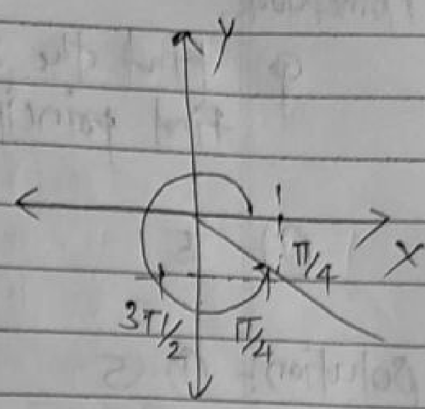
$\arg z = \frac{3\pi}{4} + 2k\pi \quad k = 0, \pm 1, \pm 2, \ldots$

$\arg z = \frac{3\pi}{4}, \frac{-5\pi}{4}, \frac{11\pi}{4}, \ldots$

here $-\pi < \frac{3\pi}{4} \leq \pi$

$\text{Arg } z = \frac{3\pi}{4}$

(4) $z = 1-i$



Solution:- $z = 1-i = (1,-1)$

Here z lies in the 4th quadrant.

$\therefore \ Arg z = \frac{3\pi}{2} + arg(|x| + i|y|)$

(OR)

$arg z = -arg(|x| + i|y|)$

$\therefore$ Here $Arg z = \frac{3\pi}{2} + arg(1+i)$

$= \frac{3\pi}{2} + \frac{\pi}{4}$

$= \frac{7\pi}{4}$

$arg z = \frac{7\pi}{4}$

$\therefore \ arg z = \frac{7\pi}{4} + 2K\pi \quad K = 0, \pm 1, \pm 2, \ldots$

here $arg z = -arg(1+i)$

$= -\pi/4$

$= \frac{-\pi}{4} + 2K\pi \quad K = 0, \pm 1, \pm 2, \ldots$

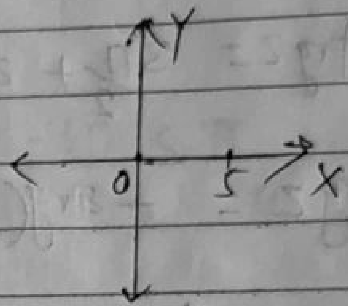
$\therefore \ Arg z = -\pi/4.$

Homework:

Q. Find the arguments of following complex numbers. Also find principle arguments.

① 5 ② 2i ③ −7 ④ −3i ⑤ 0

Solution:- ① 5



$z = 5 + 0i = (5, 0)$

Here z lie on +ve x-axis.

$\therefore \arg z = \tan^{-1}(y/x) = \tan^{-1}(0/5) = \tan^{-1}(0)$

$\arg z = 0°$

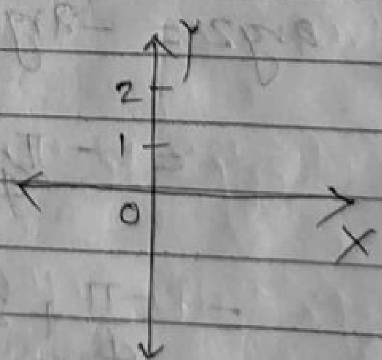
$\arg z = 0 + 2k\pi \quad , \; k = 0, \pm 1, \ldots$

$\arg z = 0, -2\pi, 2\pi$

$\therefore \operatorname{Arg} z = 0$

This is principle argument of z.

② 2i



$z = 0 + 2i = (0, 2)$

Here z lie on +ve Y-axis
ie, imaginary axis.

$\arg z = \tan^{-1}(y/x) = \tan^{-1}(2/0) = \tan^{-1}(\infty)$

$\arg z = \pi/2$

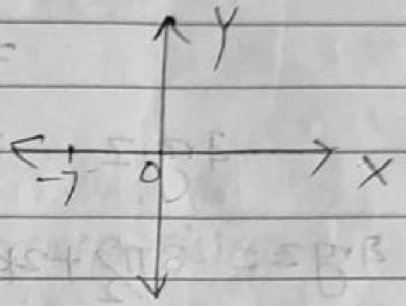
$\arg z = \frac{\pi}{2} + 2k\pi \qquad k = 0, \pm 1, \ldots$

$\arg z = \frac{\pi}{2}, \ -\frac{3\pi}{2}, \ \frac{5\pi}{2}$

$\therefore \ \operatorname{Arg} z = \frac{\pi}{2} \quad \left(\cancel{-\frac{3\pi}{2}}\right)$

This is principle argument of z.

(3) -7



$z = -7 + 0j = (-7, 0)$

Here z lie on -ve x-axis

$\therefore \ \arg z = \tan^{-1}(y/x) = \tan^{-1}\left(\frac{0}{-7}\right)$

$= \tan^{-1}(0)$ $\quad$ ∵ lie on 3rd quadrant

$\therefore \ \arg z = \pi$

$\therefore \ \arg z = \pi + 2k\pi \quad ; \ k = 0, \pm 1, \ldots$

$\arg z = \pi, \ -\pi, \ 3\pi, \ldots$

$\therefore \ \operatorname{Arg} z = \pi$

This is the argument of z.

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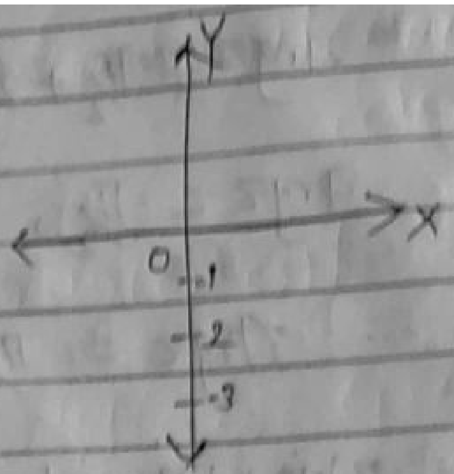
④ $-3i$



$z = 0 + (-3i) = (0, -3)$

Here z lies on -ve Y axis
i.e, imaginary axis

$\therefore \arg z = \tan^{-1}(y/x) = \tan^{-1}(-3/0)$

$= \tan^{-1}(\infty)$

$\arg z = 3\pi/2$ $\because$ lie on 3rd quadrant

$\arg z = 3\pi/2 + 2k\pi \quad k = 0, \pm 1, \ldots$

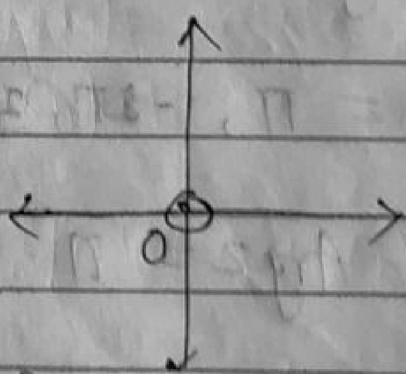
$\arg z = 3\pi/2, -\pi/2, 7\pi/2$

$\operatorname{Arg} z = 3\pi/2, -\pi/2$

This is principle argument of z.

⑤ 0

$z = 0 + 0i = (0, 0)$



Here z lie on origin O

Argument of 0 is not defined.

Note: ① If $z = x$ then z is purely real number

ⓐ If $x > 0$, then $\arg z = 0$
ⓑ If $x < 0$, then $\arg z = \pi$

② If $z = iy$ i.e, z is purely imaginary number

ⓐ If $y > 0$, then $\arg z = \pi/2$
ⓑ If $y < 0$, then $\arg z = -\pi/2$ or $3\pi/2$

• De-Moivre's Theorem:

If $z = x + iy$ then $z = r\cos\theta + ri\sin\theta$

$\therefore z = r(\cos\theta + i\sin\theta)$

where $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}(y/x)$

consider two complex numbers z_1 and z_2
i.e, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_1 \cdot z_2 = (r_1(\cos\theta_1 + i\sin\theta_1)) \cdot (r_2(\cos\theta_2 + i\sin\theta_2))$
$= r_1 r_2 [(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)]$

$= r_1 r_2 [\cos\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \, i\sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i^2 \sin\theta_1 \cdot \sin\theta_2]$

$= r_1 r_2 [\cos\theta_1 \cdot \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2]$

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$$z_1 \cdot z_2 = r_1 r_2 [(\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2) + i(\sin\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \cdot \sin\theta_2)]$$

$\because i^2 = -1$

$$z_1 \cdot z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Here, $|z_1 \cdot z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$

but we know that,

$$r_1 = |z_1|, \; r_2 = |z_2| \quad \theta_1 = \arg z \quad \theta_2 = \arg z_2$$

hence

$$|z_1 \cdot z_2| = r_1 r_2 = |z_1| \cdot |z_2|$$

& $\arg(z_1 \cdot z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$

Thus, if we multiply two complex numbers, then their moduli are multiplied and arguments are added.

In general,

if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore z_n = r_n(\cos\theta_n + i\sin\theta_n)$

then,

$$z_1 \cdot z_2 \cdots z_n = r_1 r_2 \cdots r_n [\cos(\theta_1 + \theta_2 + \cdots \theta_n) + i\sin(\theta_1 + \theta_2 \cdots + \theta_n)] \quad \cdots (1)$$

Now, put $z_1 = z_2 = \cdots = z_n = z$ where, $z = r(\cos\theta + i\sin\theta)$

then by eqⁿ (1) we get,

$\therefore z \cdot z \cdots z$ (n times) $= (r \cdot r \cdots r)[\cos(\theta+\theta+\cdots+\theta) + i\sin(\theta+\theta\cdots+\theta)]$
(n times)

Hence, we have

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$\therefore$ we have,

$$[r(\cos\theta + i\sin\theta)]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$r^n[\cos\theta + i\sin\theta]^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

This result is known as De-Moivre's theorem.

Note: $e^{i\theta} = \cos\theta + i\sin\theta$
$\therefore e^{in\theta} = \cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

- Point set in the plane:
- Neighbourhood of any point:

We know that neighbourhood of real number x_0 is an interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$ where ϵ is any positive real number.

Thus, we can define the ϵ-neighbourhood of x_0 i.e, $N_\epsilon(x_0)$

$$= \{x \in R \mid |x - x_0| < \epsilon\}.$$

But in the set of complex numbers, the nbd of complex number $z_0 = (x_0, y_0)$ is define in different ways:

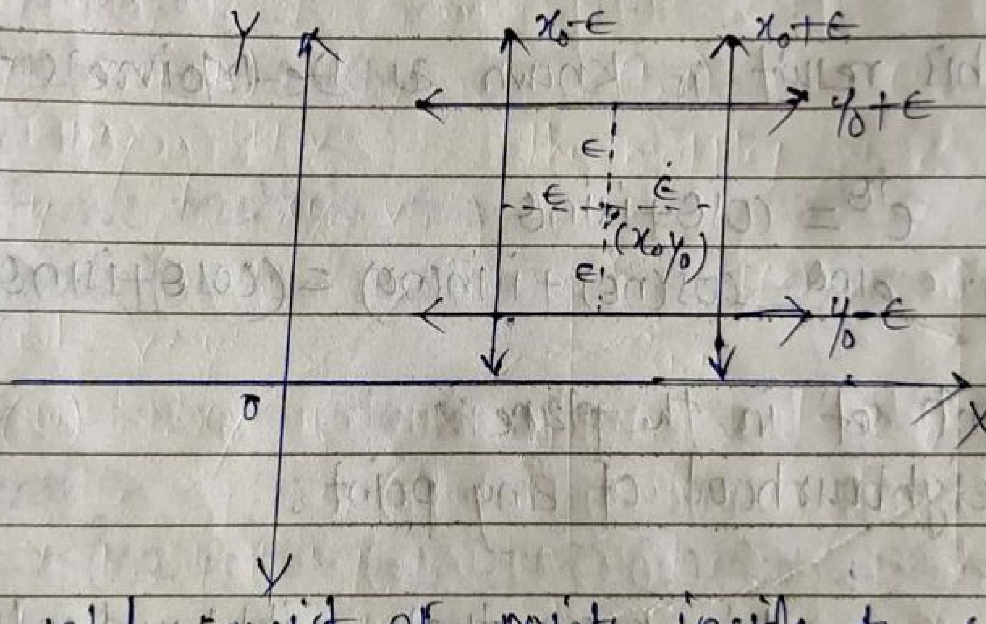
① Square ϵ-nbd of z_0
② Circular ϵ nbd of z_0

① Square ϵ-nbd of z_0

A square ϵ-nbd of z_0 is define as

$N_\epsilon(z_0) = \{ z = (x, y) \in \mathbb{C} \mid |x - x_0| < \epsilon \ \& \ |y - y_0| < \epsilon \}$

This nbd is shown in following figure:



This nbd consist of points inside the square centred at (x_0, y_0) whose sides are parallel to co-ordinate axes.

The side length of this square is 2ϵ

② Circular ϵ-nbd of z_0:

The circular ϵ-nbd of $z_0 = (x_0, y_0)$ is the set of points (x, y) whose distance from $z_0 < \epsilon$

i.e, $N_\epsilon(z_0) = \{ z=(x,y) \in \mathbb{C} \mid |z-z_0| < \epsilon \}$

$= \{ z \in \mathbb{C} \mid \sqrt{(x-x_0)^2+(y-y_0)^2} < \epsilon \}$

$= \{ z \in \mathbb{C} \mid (x-x_0)^2+(y-y_0)^2 < \epsilon^2 \}$

This nbd of z_0 represents a circle centred at z_0 with radius ϵ.

Def^n (Deleted ϵ-nbd of z_0):

The deleted ϵ-nbd of z_0 is denoted by $N'(z_0,\epsilon)$ and it is define as

$$N'(z_0,\epsilon) = \{ z \in \mathbb{C} \mid 0 < |z-z_0| < \epsilon \}$$

Here the point z_0 is punctured out from the nbd $N_\epsilon(z_0)$

Def^n (Domain):

An open connected sets is called as domain.

Def^n (Region):

Region is a domain with some, none or all of its boundary points

Note:- Any two point in a domain can be join by a polygonal line which lie in the domain.