Unit 1

Complex Variables

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* Introduction -

As the system of real numbers is not sufficient for all mathematical needs.
Ex. there is no real number which satisfies $x^2+1=0$, $x^2+9=0$.
So Euler for the first time introduced the symbol i with the property $i^2=-1$ and then Gauss introduced a number in the form $x+iy$ with $i=\sqrt{-1}$ and α, β are real ~~ann~~ numbers, is known as complex number.

* Definition

An ordered pair of real numbers such as (x, y) is termed as a complex number.
If we write $z=(x,y)$ or $z=x+iy$ where $i=\sqrt{-1}$ then
x is called the real part
& y is called the imaginary part of the complex number(z) and is denoted by,

$x = R_z$ or $R(z)$ or $Re(z)$.
$y = I_z$ or $I(z)$ or $Im(z)$.

A complex number $x+iy$ is denoted by z
ie $z = x+iy$.

* **Equality of complex numbers** (equal complex numbers)

The two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be equal if and only if $x_2 = x_1$ and $y_2 = y_1$.

* **Complex Algebra** (Addition, Subtraction, Multiplication, Division, complex conjugate).

1) Addition of complex numbers.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.
Then addition of given two complex numbers is given by,

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$= (x_1 + x_2) + i(y_1 + y_2).$$

ie In addition of complex numbers, the real parts added with real part and imaginary parts added with imaginary parts.

Ex 1) $z_1 = 7 + 2i$ $\quad z_2 = 3 + 4i$

$$\therefore z_1 + z_2 = 7 + 2i + 3 + 4i$$
$$= 10 + 6i$$

2) $z_1 = 3 - 2i$ $\quad z_2 = 3 + 5i$

$$z_1 + z_2 = 3 - 2i + 3 + 5i$$
$$= 6 + 3i$$

$\boxed{z_1 = 3 + (-2)i \text{ in } x + iy \text{ form}}$

$i^2 = -1$

2) Subtraction of complex numbers.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers and their subtraction is given by,

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$
$$= x_1 + iy_1 - x_2 - iy_2$$
$$= (x_1 - x_2) + i(y_1 - y_2)$$

ie. In subtraction, we subtract real parts from real parts and imaginary parts from imaginary parts.

or

$$z_1 - z_2 = z_1 + (-z_2)$$
$$= x_1 + iy_1 - (x_2 + iy_2)$$
$$= (x_1 - x_2) + i(y_1 - y_2).$$

$z_2 = x_2 + iy_2$
$-z_2 = -(x_2 + iy_2)$

3) Multiplication of complex numbers.

Let $z_1 = x_1 + iy_2$ and $z_2 = x_2 + iy_2$

Then
$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$
$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$
$$= x_1x_2 + i(x_1y_2 + y_1x_2) - y_1y_2$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Ex $z_1 = 7 + 2i$ $\quad z_2 = 3 + 4i$

$$z_1 \cdot z_2 = (7 + 2i)(3 + 4i)$$
$$= 7(3 + 4i) + 2i(3 + 4i)$$
$$= 21 + 28i + 6i + 8i^2$$
$$= 21 + 34i - 8$$
$$z_1 z_2 = 13 + 34i$$

$\because i^2 = -1$

4) Division of complex numbers

$z_1 = x_1 + iy_1$ $\quad z_2 = x_2 + iy_2$

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2}.$$

To simplify the division, we ~~do~~ divide numerator and denominator by the conjugate of denominator.

$$\therefore \frac{z_1}{z_2} = \frac{(x_1+iy_1)}{(x_2+iy_2)} \times \frac{(x_2-iy_2)}{(x_2-iy_2)}$$

$$= \frac{(x_1+iy_1)\cdot(x_2-iy_2)}{(x_2)^2-(iy_2)^2}$$

$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 - i^2y_1y_2}{x_2^2 - i^2y_2^2}$$

$$= \frac{x_1x_2 - ix_1y_2 + iy_1x_2 + y_1y_2}{x_2^2+y_2^2} \quad (\because i^2=-1)$$

$$= \frac{(x_1x_2+y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2+y_2^2}.$$

$$= \frac{(x_1x_2+y_1y_2)}{(x_2^2+y_2^2)} + i\frac{(y_1x_2-x_1y_2)}{(x_2^2+y_2^2)}$$

Ex. $z_1 = 7+2i$ $\quad z_2 = 3+4i$

$$\frac{z_1}{z_2} = \frac{7+2i}{3+4i}$$

$$= \frac{7+2i}{3+4i} \times \frac{3-4i}{3-4i}$$

$$= \frac{(7+2i)(3-4i)}{(3)^2-(4i)^2}$$

$$= \frac{21-28i+6i-8i^2}{9+16}$$

$$= \frac{21-22i+8}{25} = \frac{29}{25} - \frac{22}{25}i$$

5) Conjugate of complex number.

Two complex numbers which differ only in the sign of imaginary parts are called as conjugate of each other.

~~A pair of $z = x+iy$~~ &

A pair of complex numbers $x+iy$ & $x-iy$ are said to be conjugate of each other.

If z is a complex number $z = x+iy$ and its complex conjugate is given by $\bar{z} = x-iy$.

* Properties of addition of complex numbers

1) Addition is commutative

ie $z_1+z_2 = z_2+z_1$

2) Addition is associative

$z_1+(z_2+z_3) = (z_1+z_2)+z_3$.

* Properties of multiplication

1) $z_1z_2 = z_2z_1$ multiplication is commutative

2) $z_1(z_2z_3) = (z_1z_2)z_3$ multiplication is associative

3) $z_1(z_2+z_3) = z_1z_2+z_1z_3$ multiplication is distributive.

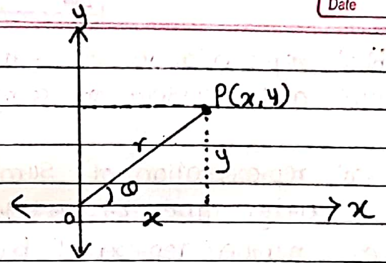
* Argand Diagram

Mathematician Argand represented a complex numbers in a diagram known as Argand diagram.

A complex number $x+iy$ can be represented by a point P whose coordinates are (x,y)

The x axis is called as real axis and the axis of y is known as imaginary axis.

The distance OP is known as modulus and the angle OP makes with x axis is argument of $z=x+iy$ complex number.



$OP = r$ = modulus of complex no.

θ = argument of complex no.

From the above Argand diagram

$\sin\theta = \frac{y}{r}$ & $\cos\theta = \frac{x}{r}$

$\Rightarrow y = r\sin\theta$ $\quad x = r\cos\theta$

$\therefore$ the complex number $z = x+iy$ becomes

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$ which is known as the polar form of complex number

as $y = r\sin\theta$ and $x = r\cos\theta$, to find out r.

let $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta+\sin^2\theta)$

$= r^2$ $\quad \because \cos^2\theta+\sin^2\theta = 1$

$\Rightarrow r = \sqrt{x^2+y^2}$

which is known as modulus of a complex no and is denoted by

$|z| = r = \sqrt{x^2+y^2}$.

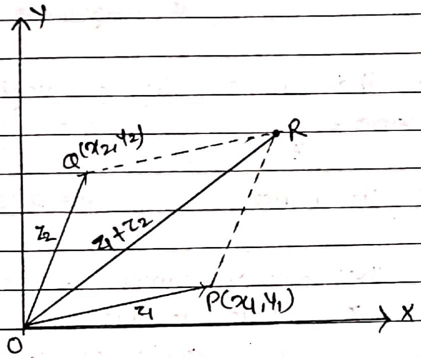
also from $y = r\sin\theta$ $\quad x = r\cos\theta$

$\tan\theta = y/x$ $\quad \theta = \tan^{-1}(y/x)$ which is

* Graphical representation of sum, difference, product and quotient of complex numbers.

1) Graphical representation of sum.

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ be two complex number represented by P and Q points on Argand diagram.



complete the parallelogram OPRQ

∴ The sum z_1+z_2 corresponds to a vector whose components are x_1+x_2 and y_1+y_2.

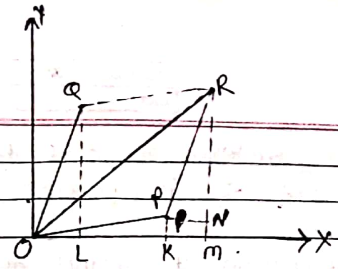
from diagram $\vec{OP} = z_1$ & $\vec{OQ} = z_2$

then $\vec{OR} = \vec{OP} + \vec{PR} = \vec{OQ} + \vec{QR} = z_1+z_2$

∴ the coordinates of R are (x_1+x_2, y_1+y_2)

Draw PK, RM, QL ||lars on OX.

also draw PN ⊥ RM.



from above diagram $OM = OK + KM = OK + KL$
$= x_1+x_2$

also, $RM = MN+NR = KP+LQ = y_1+y_2$

∴ coordinates of $R \equiv (x_1+x_2, y_1+y_2)$

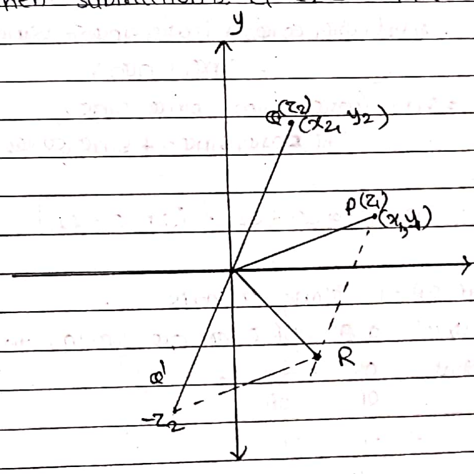
and it represents sum of complex number.

2) Graphical representation of subtraction

Let P and Q represent two numbers in diagram

$z_1 = x_1+iy_1$ & $z_2 = x_2+iy_2$

Then subtraction is $z_1 - z_2 = z_1 + (-z_2)$



$z_1 - z_2$ means addition of z_1 and $(-z_2)$

$-z_2$ is represented by OQ' formed by producing OQ to OQ' such that OQ = OQ'.

complete the parallelogram OPRQ' then sum of z_1 and $-z_2$ is represented by OR.

3). Graphical representation of product of complex numbers.

Let P & Q represents complex numbers

$z_1 = x_1 + iy_1$

$= r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2$

$= r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$

$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$ $\because i^2 = -1$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$

$= r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

Cut off OA = 1 along x-axis.

construct $\triangle ORQ$ similar to OAP.

so that $\frac{OR}{OP} = \frac{OQ}{OA}$

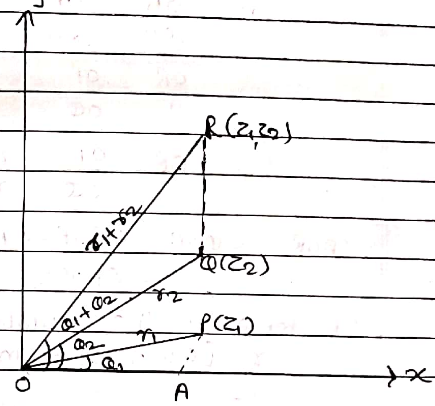
$\Rightarrow \frac{OR}{OP} = \frac{OQ}{1}$

$\Rightarrow OR = OP \cdot OQ = r_1 \cdot r_2$

Hence product of two complex no. is represented by point R.

such that $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

& $\mathrm{Arg}(z_1 \cdot z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$



4) Graphical representation of division of complex numbers.

Let P and Q represents the complex no.s

$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 - i\sin\theta_2)}$

$= \frac{r_1 r_2}{r_2 r_2} \frac{[(\cos\theta_1\cos\theta_2 + i\sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]}{(\cos^2\theta_2 + \sin^2\theta_2)}$

$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$.

cut off $OA = 1$ construct ΔOAR on OA similar to ΔOQP.

so that $\frac{OR}{OA} = \frac{OP}{OQ}$

$\frac{OR}{1} = \frac{OP}{OQ}$

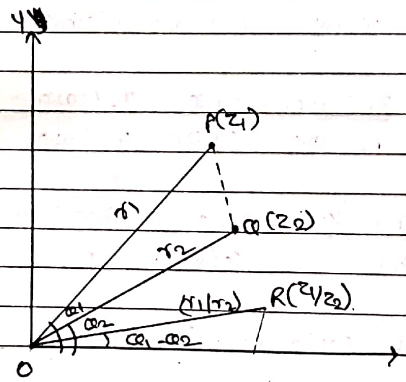
$OR = \frac{OP}{OQ} = \frac{r_1}{r_2}$

$\angle AOR = \angle QOP = \angle AOP - \angle AOQ$
$= \theta_1 - \theta_2$

$\therefore R$ represents the number
$\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

Hence the complex no. $\frac{z_1}{z_2}$ is represented by point R.

(i) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ & (ii) $Arg\left(\frac{z_1}{z_2}\right) = Arg(z_1) - Arg(z_2)$

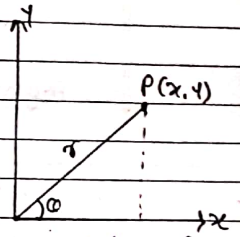


* Properties of moduli, arguments and geometry of complex numbers.

Modulus and Argument.

Let $z = x + iy$ be a complex number.

$x = r\cos\theta$ — ①
$y = r\sin\theta$ — ②
$\therefore x^2 + y^2 = r^2$
$\therefore r = \sqrt{x^2+y^2}$



Then r is called as the modulus of or absolute value of the complex number $x+iy$, and is denoted by $|z|$ or $|x+iy|$

from ① and ②: $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\Rightarrow \frac{y}{x} = \tan\theta$

$\theta = \tan^{-1}(y/x)$

or from diagram $\tan\theta = \frac{y}{x}$

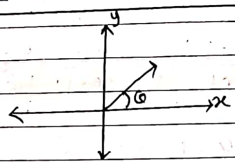
$\theta = \tan^{-1}(y/x)$

The angle θ is called as argument or amplitude of the complex number $x+iy$ and is denoted by $\arg(x+iy)$ or $\arg(z)$. The values of θ lying in the range $-\pi < \theta \le \pi$ is called as principal value of the argument.

The principal value of θ is written either between 0 to π or between 0 to $-\pi$.

$z = 2+2i$
$x+iy$

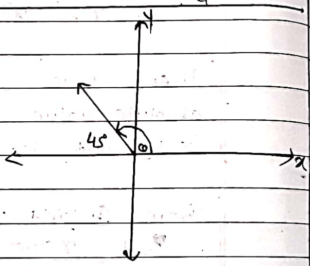
$|z| = \sqrt{(2)^2+(2)^2}$
$= \sqrt{4+4}$
$= \sqrt{8}$



Argument, $\theta = \tan^{-1}(y/x) = \tan^{-1}(2/2) = \tan^{-1}(1)$
$\Rightarrow \theta = 45^\circ = \frac{\pi}{4}$ radians

$z = 2-2i$

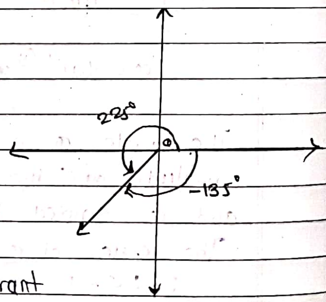
$|z| = \sqrt{(2)^2+(-2)^2}$
$= \sqrt{4+4}$
$= \sqrt{8}$



~~Arg~~ $\theta = \tan^{-1}(y/x)$
$= \tan^{-1}(-2/2)$
$= \tan^{-1}(-1)$
$\theta = 180 - 45^\circ = 135^\circ$

$z = -2-2i$

$|z| = \sqrt{(-2)^2+(-2)^2}$
$= \sqrt{4+4}$
$= \sqrt{8}$



$\theta = \tan^{-1}(y/x)$
$= \tan^{-1}(-2/-2)$
$= \tan^{-1}(1)$
$= 45^\circ$

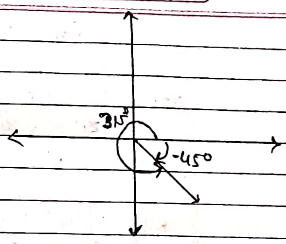
but in third quadrant
$\therefore \theta = 180+45 = 225^\circ$ in anticlockwise dir^n
Or ~~$\theta = -180-45 = -225^\circ$~~
$45-180^\circ = -135^\circ$ in clockwise dir^n

$z = -2+2i$

$|z| = \sqrt{(-2)^2+(2)^2}$
$= \sqrt{4+4} = \sqrt{8}$

$\theta = \tan^{-1}(2/-2)$
$= \tan^{-1}(-1)$
$= -45^\circ$

or $\theta = 360-45$
$= 315^\circ$



Properties of moduli and argument.

① If z_1 and z_2 are any two complex numbers prove that
$|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left[|z_1|^2+|z_2|^2\right]$

Let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$
be two complex numbers.
then $z_1+z_2 = (x_1+x_2)+i(y_1+y_2)$
$|z_1+z_2| = \sqrt{(x_1+x_2)^2+(y_1+y_2)^2}$
$\therefore |z_1+z_2|^2 = (x_1+x_2)^2+(y_1+y_2)^2$

also $z_1-z_2 = (x_1+iy_1)-(x_2+iy_2)$
$= (x_1-x_2)+i(y_1-y_2)$
$|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
$|z_1-z_2|^2 = (x_1-x_2)^2+(y_1-y_2)^2$

LHS $= |z_1+z_2|^2+|z_1-z_2|^2$
$= (x_1+x_2)^2+(y_1+y_2)^2+(x_1-x_2)^2+(y_1-y_2)^2$
$= x_1^2+2x_1x_2+x_2^2+y_1^2+y_2^2+2y_1y_2+x_1^2+x_2^2$
$-2x_1x_2+y_1^2+y_2^2-2y_1y_2$

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$= 2(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ — ①

Now to solve RHS.

$|z_1| = \sqrt{x_1^2+y_1^2}$

and $|z_2| = \sqrt{x_2^2+y_2^2}$

$RHS = 2[|z_1|^2 + |z_2|^2]$

$= 2[(x_1^2+y_1^2) + (x_2^2+y_2^2)]$

$= 2(x_1^2 + x_2^2 + y_1^2 + y_2^2)$ — ②

from ① & ② LHS = RHS.

$\therefore |z_1+z_2|^2 + |z_1-z_2|^2 = 2[|z_1|^2+|z_2|^2]$

② If z_1 and z_2 are two complex numbers such that $|z_1+z_2| = |z_1-z_2|$ then prove that $\arg z_1 - \arg z_2 = \pi/2$.

→

$z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

To solve,

$|z_1+z_2| = |z_1-z_2|$

let $z_1+z_2 = x_1+iy_1 + x_2+iy_2$

$= (x_1+x_2) + i(y_1+y_2)$

$|z_1+z_2| = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$

and $z_1 - z_2 = (x_1+iy_1) - (x_2+iy_2)$

$= (x_1-x_2) + i(y_1-y_2)$

$|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$\therefore |z_1+z_2| = |z_1-z_2|$ becomes

$(x_1+x_2)^2 + (y_1+y_2)^2 = (x_1-x_2)^2 + (y_1-y_2)^2$

$\Rightarrow x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + y_2^2 + 2y_1y_2 = x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2$

$\Rightarrow 2x_1x_2 + 2y_1y_2 + 2x_1x_2 + 2y_1y_2 = 0$

$\Rightarrow 4x_1x_2 + 4y_1y_2 = 0$

$\Rightarrow x_1x_2 + y_1y_2 = 0$

Now $\arg z_1 - \arg z_2 = \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{y_2}{x_2}\right)$

$= \tan^{-1}\left[\frac{(y_1/x_1) - (y_2/x_2)}{1+\left(\frac{y_1}{x_1}\right)\cdot\left(\frac{y_2}{x_2}\right)}\right]$

$= \tan^{-1}\left[\frac{x_2y_1 - x_1y_2}{x_1x_2 + y_1y_2}\right]$

$= \tan^{-1}\left(\frac{x_2y_1 \; x_1y_2}{0}\right) = \tan^{-1}(\infty)$

$= \pi/2$.

$\boxed{\therefore \arg z_1 - \arg z_2 = \frac{\pi}{2}}$

③ $|z| = 0$ iff $z = 0$

let $z = 0 + 0i$ be a complex number

then $|z| = \sqrt{0^2+0^2} = 0$

$\therefore$ if $z = 0$ then $|z| = 0$

④ $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

$z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2 y_1y_2$

$= x_1x_2 + i(x_1y_2 + y_1x_2) - y_1y_2$

$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

$|z_1 \cdot z_2| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2}$

$= \sqrt{(x_1x_2)^2 - 2x_1x_2y_1y_2 + (y_1y_2)^2 + (x_1y_2)^2 + (y_1x_2)^2 + 2y_1x_1x_2y_2}$

$= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (y_1x_2)^2}$ — ①

also $|z_1| = \sqrt{x_1^2 + y_1^2}$

$|z_2| = \sqrt{x_2^2 + y_2^2}$

$|z_1| \cdot |z_2| = \sqrt{(x_1^2 + y_1^2)} \cdot \sqrt{(x_2^2 + y_2^2)}$

$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$

$= \sqrt{x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2}$

$= \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (y_1x_2)^2}$ — ②

from ① & ②

$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

⑤ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

LHS = $\left|\frac{z_1}{z_2}\right|$

as $\frac{z_1}{z_2} = \frac{(x_1x_2 + y_1y_2)}{(x_2^2 + y_2^2)} + i\frac{(y_1x_2 - x_1y_2)}{(x_2^2 + y_2^2)}$

$\left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(x_1x_2 + y_1y_2)^2}{(x_2^2 + y_2^2)^2} + \frac{(y_1x_2 - x_1y_2)^2}{(x_2^2 + y_2^2)^2}}$

$= \sqrt{\frac{(x_1x_2)^2 + 2x_1x_2y_1y_2 + (y_1y_2)^2 + (y_1x_2)^2 + (x_1y_2)^2 - 2x_1x_2y_1y_2}{(x_2^2 + y_2^2)^2}}$

$= \sqrt{\frac{(x_1x_2)^2 + (y_1y_2)^2 + (x_1y_2)^2 + (y_1x_2)^2}{(x_2^2 + y_2^2)^2}}$

$\frac{|z_1|}{|z_2|} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}}$

$= \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} \times \frac{\sqrt{x_2^2 - y_2^2}}{\sqrt{x_2^2 - y_2^2}}$

on solving we get $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

IMP

⑥ $|z_1 + z_2| \le |z_1| + |z_2|$

$z_1 = x_1 + iy_1$ $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$

$|z_1| = \sqrt{x_1^2 + y_1^2}$

$|z_2| = \sqrt{x_2^2 + y_2^2}$.

LHS = $|z_1 + z_2|$

$= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$

$= \sqrt{x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2}$

$= \sqrt{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2(x_1x_2 + y_1y_2)}$

Now $|z_1+z_2|^2 \leq (x_1^2+y_1^2) + (x_2^2+y_2^2) + 2(x_1x_2+y_1y_2)$

$\leq (x_1^2+y_1^2) + (x_2^2+y_2^2) + 2\sqrt{(x_1x_2+y_1y_2)^2}$

$\leq |z_1|^2 + |z_2|^2 + 2\sqrt{(x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2)}$

$\leq |z_1|^2 + |z_2|^2 + 2\sqrt{(x_1^2+y_1^2)+(x_2^2+y_2^2)}$

$\leq |z_1|^2 + |z_2|^2 + 2|z_1|.|z_2|$

$|z_1+z_2|^2 \leq [|z_1|+|z_2|]^2$

$\Rightarrow \quad |z_1+z_2| \leq |z_1|+|z_2|$. Hence proved

(4) $|z_1-z_2| \geq |z_1|-|z_2|$

let $|z_1| = |(z_1-z_2)+z_2|$

$\leq |z_1-z_2|+|z_2|$

$\therefore |z_1|-|z_2| \leq |z_1-z_2|$

$\Rightarrow \quad |z_1-z_2| \geq |z_1|-|z_2|$

* Properties of conjugate : (solve)

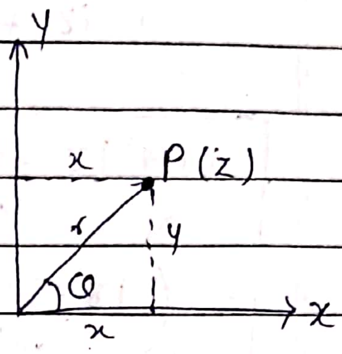
1. $\bar{\bar{z}} = z$
2. $(\overline{z_1+z_2}) = \bar{z}_1 \pm \bar{z}_2$
3. $(\overline{z_1z_2}) = \bar{z}_1 \cdot \bar{z}_2$
4.
5. $z\bar{z} = |z|^2$
6. $z+\bar{z} = 2\,Re(z)$
7. $z-\bar{z} = 2i\,Im(z)$
8. $\frac{1}{z} = \frac{|\bar{z}|}{|z|^2}$

(x,y) (r,θ)

* Rectangular, Polar and Exponential form of complex number.

$z = x + iy$ be a complex no.
and it is a rectangular form of complex no.

The point P represents a complex number in rectangular form.



Polar form

from the above Argand diagram

$x = r\cos\theta$ $y = r\sin\theta$

$\therefore$ $z = x + iy$ becomes

$= r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

which is known as the polar form of the complex number.

Exponential form,

as we know,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

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$z = \cos z + i\sin z$

$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)$

$= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \dots$

$= 1 + iz + \frac{i^2z^2}{2!} + \frac{i^3z^3}{3!} + \dots = e^{iz}$

$= 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \dots$

$= \left(1 - \frac{z^2}{2!} + \dots\right) + i\left(z - \frac{z^3}{3!} + \dots\right)$

$=$

$\therefore \boxed{e^{iz} = \cos z + i\sin z}$ is the exponential form of complex no.

* De-Moiver's thm.

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof →

$e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$

$e^{i\theta n} = (\cos\theta + i\sin\theta)^n$

$= \cancel{\cos n\theta}$

$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^n$

$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

hence proved.

Unit 2

Vector Analysis

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* Properties of argument:

1) The argument of product of two complex number is equal to sum of their argument.

$\arg(z_1 z_2) = \theta_1 + \theta_2$.

→

$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1 \cdot z_2 = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2[\cos\theta_1\cos\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2]$

$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2)]$

$z_1 \cdot z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$\arg(z_1 \cdot z_2) = \theta_1 + \theta_2$.

2) The argument of quotient of two complex number is equal to difference of their argument.

$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{r_2(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 - i\sin\theta_2)}$

$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\right]$

$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - \cos\theta_1\, i\sin\theta_2 - i^2\sin\theta_1\sin\theta_2}{(\cos\theta_2)^2 - (i\sin\theta_2)^2}\right]$$

$\Rightarrow 1$

$$= \frac{r_1}{r_2}\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$$

$\therefore \arg\left(\frac{z_1}{z_2}\right) = (\theta_1-\theta_2).$

* Properties of conjugate

1) $\bar{\bar{z}} = z$

let $z = x+iy$ —①

$\bar{z} = x-iy$

$\bar{\bar{z}} = x+iy$ —②

from ①&② $\bar{\bar{z}} = z$

2) $\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$

$z_1 = x_1+iy_1 \qquad z_2 = x_2+iy_2$

$z_1+z_2 = (x_1+x_2) + i(y_1+y_2)$

LHS $= \overline{(z_1+z_2)} = (x_1+x_2) - i(y_1+y_2)$

RHS. $= \bar{z}_1 + \bar{z}_2$

$= (x_1-iy_1) + (x_2-iy_2)$

$= (x_1+x_2) - i(y_1+y_2)$.

from ① & ②.

$\overline{(z_1+z_2)} = \bar{z}_1 + \bar{z}_2$.

2) $\cancel{z = -\bar{z}}$

$\cancel{z = x+iy}$

$\cancel{\bar{z} = x-iy}$

$\cancel{-\bar{z} = -(x-iy) = -x+iy}$

3) $z\bar{z} = |z|^2 = |\bar{z}|^2$

$z = x+iy$

$\bar{z} = x-iy$

$z\bar{z} = (x+iy)(x-iy)$

$= x^2 - ixy + ixy - i^2y^2$

$= x^2+y^2$ —①

$|z| = \sqrt{x^2+y^2}$ also $|\bar{z}| = \sqrt{x^2+y^2}$

$\Rightarrow |z|^2 = (x^2+y^2)$ $|\bar{z}|^2 = \sqrt{x^2+y^2}$ —②

from ① and ②,

$z\bar{z} = |z|^2 = |\bar{z}|^2$.

4) $z+\bar{z} = 2\,Re(z)$

$z = x+iy$ (x - real part, y - imaginary part)

$\bar{z} = x-iy$

$z+\bar{z} = x+iy+x-iy$

$= 2x$

$= 2\,Re(z)$.

5) $z - \bar{z} = 2i\,Im(z)$

$z = x+iy \qquad \bar{z} = x-iy$

LHS $= z-\bar{z} = (x+iy)-(x-iy) = x+iy-x+iy$

$= 2iy$.

$= 2i\,Im(z)$.

$=$ RHS