

WAVE

Introduction :-

We observed, that if we throw a stone on the surface of a lake, a disturbance is produced in the water and it travels uniformly in all directions in the form of ripples. Thus we see that a disturbance created at one point due to the throwing of a stone in the water travelled in all directions through the water which acted as a medium.

Similarly energy can be transferred in the form of wave over large distances, through material medium. Waves are of different kind such as waves on water, sound waves, light waves, radio waves etc.

These are the mechanical waves can be classified into two types :-

- i) Transverse waves
- ii) Longitudinal waves

i) Transverse Waves :-

The waves in which the particles of the medium vibrate in a direction, perpendicular to the direction of propagation of waves is called transverse waves.

Eg :- i) Waves on surface of water.

ii) Waves on string.

11) Longitudinal Waves:-

The waves in which the particles of the medium vibrate in a direction parallel to the direction of propagation of waves are called longitudinal waves.

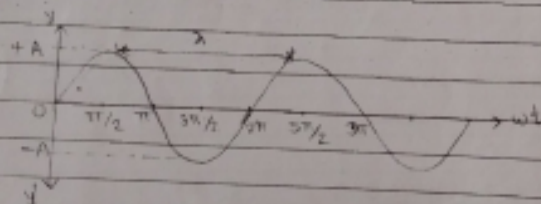
Eg:- 1) Sound waves in air.

Let's now define certain quantities, in terms of which waves motion is described.

Amplitude:-

The maximum displacement of any particle from its mean position is called amplitude (a) of the wave.

SI unit:- metre (m).



Period:-

The time taken (T) for any particle to complete one vibration is called period of the wave.

SI unit:- Second (s)

Frequency:-

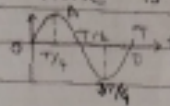
The number of vibrations per second by a particle is called frequency (n). or (ν)

$$\nu = 1/T$$

SI unit:- hertz (Hz)

phase :-

The state of vibration of a particle at any instant of time is called phase.

velocity :-

The distance travelled by the waves in one second is called velocity (v).

$$v = \frac{\text{displacement}}{\text{time}} = \frac{\lambda}{T}$$

$$v = n\lambda$$

Wavelength :-

The distance between two consecutive particles of the medium which are in the same phase is called wavelength (λ).

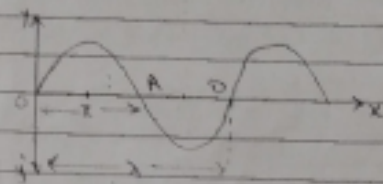
Simple harmonic progressive wave :-

The waves are continuously travel in a given direction. The particles of the medium start vibrating about their mean position. When these waves start travelling through the medium, if these vibrations are simple harmonic vibrations, the waves are called simple harmonic progressive waves.

if there is a phase difference b/w the vibrations of different particles of the medium through the all particles vibrate with the same amplitude and period.

Lets consider a simple harmonic progressive wave travelling in the positive direction of x -axis

let y represent the displacement of each particle and x represent its position in the medium.



Suppose that the particle at the origin is at the mean position. Then at any instant of time t , the displacement of the particle at the origin O is given by

$$y = a \sin \omega t \quad \dots \dots \textcircled{1}$$

Where,

' a ' is a amplitude and

' ω ' is a angular velocity of SHM

Now,

The consider a particle A situated at a distance x from O . This particle lags behind in phase, as compared to the particle at O . Therefore the displacement of the particle A, at time t is given by

$$y = a \sin (\omega t - \delta) \quad \dots \dots \textcircled{2}$$

Where, ' δ ' represent the phase lag of particle A with respect to O .

Since the path difference of λ corresponds to a phase difference of 2π radians.

a path difference of x will correspond to a phase difference of $\frac{2\pi x}{\lambda}$.

Thus $\phi = \frac{2\pi x}{\lambda}$

The displacement of the particle A is given by,

$$y = a \sin \left(\omega t - \frac{2\pi x}{\lambda} \right) \quad \text{--- (iii)}$$

If T is the period of the wave then,

$$\omega = \frac{2\pi}{T}$$

$$\therefore y = a \sin \left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda} \right)$$

$$\therefore y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \quad \text{--- (iv)}$$

The eqⁿ (iii) is the super progressive wave.

The eqⁿ (iv) is different from in below.

n is the frequency of wave. $n = \frac{1}{T}$ in using (iv) we get

$$y = a \sin 2\pi \left(nt - \frac{x}{\lambda} \right) \quad \text{--- (v)}$$

$$y = a \sin 2\pi n \left(t - \frac{x}{n\lambda} \right)$$

But $v = n\lambda$, the velocity of the wave

$$y = a \sin 2\pi n \left(t - \frac{x}{v} \right) \quad \text{--- (vi)}$$

$$(\because \omega = 2\pi n)$$

$$\therefore y = a \sin \omega \left(t - \frac{x}{v} \right) \quad \text{--- (VII)}$$

$$\therefore y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{--- (VIII)}$$

All the above equations can be represent a one-dimensional, simple harmonic, progressive wave, travelling in the direction of the positive axis.

If the wave is travelling in the direction of the negative x-axis then the eqⁿ (VIII) can be written as

$$y = a \sin 2\pi \left(\frac{t}{T} + \frac{x}{\lambda} \right) \quad \text{--- (IX)}$$

$$\therefore y = a \sin \frac{2\pi}{\lambda} (vt + x) \quad \text{--- (X)}$$

✦ particle velocity and Wave Velocity :-

The eqⁿ for simple harmonic progressive waves is given by

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \text{--- (1)}$$

Here,

$v \rightarrow$ velocity of wave

$y \rightarrow$ displacement of the particle

Then the velocity of particle is

$$u = \frac{dy}{dx}$$

eqⁿ ① is diffⁿ wrto t

$$u = \frac{dy}{dt} = \frac{d}{dt} \left[\frac{a \sin 2\pi (vt-x)}{\lambda} \right]$$

$$= \frac{a \cos 2\pi (vt-x) \cdot 2\pi v}{\lambda} - 0$$

$$= \frac{2\pi av}{\lambda} \cos 2\pi (vt-x)$$

$$u = \frac{2\pi av}{\lambda} \cos 2\pi (vt-x) \dots \dots \textcircled{2}$$

The maximum velocity of the particle velocity is

$$u_{max} = \frac{2\pi av}{\lambda} \dots \dots \textcircled{3}$$

Particle velocity = Wave velocity $\times$ slope of displacement curve.

$$u_{max} = \frac{2\pi av}{\lambda} \times \text{Wave velocity}$$

To find the particle accelⁿ:-

Diffⁿ eqⁿ ② wrto t, we get

$$f = \frac{d^2y}{dt^2}$$

$$f = \frac{d}{dt} \left[\frac{2\pi v}{\lambda} \cdot \frac{a \cos 2\pi (vt-x)}{\lambda} \right]$$

$$= \frac{2\pi v}{\lambda} \cdot - \frac{a \sin 2\pi (vt-x) \cdot 2\pi v}{\lambda} - 0$$

$$\frac{d^2y}{dt^2} = f = - \frac{4\pi^2 v^2}{\lambda^2} a \sin 2\pi (vt-x)$$

f $\rightarrow$ acceleration of the particle.

$$F = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) y \quad \dots \dots \textcircled{iv}$$

put $y = a$

$$F = -\left(\frac{4\pi^2 v^2}{\lambda^2}\right) a \quad \dots \dots \textcircled{v}$$

eqⁿ $\textcircled{v}$ is the acceleration particle is maximum.

The negative sign in show that the acceleration of particle is directed in towards mean position.

Diff' eqⁿ $\textcircled{iii}$ wrto 'x'. we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[a \sin \frac{2\pi}{\lambda} (vt - x) \right]$$

$$\frac{dy}{dx} = a \cos \frac{2\pi}{\lambda} (vt - x) \left(-1 \cdot \frac{2\pi}{\lambda} \right)$$

$$\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots \dots \textcircled{vi}$$

The $\frac{dy}{dx}$ is the represent slope of the displacement of the wave.

from eqⁿ $\textcircled{iv}$ & $\textcircled{vi}$

$$\frac{dy}{dt} = \frac{dy}{dx}$$

$$\frac{204\pi}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

$$v \frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x)$$

$$v = -1$$

$$\frac{dy}{dt} = -v \frac{dy}{dx} \quad \dots \dots \textcircled{iii}$$

particle velocity = wave velocity $\times$ slope of the displacement curve at that instant

* Differential equation of Wave Motion:-

The general eqⁿ of simple harmonic progressive wave is -

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots \dots \textcircled{i}$$

then, the diffⁿ wrt to 't' in eqⁿ ①, we get,

$$\frac{dy}{dt} = a \cos \frac{2\pi}{\lambda} (vt - x) \cdot \frac{2\pi}{\lambda} v = 0$$

$$\frac{dy}{dt} = \frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots \dots \textcircled{ii}$$

then, again diffⁿ eqⁿ ② wrt to 't', we get

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left[\frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]$$

$$= - \frac{2\pi a v}{\lambda} \sin \frac{2\pi}{\lambda} (vt - x) \cdot \frac{2\pi v}{\lambda}$$

$$\frac{d^2y}{dt^2} = - \frac{4\pi^2 a v^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots \dots \textcircled{iii}$$

To find the wave velocity:

diff eqⁿ ① wr.to 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx} \left[a \sin \frac{2\pi}{\lambda} (vt-x) \right]$$

$$\frac{dy}{dx} = a \cos \frac{2\pi}{\lambda} (vt-x) \cdot \frac{2\pi}{\lambda} (-1)$$

$$\boxed{\frac{dy}{dx} = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt-x)} \quad \dots\dots \textcircled{2}$$

Again diff eqⁿ ② wr.to 'x'

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[-\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt-x) \right]$$

$$= -\frac{2\pi a}{\lambda} -\sin \frac{2\pi}{\lambda} (vt-x) \cdot \frac{2\pi}{\lambda} (-1)$$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt-x)} \quad \dots\dots \textcircled{3}$$

We know that the rate of change of compression wr.to distance.

from eqⁿ ② & ③

$$\frac{dy}{dt} = \frac{dy}{dx}$$

$$\frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt-x) = -\frac{2\pi a}{\lambda} \cos \frac{2\pi}{\lambda} (vt-x)$$

$$v = -1$$

$$\frac{dy}{dt} = -v \frac{dy}{dx} \quad \dots\dots (vi)$$

From (ii) + Eqn (vi)

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}$$

$$\frac{-4\pi^2 v^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt+x) = \frac{-4\pi^2 a}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt+x)$$

$$v^2 = 1$$

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2} \quad \dots\dots (vii)$$

The above eqn can be written as

$$\frac{d^2y}{dt^2} = k \frac{d^2y}{dx^2}$$

Here, $k = v^2$

$$\sqrt{k} = v$$

Thus, knowing the value of k is the value of the wave velocity can be calculated.

* Physical condition due to disturbance q_1 at some point in the medium is propagated motion of matter is propagated.

* If n particles in medium, then $\frac{1}{n}$ sec required to travel from one particle to the another particle.

→* Define :-

a) Compressions :-

The particle moves in the direction as the propagation of wave and it is an a region of compression.

b) Rarefaction :-

When the particle moves from opposit direction of propagation of wave, it is an region of rarefaction.

→* Energy of a progressive wave :-

i) In this case of a progressive wave, new waves are continuously formed the head of the wave. This means that there is a continuous transfer of energy in the direction of propagation of the wave.

ii) This energy supplied from the source. The energy transferred per second and also corresponds to the energy possessed by the particles in a length v , where $v \rightarrow$ velocity of the wave.

i) The energy of the particles is partly kinetic & partly potential. The kinetic energy is due to the velocity of the vibrating particles. For a particle executing simple harmonic motion, the velocity is maximum at the mean position and it is zero at the extreme positions.

ii) Consequently the kinetic energy of the particle at the mean position. At the extreme positions, the potential energy of the particle is maximum and at the mean position the potential energy is minimum.

Analytical Treatment:-

The equation of simple harmonic wave is

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad \dots\dots \textcircled{1}$$

The particle velocity (v) at any instant obtained by diff'g $\textcircled{1}$ equation wrto t ,

$$v = \frac{dy}{dt} = a \cos \frac{2\pi}{\lambda} (vt - x) \cdot \frac{2\pi}{\lambda} (v)$$

$$\frac{dy}{dt} = \frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \quad \dots\dots \textcircled{2}$$

The acceleration of the particle at instant,

$$f = \frac{dv}{dt}$$

$$f = \frac{d^2y}{dt^2}$$

diff'g eqⁿ $\textcircled{2}$ wrto t we get

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left[\frac{2\pi av}{\lambda} \cos \frac{2\pi}{\lambda} (vt - x) \right]$$

$$= \frac{2\pi av}{\lambda} - \sin \frac{2\pi}{\lambda} (vt-x) \cdot \frac{2\pi}{\lambda} (v)$$

$$\boxed{\frac{d^2 y}{dt^2} = -\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt-x)} \quad \dots \dots (iii)$$

The negative sign shows that acceleration is directed towards the mean position.

Potential Energy :-

To move the particle from its mean position to a distance y , work has to be done against acceleration.

Work done for a displacement $dy = F dy$

Let, ρ be the density of the medium.

Work done per unit volume for a displacement dy

$$dy = \rho \left[\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt-x) \right] dy$$

The total workdone for the displacement y

$$y = \int_0^y \left[\frac{4\pi^2 av^2}{\lambda^2} \sin \frac{2\pi}{\lambda} (vt-x) \right] dy$$

$$y = \int_0^y \left[\frac{4\pi^2 \rho v^2}{\lambda^2} \cdot y \right] dy$$

$$\left[\because y = a \sin \frac{2\pi}{\lambda} (vt-x) \right]$$

∴ the potential energy per unit volume

$$= \left(\frac{4\pi^2 \rho v^2}{\lambda^2} \right) \int_0^y y \, dy$$

$$= \frac{4\pi^2 \rho v^2}{\lambda^2} \times \frac{y^2}{2}$$

$$= \frac{2\pi^2 \rho v^2 y^2}{\lambda^2}$$

$$= \frac{2\pi^2 \rho v^2}{\lambda^2} \cdot \frac{a^2 \sin^2 \frac{2\pi}{\lambda} (vt-x)}{1}$$

$$\boxed{PE = \frac{2\pi^2 \rho v^2 a^2 \sin^2 \frac{2\pi}{\lambda} (vt-x)}{\lambda^2}} \quad \dots\dots (iv)$$

Kinetic energy :-

∴ the kinetic energy is per unit volume

$$KE = \frac{1}{2} \rho v^2$$

$$KE = \frac{1}{2} \rho \left[\frac{2\pi a v}{\lambda} \cos \frac{2\pi}{\lambda} (vt-x) \right]^2$$

$$KE = \frac{1}{2} \cdot \frac{4\pi^2 a^2 v^2}{\lambda^2} \cdot \rho \cos^2 \left[\frac{2\pi}{\lambda} (vt-x) \right]$$

$$\boxed{K.E = \frac{2\pi^2 a^2 v^2 \rho}{\lambda^2} \cos^2 \left[\frac{2\pi}{\lambda} (vt-x) \right]}$$

..... (v)

The total energy per unit volume.

$$E = P.E + K.E$$

$$E = \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \sin^2 \left[\frac{2\pi}{\lambda} (vt - x) \right] + \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \cos^2 \left[\frac{2\pi}{\lambda} (vt - x) \right]$$

$$E = \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \left[\sin^2 \frac{2\pi}{\lambda} (vt - x) + \cos^2 \frac{2\pi}{\lambda} (vt - x) \right]$$

$$E = \frac{2\pi^2 v^2 a^2 \rho}{\lambda^2} \quad \text{--- (i)} \quad \left[\because \sin^2 x + \cos^2 x = 1 \right]$$

But $v = n\lambda$

$$E = 2\pi^2 \rho n^2 a^2 \quad \text{--- (ii)}$$

The kinetic energy per unit volume and the potential energy volume are equal to the half of the total energy per unit volume.

$$\therefore \text{The K.E per unit volume} = \pi^2 \rho n^2 a^2$$

$$\therefore \text{The P.E per unit volume} = \pi^2 \rho n^2 a^2$$

$$\begin{aligned} \text{The energy transfer per unit area per second} \\ &= E \times v \\ &= 2\pi^2 \rho n^2 a^2 v \end{aligned}$$

The energy is transfer per second is called energy current per unit area of a cross section.

Q6

Potential Energy:-

$$Sp = \frac{1}{2} \times \text{stress} \times \text{strain}$$

$$\text{stress} = E \times \text{strain}$$

$$Sp = \frac{1}{2} \times E \times \text{strain} \times \text{stress} \cdot \delta x$$

$$= \frac{1}{2} \times E \times \left(\frac{dy}{dx}\right) \times \left(\frac{dy}{dx}\right) \delta x$$

$$Sp = \frac{1}{2} \times E \times \left(\frac{dy}{dx}\right)^2 \delta x \quad \dots \text{--- (i)}$$

$$Sp = \frac{1}{2} \times v^2 \rho \cdot \frac{4\pi^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{v^2 \rho}{2} \cdot \frac{4\pi^2 a^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{v^2 \rho}{2} \cdot \frac{4\pi^2 a^2}{\lambda^2} \int \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{v^2 \rho}{2} \cdot \frac{4\pi^2 a^2}{\lambda^2} \cdot \frac{1}{2} \int 2 \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$[2 \cos^2 \theta = 1 + \cos 2\theta]$$

$$= \frac{v^2 \rho}{2} \cdot \frac{4\pi^2 a^2}{\lambda^2} \cdot \frac{1}{2} \int 1 + \cos \frac{4\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{v^2 \rho}{4} \cdot \frac{4\pi^2 a^2}{\lambda^2} \left[\int 1 \delta x + \int \cos \frac{4\pi}{\lambda} (vt - x) \delta x \right]$$

$$= \frac{v^2 \rho}{4} \cdot \frac{4\pi^2 a^2}{\lambda^2} \left[x + -\sin \frac{4\pi}{\lambda} (vt - x) \right]$$

$$\delta p = -\frac{\rho}{\lambda} \frac{4\pi^2 a^2}{\lambda^2} x + \text{C.O.D}$$

$$[\sin 4\pi = 0]$$

$$\boxed{\delta p = \frac{\rho}{4} \left(\frac{-2\pi a v}{\lambda} \right)^2} \dots \dots \textcircled{vi}$$

$$PE = \frac{1}{2} \rho \delta x \left(\frac{dy}{dt} \right)^2$$

$$= \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^l \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \frac{1}{2} \int_0^l 2 \cos^2 \frac{2\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{1}{2} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \frac{1}{2} \int_0^l [1 + \cos \frac{4\pi}{\lambda} (vt - x)] \delta x$$

$$[2 \cos^2 \theta = 1 + \cos 2\theta]$$

$$= \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^l 1 \delta x + \int_0^l \cos \frac{4\pi}{\lambda} (vt - x) \delta x$$

$$= \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 \left[x \right]_0^l + \left[\frac{\sin \frac{4\pi}{\lambda} (vt - x)}{\frac{4\pi}{\lambda}} \right]_0^l$$

$$= \frac{1}{4} \rho \left(\frac{2\pi a v}{\lambda} \right)^2 l - 0$$

$$[\sin 4\pi = 0]$$

$$P.E = \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 \quad \dots (vii)$$

$$P.E = \rho \pi^2 n^2 a^2$$

Kinetic Energy :-

$$\Delta E = \frac{2\pi^2 \rho a^2 v^2}{\lambda^2} \cos^2 \frac{2\pi}{\lambda} (vt - x) \quad \dots (viii)$$

$$= F \times (dy)^2$$

$$= \frac{\rho}{2} \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^y \sin^2 \frac{2\pi}{\lambda} (vt - x) dy$$

$$= \frac{\rho}{2} \left(\frac{2\pi a v}{\lambda} \right)^2 \cdot \frac{1}{2} \int_0^y 2 \sin^2 \frac{2\pi}{\lambda} (vt - x) dy$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^y 2 \sin^2 \frac{2\pi}{\lambda} (vt - x) dy$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^y [1 - \cos \frac{4\pi}{\lambda} (vt - x)] dy$$

$[2 \sin^2 \theta = 1 - \cos 2\theta]$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 \int_0^y 1 dy - \int_0^y \cos \frac{4\pi}{\lambda} (vt - x) dy$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 \left[y \right]_0^y + \frac{\sin \frac{4\pi}{\lambda} (vt - x)}{\frac{4\pi}{\lambda}}$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 y_0^4 + 0$$

$$[\sin 4\pi = 0]$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 y$$

$$K.E = \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2$$

----- (vi)

$$K.E = \pi^2 \rho n^2 a^2$$

$$T.E = P.E + K.E$$

$$= \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2 + \frac{\rho}{4} \left(\frac{2\pi a v}{\lambda} \right)^2$$

$$= \frac{1}{4} + \frac{1}{4} \left[\rho \left(\frac{2\pi a v}{\lambda} \right)^2 \right]$$

$$T.E = \frac{\rho}{2} \left(\frac{2\pi a v}{\lambda} \right)^2$$

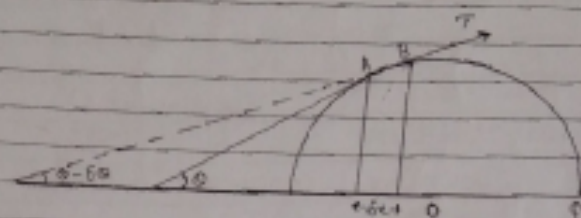
or

$$T.E = K.E + P.E$$

$$= \pi^2 \rho n^2 a^2 + \pi^2 \rho n^2 a^2$$

$$T.E = 2\pi^2 \rho n^2 a^2$$

Equation of Motion of a vibrating string:-



Consider a string PQ stretched by a tension T. Let the string be plucked at the centre O and left free. The string vibrates transversely. These vibrations are simple harmonic in nature. The force tending to bring any element of the string back to the equilibrium position is the component of tension acting at right angles to PQ.

Let the undisplaced position of the string correspond to x-axis and let the displacement be along the y-axis. Consider a small element AB of the length Δx . The tangents drawn at A and B make angles θ and $\theta + \delta\theta$ with the x-axis. The tension at A is resolved into two rectangular components. The downward component of tension at A = $T \sin \theta$.

As θ is small,

$$\sin \theta = \tan \theta$$

$\therefore$ Downward Component of tension.
at A = $T \tan \theta$

But

$$\tan \theta = \frac{dy}{dx} \quad \text{at } A$$

(ie slope at A)

∴ Downward component of tension at A

$$= T \frac{dy}{dx} \quad \dots\dots \textcircled{1}$$

The rate of change of slope with respect to the length of the element,

$$= \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$$= \frac{d^2y}{dx^2}$$

The change in slope for a distance δx

$$= \frac{d^2y}{dx^2} \cdot \delta x$$

The slope at the point B

$$\tan (\theta - \theta_0) = \frac{dy}{dx} + \frac{d^2y}{dx^2} \cdot \delta x$$

The upward component of tension acting at B
 $= T \tan (\theta - \delta\theta)$

$$= T \left[\frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot \delta x \right] \dots\dots (ii)$$

From eqⁿ (i) and (ii), resultant downward tension,

$$T = T \frac{dy}{dx} - T \left(\frac{dy}{dx} - \frac{d^2y}{dx^2} \cdot \delta x \right)$$

$$F = T \frac{d^2y}{dx^2} \cdot \delta x \dots\dots (iii)$$

Let,

m be the mass of the string per unit length.

mass of the element = $m \delta x$

Acceleration of the element in the direction of y-axis

$$\frac{d^2y}{dt^2}$$

Force acting on the element = mass $\times$ accelⁿ

$$F = m \delta x \cdot \frac{d^2y}{dt^2} \dots\dots (iv)$$

From eqⁿ (iii) and (iv)

$$m \delta x \cdot \frac{d^2y}{dt^2} = T \frac{d^2y}{dx^2} \cdot \delta x$$

$$\frac{d^2y}{dt^2} = \frac{T}{m} \frac{d^2y}{dx^2} \dots\dots (v)$$

This is similar to the differential equation of wave motion.

$$\frac{d^2 y}{dt^2} = v^2 \frac{d^2 y}{dx^2} \quad \dots\dots (v)$$

Here, v is the velocity of propagation of the wave.

Comparing equation (v) and (vi).

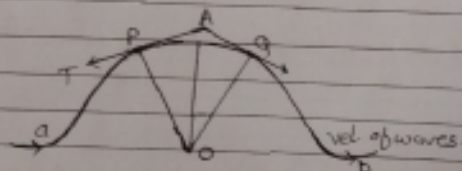
$$v^2 = \frac{T}{m}$$

$$v = \sqrt{\frac{T}{m}}$$

$$v = n\lambda$$

$$\dots\dots (vii)$$

Velocity of Transverse Waves along a string



Let us now derive, from elementary principles, an expression for the velocity v with which a transverse wave pulse travels along a stretched string.

Eg:- might be the pulse set up in a violin string by light plucking or that sent along a taut rope giving one end a quick sideways jerk. We shall assume that the string, wire or rope is perfectly flexible, and uniform and that the displacements are small and the effect of reflections at the ends is neglected.

Suppose the wave pulse travels along a string from left to right with velocity v . Let us imagine the entire string to be moved from right to left with the same speed so, that the wave form remains fixed while.

The particles comprising the string successively round the curve.

Consider a small position PQ of the string at the crest and having a length Δx . It forms the arc of a circle of which O is the centre, such that

$$\angle POQ = \theta$$

The mass of this portion of the string is $m \Delta x$, if m be the mass per unit length, and since it is moving in a circle with speed v , the magnitude of the centripetal force acting on it is -

$$m \Delta x \times \frac{v^2}{R}$$

Where, R - is the radius of the circle.

This force acts along AO , which bisects the angle POQ and is produced by the resultant of the tension

T acting tangentially at the ends of the portion PQ . Resolving the tensions along AO and perpendicular to it, we find the sum of the components of the tensions along AO is $2T \cos \angle PAO = 2T \sin \frac{\theta}{2}$

Where as the components perpendicular to AO being equal and into opposite directions cancel out.

Since θ is a very small

We may write $\frac{\theta}{2}$ then

$\sin \frac{\theta}{2}$ and the resultant force due to the tension is, ²

therefore equal to 10

As this resultant force is supplying the necessary centripetal acceleration to the portion PQ to go round the curve, we have

$$T\theta = m\delta x \cdot \frac{v^2}{R}$$

$$\text{put } \theta = \frac{\delta x}{R}$$

$$T \cdot \frac{\delta x}{R} = m\delta x \frac{v^2}{R}$$

$$T = mv^2$$

$$\frac{T}{m} = v^2$$

$$\therefore v = \sqrt{\frac{T}{m}}$$

Thus the transverse wave will travel along a string with velocity

$$\therefore v = \sqrt{\frac{T}{m}}$$

Where,

$T \rightarrow$ stretching force

$m \rightarrow$ mass per unit length

GURUKUL Page No.
Date

Frequency and Period of Vibration of a string:-

Let us consider a string fixed at a both ends. if it is plucked by the application of a simple harmonic force. Two humps start one in either direction but on arrival at the fixed end it is unable to draw the fixed support. to its side and thus it causes a rebound and the hump is reflected.

So a pulse starts in a reverse directions and goes to the other end, gets reflected there and arrives at the original point after traversing twice the length of the string. If during this time the point has executed one complete vibration. the harmonic force communicated to the point will find the incoming wave to be in phase with it. The wave will be reinforced and vibrations will thus be maintained till the loss of energy due to friction and dissipation brings it to rest.

Here then during one time period the distance traversed by the waves is twice the length of the string.

$$\text{so that, } \lambda = 2l$$

where, l is the length of the string

$$\text{Now } v = n\lambda$$

$$n = \frac{v}{\lambda}$$

$$n = \frac{v}{2l}$$

$$n = \frac{1}{2l} \sqrt{\frac{T}{m}}$$

The period of vibration t of the fundamental mode of vibration given by

$$t = \frac{1}{n} = 2l \sqrt{\frac{T}{m}}$$



The Laws of vibration of strings:-

The laws of vibrations of strings embodied in eqⁿ (i) were first discovered by Martin Mersenne of Paris in 1636.

i) Law of length:-

For a given tension the frequency of a string varies inversely as the length

$$n \propto \frac{1}{L}$$

ii) Law Tension:-

The frequency varies directly proportional to square root of the stretching force.

$$n \propto \sqrt{T}$$

iii) Law of mass:-

The fundamental frequency is inversely proportional to square root of mass per unit length.

$$n \propto \frac{1}{\sqrt{m}}$$